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SOME RESULTS ON RANDOM COINCIDENCE POINTS OF COMPLETELY RANDOM OPERATORS

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Abstract The purpose of this paper is to examine the notion of completely random operators and to present some results on the existence of random coincidence points of completely random operators. Some applications to random fixed point theorems and random equations are given.

Keywords Random operators · Completely random operators · Random fixed points · Random coincidence points

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, X, Y be separable metric spaces, and $f: \Omega \times X \to Y$ be a random operator in the sense that for each fixed x in X, the mapping $f(\cdot, x): \omega \mapsto f(\omega, x)$ is measurable. The random operator f is said to be continuous if for each ω in Ω, the mapping $f(\omega, \cdot): x \mapsto f(\omega, x)$ is continuous. An X-valued random variable ξ is said to be a random fixed point of the random operator $f: \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = \xi(\omega)$ a.s., and an X-valued random variable ξ is said to be a random coincidence point of the random operators $f, g: \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = g(\omega, \xi(\omega))$ a.s.

The theory of random fixed points and random coincidence points is an important topic of stochastic analysis and has been investigated by various authors (see, e.g., [2–6, 8–12]).

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For continuous random operators, in [14, Theorem 2.3], it was shown that if X, Y are Polish spaces and f, g are continuous random operators, the random equation $f(\omega, x) = g(\omega, x)$ has a random solution if and only if the deterministic equation $f(\omega, \cdot) = g(\omega, \cdot)$ has a solution for almost all ω. From this it follows that if X is a Polish space and $f, g: \Omega \times X \to X$ are two continuous random operators, then f, g have a random coincidence point if and only if for almost all ω, the deterministic mappings $f(\omega, \cdot)$ and $g(\omega, \cdot)$ have a coincidence point. Therefore, the results on the random coincidence points follow immediately from the results on the corresponding deterministic coincidence points.

In this paper, we are concerned with a mapping $\Phi: L_0^X(\Omega) \to L_0^Y(\Omega)$. Since a random operator f can be viewed as an action that transforms each deterministic input x in X into a random output $f(x)$ in $L_0^Y(\Omega)$, whereas $\Phi: L_0^X(\Omega) \to L_0^Y(\Omega)$ can be viewed as an action that transforms each random input u in $L_0^X(\Omega)$ into a random output Φu in $L_0^Y(\Omega)$, we call Φ a completely random operator. In Sect. 2, we present some properties of completely random operators. Section 3 deals with the notion of a random coincidence point of completely random operators and gives some conditions ensuring the existence of a random coincidence point of completely random operators. It should be noted that the existence of a random coincidence point of completely random operators does not follow from the existence of the corresponding deterministic coincidence point theorem as in the case of random operators. In Sect. 4, some applications to random fixed point theorems and random equations are presented.

2 Some properties of completely random operators

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and X be a separable Banach space. A mapping $\xi: \Omega \to X$ is called an X-valued random variable if ξ is $(\mathcal{F}, \mathcal{B}(X))$-measurable, where $\mathcal{B}(X)$ denotes the Borel σ-algebra of X. The set of all (equivalent classes) X-valued random variables is denoted by $L_0^X(\Omega)$, and it is equipped with the topology of convergence in probability. For each $p > 0$, the set of X-valued random variables ξ such that $E\|\xi\|^p < \infty$ is denoted by $L_p^X(\Omega)$.

First, recall the following (see, e.g., [13]).

Definition 1 Let X, Y be two separable Banach spaces.

1. A mapping $f: \Omega \times X \to Y$ is said to be a random operator if for each fixed x in X, the mapping $\omega \mapsto f(\omega, x)$ is measurable.
2. The random operator $f: \Omega \times X \to Y$ is said to be continuous if for each ω in Ω, the mapping $x \mapsto f(\omega, x)$ is continuous.
3. Let $f, g: \Omega \times X \to Y$ be two random operators. The random operator g is said to be a modification of f if for each x in X, we have $f(\omega, x) = g(\omega, x)$ a.s.

   Note that the exceptional set may depend on x.

The notion of a completely random operator is defined as follows.

Definition 2 Let X, Y be two separable Banach spaces.

1. A mapping $\Phi: L_0^X(\Omega) \to L_0^Y(\Omega)$ is called a completely random operator.
2. A completely random operator Φ is said to be continuous if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ a.s., we have $\lim \Phi u_n = \Phi u$ a.s.

3. A completely random operator Φ is said to be continuous in probability if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ in probability, we have $\lim \Phi u_n = \Phi u$ in probability.
4. A completely random operator Φ is said to be an extension of a random operator $f\colon \Omega \times X \to Y$ if for each x in X,

$$\Phi x(\omega) = f(\omega, x) \quad \text{a.s.},$$

where for each x in X, x denotes the random variable u in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

Theorem 1 *Let $f : \Omega \times X \to Y$ be a random operator admitting a continuous modification. Then there exists a continuous completely random operator $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ such that Φ is an extension of f.*

Proof Let g be a continuous modification of f. Define $\Phi\colon L_0^X(\Omega) \to L_0^Y(\Omega)$ by

$$\Phi u(\omega) = g\big(\omega, u(\omega)\big) \tag{1}$$

for each random variable u in $L_0^X(\Omega)$. This mapping is well defined. Indeed, by [7, Theorem 6.1], $g\colon \Omega \times X \to Y$ is measurable, and hence $\omega \mapsto g(\omega, u(\omega))$ is measurable. Next, we have to show that if h is another continuous modification of f, then

$$g\big(\omega, u(\omega)\big) = h\big(\omega, u(\omega)\big) \quad \text{a.s.}$$

By the separability of X there exists a sequence (x_n) dense in X. For each x_n, there exists a set Ω_n of probability one such that $g(\omega, x_n) = h(\omega, x_n)$ for all ω in Ω_n. Let $\Omega_0 = \bigcap_{n=1}^{\infty} \Omega_n$. Clearly, Ω_0 has probability one, and we have

$$g(\omega, x_n) = h(\omega, x_n) \quad \forall \omega \in \Omega_0 \ \forall n. \tag{2}$$

Fix ω in Ω_0. By the density of (x_n) in X, there exists a subsequence (x_{n_k}) converging to $u(\omega)$. By the continuity of the mappings $x \mapsto g(\omega, x)$ and $x \mapsto h(\omega, x)$ we have

$$\lim_{k\to\infty} g(\omega, x_{n_k}) = g\big(\omega, u(\omega)\big), \qquad \lim_{k\to\infty} h(\omega, x_{n_k}) = h\big(\omega, u(\omega)\big). \tag{3}$$

By (2) and (3) we conclude that $h(\omega, \xi(\omega)) = g(\omega, \xi(\omega))$ for all ω in Ω_0, as claimed.

By (1) it is easy to show that the completely random operator Φ is continuous and is an extension of f. □

Theorem 2 *Let $\Phi, \Psi : L_0^X(\Omega) \to L_0^Y(\Omega)$ be probabilistic completely random operators, and Ψ be continuous in probability. Assume that there exists a positive random variable $k(\omega)$ such that for each pair u, v in $L_0^X(\Omega)$ and all $t > 0$, we have*

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(k(\omega)\|\Psi u - \Psi v\| > t\big). \tag{4}$$

Then Φ is also continuous in probability.

Proof For all u, v in $L_0^X(\Omega)$, we have

$$\begin{aligned} P\big(\|\Phi u - \Phi v\| > t\big) &\le P\big(k(\omega)\|\Psi u - \Psi v\| > t\big) \\ &= P\big(k(\omega)\|\Psi u - \Psi v\| > t, \|\Psi u - \Psi v\| \le r\big) + P\big(\|\Psi u - \Psi v\| > r\big) \\ &\le P\big(rk(\omega) > t\big) + P\big(\|\Psi u - \Psi v\| > r\big) \\ &= P\big(k(\omega) > t/r\big) + P\big(\|\Psi u - \Psi v\| > r\big). \end{aligned}$$

Suppose that p-$\lim u_n = u$. Then we have

$$P\big(\|\Phi u_n - \Phi u\| > t\big) \le P\big(k(\omega) > t/r\big) + P\big(\|\Psi u_n - \Psi u\| > r\big).$$

So, for each $r > 0$,

$$\begin{aligned} \limsup_n P\big(\|\Phi u_n - \Phi u\| > t\big) &\le P\big(k(\omega) > t/r\big) + \limsup_n P\big(\|\Psi u_n - \Psi u\| > t\big) \\ &= P\big(k(\omega) > t/r\big). \end{aligned}$$

Letting $r \to 0$, we get

$$\limsup_n P\big(\|\Phi u_n - \Phi u\| > t\big) = 0.$$

Therefore, Φ is continuous in probability. □

3 Random coincidence points of completely random operators

Let f, g: $\Omega \times X \to X$ be random operators. Recall (see, e.g., [1, 2, 4, 12]) that an X-valued random variable ξ is said to be a random fixed point of the random operator f if

$$f\big(\omega, \xi(\omega)\big) = \xi(\omega) \quad \text{a.s.}$$

An X-valued random variable u^* is said to be a random coincidence point of two random operators f, g if

$$f\big(\omega, u^*(\omega)\big) = g\big(\omega, u^*(\omega)\big) \quad \text{a.s.}$$

Assume that f, g are continuous. Then, by Theorem 1 the mappings Φ, Ψ: $L_0^X(\Omega) \to L_0^X(\Omega)$ defined respectively by

$$\Phi u(\omega) = f\big(\omega, u(\omega)\big), \qquad \Psi u(\omega) = g\big(\omega, u(\omega)\big)$$

are completely random operators extending f and g, respectively. For each random fixed point ξ of f, we get

$$\Phi\xi(\omega) = \xi(\omega) \quad \text{a.s.},$$

and for each random coincidence point u^* of two random operators f, g, we have

$$\Phi u^*(\omega) = \Psi u^*(\omega) \quad \text{a.s.}$$

This leads us to the next definition.

Definition 3

1. Let Φ: $L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An X-valued random variable ξ in $L_0^X(\Omega)$ is called a random fixed point of Φ if

$$\Phi\xi = \xi.$$

2. Let $\Phi_1, \Phi_2, \dots, \Phi_n$: $L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators. An X-valued random variable u^* in $L_0^X(\Omega)$ is called a random coincidence point of $\Phi_1, \Phi_2, \dots, \Phi_n$ if

$$\Phi_1 u^* = \Phi_2 u^* = \cdots = \Phi_n u^*. \tag{5}$$

We are going to present some conditions ensuring the existence of a random coincidence point of completely random operators.

Theorem 3 *Let $\Phi, \Psi : L_0^X(\Omega) \to L_0^X(\Omega)$ be random operators, and $f : [0, \infty) \to [0, \infty)$ be a mapping such that $f(t) = 0$ if and only if $t = 0$ and $f(t) < t$ for all $t > 0$. For each $t > 0$, define*

$$h(t) = \inf_{s \ge t} \frac{f(s)}{s}. \tag{6}$$

Assume that $h(t) > 0$ for all $t > 0$ and

(a) *$\Psi(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset \Psi(L_0^X(\Omega))$;*
(c) *for each pair u, v in $L_0^X(\Omega)$ and all $t > 0$, we have*

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(\|\Psi u - \Psi v\| - f\big(\|\Psi u - \Psi v\|\big) > t\big). \tag{7}$$

Then Φ, Ψ have a random coincidence point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that

$$M = E\|\Phi u_0 - \Psi u_0\|^p < \infty. \tag{8}$$

Proof If Φ, Ψ have a coincidence point u^*, then (8) holds with $u_0 = u^*$ for any $p > 0$.

Conversely, suppose that $E\|\Phi u_0 - \Psi u_0\|^p < \infty$ for some random variable u_0 in $L_0^X(\Omega)$ and $p > 0$. By assumption (b) there exists a random variable u_1 in $L_0^X(\Omega)$ such that $\Psi u_1 = \Phi u_0$. Again, there exists a random variable u_2 in $L_0^X(\Omega)$ such that $\Psi u_2 = \Phi u_1$. By induction, there exists a sequence (u_n) in $L_0^X(\Omega)$ such that

$$\Psi u_n = \Phi u_{n-1}, \quad n = 1, 2, \dots. \tag{9}$$

We will show that (ξ_n) given by $\xi_n = \Psi u_n = \Phi u_{n-1}$ $(n = 1, 2, \dots)$ is a Cauchy sequence in $L_0^X(\Omega)$. Define the function $g(t)$, $t > 0$, by

$$g(t) = 1 - \frac{f(t)}{t}.$$

So, we have

$$f(t) = \big(1 - g(t)\big)t.$$

Since $f(t) > 0$ for all $t > 0$, we get $g(t) < 1$ for all $t > 0$. For any u, v in $L_0^X(\Omega)$, we have

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(\|\Psi u - \Psi v\| - f\big(\|\Psi u - \Psi v\|\big) > t\big).$$

Equivalently,

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(g\big(\|\Psi u - \Psi v\|\big)\|\Psi u - \Psi v\| > t\big). \tag{10}$$

Fix $t > 0$. For all $s \ge t > 0$, we have

$$g(s) = 1 - \frac{f(s)}{s} \le 1 - h(t) = q(t).$$

Since $g(t) < 1$, we get

$$\big\{g\big(\|\Psi u - \Psi v\|\big)\|\Psi u - \Psi v\| > t\big\} \subset \big\{\|\Psi u - \Psi v\| > t\big\}.$$

Hence,

$$\begin{aligned}
P\big(\|\Phi u - \Phi v\| > t\big) &\le P\big(g\big(\|\Psi u - \Psi v\|\big)\|\Psi u - \Psi v\| > t\big)\\
&= P\big(g\big(\|\Psi u - \Psi v\|\big)\|\Psi u - \Psi v\| > t, \|\Psi u - \Psi v\| > t\big)\\
&\le P\big(q(t)\|\Psi u - \Psi v\| > t, \|\Psi u - \Psi v\| > t\big)\\
&\le P\big(q(t)\|\Psi u - \Psi v\| > t\big)\\
&= P\big(\|\Psi u - \Psi v\| > t/q(t)\big) = P\big(\|\Psi u - \Psi v\| > t/q\big),
\end{aligned}$$

where $q = q(t)$. Note that $q < 1$ since $h(t) > 0$.

Thus, for each n, we obtain

$$\begin{aligned}
P\big(\|\xi_{n+1} - \xi_n\| > t\big) &= P\big(\|\Phi u_n - \Phi u_{n-1}\| > t\big)\\
&\le P\big(\|\Psi u_n - \Psi u_{n-1}\| > t/q\big)\\
&= P\big(\|\xi_n - \xi_{n-1}\| > t\big).
\end{aligned}$$

By induction and the Chebyshev inequality we get

$$\begin{aligned}
P\big(\|\xi_{n+1} - \xi_n\| > t\big) &\le P\big(\|\xi_n - \xi_{n-1}\| > t/q\big)\\
&\le \cdots\\
&\le P\big(\|\xi_2 - \xi_1\| > t/q^{n-1}\big)\\
&= P\big(\|\Phi u_1 - \Phi u_0\| > t/q^{n-1}\big)\\
&\le P\big(\|\Psi u_1 - \Psi u_0\| > t/q^n\big)\\
&= P\big(\|\Phi u_0 - \Psi u_0\| > t/q^n\big)\\
&\le E\|\Phi u_0 - \Psi u_0\|^p \frac{(q^n)^p}{t^p} = M\frac{(q^n)^p}{t^p}.
\end{aligned}$$

Let $r = \frac{x}{q}$, where $q < x < 1$. Then $r > 1$ and

$$(r-1)\left(\frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^m}\right) + \frac{1}{r^m} = 1 \quad \forall m \geq 1.$$

Thus, for any $t > 0$ and m, n in N, we have

$$\begin{aligned}
P\big(\|\xi_{n+m} - \xi_n\| > t\big) &\leq P\big(\|\xi_{n+m} - \xi_n\| > \big(1 - 1/r^m\big)t\big) \\
&\leq P\big(\|\xi_{n+m} - \xi_{n+m-1}\| > t(r-1)/_{r^m}\big) + \cdots \\
&\quad + P\big(\|\xi_{n+1} - \xi_n\| > t(r-1)/_r\big) \\
&\leq \frac{M}{[(r-1)t]^p}\big[\big(r^m\big)^p\big(q^{n+m-1}\big)^p + \cdots + r^p\big(q^n\big)^p\big] \\
&= \frac{M}{[(r-1)t]^p}\big(q^n\big)^p r^p\big[(qr)^{p(m-1)} + \cdots + (qr)^p + 1\big] \\
&= \frac{M}{[(r-1)t]^p}\big(q^n\big)^p r^p \frac{1 - (qr)^{mp}}{1 - (qr)^p} \\
&< \frac{Mr^p}{[(r-1)t]^p[1 - (qr)^p]}\big(q^p\big)^n,
\end{aligned}$$

which tends to 0 as $n \to \infty$. It follows that (ξ_n) is a Cauchy sequence in $L_0^X(\Omega)$. Hence, there exists ξ in $L_0^X(\Omega)$ such that $p\text{-}\lim \xi_n = \xi$. By assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $\Psi u^* = \xi$. So we have

$$\begin{aligned}
P\big(\big\|\xi_{n+1} - \Phi u^*\big\| > t\big) &= P\big(\big\|\Psi u_{n+1} - \Phi u^*\big\| > t\big) \\
&= P\big(\big\|\Phi u_n - \Phi u^*\big\| > t\big) \\
&\leq P\big(\big\|\Psi u_n - \Psi u^*\big\| - f\big(\big\|\Psi u_n - \Psi u^*\big\|\big) > t\big) \\
&\leq P\big(\big\|\Psi u_n - \Psi u^*\big\| > t\big) = P\big(\|\xi_n - \xi\| > t\big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
P\big(\big\|\xi - \Phi u^*\big\| > t\big) &\leq P\big(\|\xi - \xi_{n+1}\| > t/2\big) + P\big(\big\|\xi_{n+1} - \Phi u^*\big\| > t/2\big) \\
&\leq P\big(\|\xi - \xi_{n+1}\| > t/2\big) + P\big(\|\xi_n - \xi\| > t/2\big).
\end{aligned}$$

Letting $n \to \infty$, we get $P(\|\xi - \Phi u^*\| > t) = 0$ for all $t > 0$, which implies that $\Phi u^* = \xi = \Psi u^*$. Hence, u^* is a random coincidence point of Φ, Ψ. □

Corollary 1 *Let Φ, Ψ be completely random operators satisfying conditions* (a) *and* (b) *stated in Theorem 3. Assume that there exists a number q in* $(0, 1)$ *such that*

$$P\big(\|\Phi u - \Phi v\| > t\big) \leq P\big(\|\Psi u - \Psi v\| > t/q\big) \tag{11}$$

for all random variables u, v in $L_0^X(\Omega)$ and $t > 0$. Then Φ, Ψ have a random coincidence point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that (8) *holds.*

Proof Consider the functions $f(t)=(1-q)t$ and $h(t)=1-q>0$. Then $f(t)$ satisfies the conditions stated in Theorem 3. □

Remark 1 The next example shows that a random coincidence point of Φ and Ψ in Theorem 3 needs not to be unique.

Example 1 Define two completely random operators $\Phi, \Psi\colon L_0^R(\Omega) \to L_0^R(\Omega)$ by

$$\Phi u = q|u| + \eta, \qquad \Psi u = |u|,$$

where η is a positive random variable, and q is in $(0, 1)$. It is easy to check that Φ, Ψ satisfy all assumptions of Theorem 3 with $f(t)=(1-q)t$. On the other hand, Φ and Ψ have two random coincidence points $u_1^* = \frac{1}{1-q}\eta$ and $u_2^* = -\frac{1}{1-q}\eta$.

Theorem 4 *Let $f:[0,\infty)\to[0,\infty)$ be a function such that $f(0)=0$ and $f(t)<t$. Put*

$$h(t) = \inf_{s\geq t}\frac{f(s)}{s} \quad \forall t>0. \tag{12}$$

Assume that $h(t)>0$ for all $t>0$, Φ, Ψ, and Θ are three probabilistic completely random operators satisfying the following conditions

(a) *$\Theta(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset \Theta(L_0^X(\Omega))$, $\Psi(L_0^X(\Omega)) \subset \Theta(L_0^X(\Omega))$;*
(c) *for any random variables u, v in $L_0^X(\Omega)$ and $t>0$, we have*

$$P\big(\|\Phi u - \Psi v\| > t\big) \leq P\big(\|\Theta u - \Theta v\| - f\big(\|\Theta u - \Theta v\|\big) > t\big). \tag{13}$$

Then Φ, Ψ, and Θ have a random coincidence point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p>0$ such that

$$M = E\|\Phi u_0 - \Theta u_0\|^p < \infty. \tag{14}$$

Proof If Φ, Ψ, and Θ have a random coincidence point u^*, then (14) holds with $u_0 = u^*$ for any $p>0$.

Conversely, suppose that $E\|\Phi u_0 - \Theta u_0\|^p < \infty$ for some random variable u_0 in $L_0^X(\Omega)$ and $p>0$. By assumption (b) there exists a random variable u_1 in $L_0^X(\Omega)$ such that $\Theta u_1 = \Phi u_0$. Again, there exists a random variable u_2 in $L_0^X(\Omega)$ such that $\Theta u_2 = \Psi u_1$. By induction there exists a sequence (u_n) in $L_0^X(\Omega)$ such that

$$\begin{aligned} &\Theta u_1 = \Phi u_0, \qquad \Theta u_2 = \Psi u_1, \ldots, \qquad \Theta u_{2n+1} = \Phi u_{2n}, \\ &\Theta u_{2n+2} = \Psi u_{2n+1}, \quad n=1,2,\ldots. \end{aligned} \tag{15}$$

We will show that (ξ_n) given by $\xi_n = \Theta u_{n-1}$ $(n=2,3,\ldots)$ in (15) is a Cauchy sequence in $L_0^X(\Omega)$. Define the function $g(t)$, $t>0$, by

$$g(t) = 1 - \frac{f(t)}{t}.$$

We have

$$f(t) = \big(1-g(t)\big)t.$$

Since $f(t) > 0$ for all $t > 0$, we get $g(t) < 1$ for all $t > 0$. For any random variables u, v in $L_0^X(\Omega)$, we have

$$P\big(\|\Phi u - \Psi v\| > t\big) \leq P\big(\|\Theta u - \Theta v\| - f(\|\Theta u - \Theta v\|) > t\big).$$

Equivalently,

$$P\big(\|\Phi u - \Psi v\| > t\big) \leq P\big(g\big(\|\Theta u - \Theta v\|\big)\|\Theta u - \Theta v\| > t\big). \tag{16}$$

Fix $t > 0$. For all $s \geq t > 0$, we have

$$g(s) = 1 - \frac{f(s)}{s} \leq 1 - h(t) = q(t).$$

Since $g(t) < 1$, we get

$$\big\{g\big(\|\Theta u - \Theta v\|\big)\|\Theta u - \Theta v\| > t\big\} \subset \big\{\|\Theta u - \Theta v\| > t\big\}.$$

Hence,

$$\begin{aligned}
P\big(\|\Phi u - \Psi v\| > t\big) &\leq P\big(g\big(\|\Theta u - \Theta v\|\big)\|\Theta u - \Theta v\| > t\big) \\
&= P\big(g\big(\|\Theta u - \Theta v\|\big)\|\Theta u - \Theta v\| > t, \|\Theta u - \Theta v\| > t\big) \\
&\leq P\big(q(t)\|\Theta u - \Theta v\| > t, \|\Theta u - \Theta v\| > t\big) \\
&\leq P\big(q(t)\|\Theta u - \Psi v\| > t\big) \\
&= P\big(\|\Theta u - \Theta v\| > t/q(t)\big) = P\big(\|\Theta u - \Theta v\| > t/q\big),
\end{aligned}$$

where $q = q(t)$. Note that $q < 1$ since $h(t) > 0$.

Then, for each n, we obtain

$$\begin{aligned}
P\big(\|\xi_{2n+2} - \xi_{2n+1}\| > t\big) &= P\big(\|\Phi u_{2n} - \Psi u_{2n-1}\| > t\big) \\
&\leq P\big(\|\Theta u_{2n} - \Theta u_{2n-1}\| > t/q\big) \\
&= P\big(\|\xi_{2n+1} - \xi_{2n}\| > t/q\big)
\end{aligned}$$

and

$$\begin{aligned}
P\big(\|\xi_{2n+1} - \xi_{2n}\| > t\big) &= P\big(\|\Phi u_{2n-2} - \Psi u_{2n-1}\| > t\big) \\
&\leq P\big(\|\Theta u_{2n-2} - \Theta u_{2n-1}\| > t/q\big) \\
&= P\big(\|\xi_{2n} - \xi_{2n-1}\| > t/q\big).
\end{aligned}$$

By induction and by the Chebyshev inequality we get

$$\begin{aligned}
P\big(\|\xi_{n+1} - \xi_n\| > t\big) &\leq P\big(\|\xi_n - \xi_{n-1}\| > t/q\big) \\
&\leq \cdots \\
&\leq P\big(\|\xi_3 - \xi_2\| > t/q^{n-2}\big) \\
&= P\big(\|\Psi u_1 - \Phi u_0\| > t/q^{n-1}\big)
\end{aligned}$$

$$\begin{aligned}&\le P\big(\|\Theta u_1-\Theta u_0\|>t/q^{n-1}\big)\\&=P\big(\|\Phi u_0-\Theta u_0\|>t/q^{n-1}\big)\\&\le E\|\Phi u_0-\Theta u_0\|^p\frac{(q^{n-1})^p}{t^p}=M\frac{(q^{n-1})^p}{t^p}.\end{aligned}$$

Let $r=\frac{x}{q}$, where $q<x<1$. Then $r>1$ and

$$(r-1)\left(\frac{1}{r}+\frac{1}{r^2}+\cdots+\frac{1}{r^m}\right)+\frac{1}{r^m}=1\qquad\forall m\ge 1.$$

Thus, for any $t>0$, $n\ge 2$, and m in N, we have

$$\begin{aligned}P\big(\|\xi_{n+m}-\xi_n\|>t\big)&\le P\left(\|\xi_{n+m}-\xi_n\|>\left(1-\frac{1}{r^m}\right)t\right)\\&\le P\big(\|\xi_{n+m}-\xi_{n+m-1}\|>t(r-1)/_{r^m}\big)\\&\quad+\cdots+P\big(\|\xi_{n+1}-\xi_n\|>t(r-1)/_r\big)\\&\le\frac{M}{[(r-1)t]^p}\big[\big(r^m\big)^p\big(q^{n+m-2}\big)^p+\cdots+r^p\big(q^{n-1}\big)^p\big]\\&=\frac{M}{[(r-1)t]^p}\big(q^{n-1}\big)^pr^p\big[(qr)^{p(m-1)}+\cdots+(qr)^p+1\big]\\&=\frac{M}{[(r-1)t]^p}\big(q^{n-1}\big)^pr^p\frac{1-(qr)^{m-1p}}{1-(qr)^p}\\&<\frac{Mr^p}{[(r-1)t]^p[1-(qr)^p]}\big(q^p\big)^{n-1},\end{aligned}$$

which tends to 0 as $n\to\infty$. It implies that (ξ_n) is a Cauchy sequence in $L_0^X(\Omega)$. Hence, there exists ξ in $L_0^X(\Omega)$ such that p-$\lim\xi_n=\xi$. By assumption (a) there exists u^* in $L_0^X(\Omega)$ such that $\Theta u^*=\xi$. So we have

$$\begin{aligned}P\big(\big\|\Theta u_{2n+1}-\Psi u^*\big\|>t\big)&=P\big(\big\|\Phi u_{2n}-\Psi u^*\big\|>t\big)\\&\le P\big(\big\|\Theta u_{2n}-\Theta u^*\big\|>t\big)\\&\le P\big(\big\|\Theta u_{2n}-\Theta u^*\big\|-f\big(\big\|\Theta u_{2n}-\Theta u^*\big\|\big)>t\big)\\&\le P\big(\big\|\Theta u_{2n}-\Theta u^*\big\|>t/q\big)\\&=P\big(\|\xi_{2n+1}-\xi\|>t/q\big).\end{aligned}$$

Letting $n\to\infty$, we get $P(\|\xi-\Psi u^*\|>t)=0$, which implies that $\Psi u^*=\xi$ a.s. By the same proof we have $\Phi u^*=\xi$ a.s. Hence, u^* is a random coincidence point of Φ,Ψ, and Θ. □

Theorem 5 *Let Φ,Ψ: $L_0^X(\Omega)\to L_0^X(\Omega)$ be completely random operators*, f: $[0,\infty)\to[0,\infty)$ *be a continuous*, *increasing function such that* $f(0)=0$ *and* $\lim_{t\to\infty}f(t)=\infty$, *and* q *be a positive number in* $(0,1)$. *Assume that*

(a) *$\Psi(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset \Psi(L_0^X(\Omega))$;*
(c) *for any u, v in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P\big(\|\Phi u - \Phi v\| > f(t)\big) \le P\big(\|\Psi u - \Psi v\| > f(t/q)\big). \tag{17}$$

The following assertions are valid:

1. *If Φ, Ψ have a random coincidence point, then there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$M = \sup_{t>0} t^p P\big(\|\Phi u_0 - \Psi u_0\| > f(t)\big) < \infty. \tag{18}$$

2. *If there exists a number c in $(q, 1)$ such that*

$$\sum_{n=1}^{\infty} f\big(c^n\big) < \infty, \tag{19}$$

then condition (18) is sufficient for Φ, Ψ to have a random coincidence point.
3. *If for all $t, s > 0$,*

$$f(t+s) \ge f(t) + f(s), \tag{20}$$

then condition (18) is also sufficient for Φ, Ψ to have a random coincidence point.

Proof Let $g = f^{-1}$ be the inverse function of f. Then g: $[0, \infty) \to [0, \infty)$ is increasing with $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$. Condition (17) is equivalent to

$$P\big(g\big(\|\Phi u - \Phi v\|\big) > t\big) \le P\big(g\big(\|\Psi u - \Psi v\|\big) > t/q\big). \tag{21}$$

Let u_0 be a random variable in $L_0^X(\Omega)$ such that (18) holds. By assumption (b) there exists a random variable u_1 in $L_0^X(\Omega)$ such that $\Psi u_1 = \Phi u_0$. Again, there exists a random variable u_2 in $L_0^X(\Omega)$ such that $\Psi u_2 = \Phi u_1$. By induction there exists a sequence (u_n) in $L_0^X(\Omega)$ such that

$$\Psi u_n = \Phi u_{n-1}, \quad n = 1, 2, \ldots. \tag{22}$$

Put $\xi_n = \Psi u_n = \Phi u_{n-1}, n = 1, 2, \ldots$. From (21) it follows that

$$\begin{aligned} P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > t\big) &= P\big(g\big(\|\Phi u_n - \Phi u_{n-1}\|\big) > t\big) \\ &\le P\big(g\big(\|\Psi u_n - \Psi u_{n-1}\|\big) > t/q\big) \\ &= P\big(g\big(\|\xi_n - \xi_{n-1}\|\big) > t/q\big). \end{aligned}$$

By induction, for each n, we obtain

$$\begin{aligned} P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > t\big) &\le P\big(g\big(\|\Psi u_1 - \Psi u_0\|\big) > t/q^n\big) \\ &= P\big(g\big(\|\Phi u_0 - \Psi u_0\|\big) > t/q^n\big). \end{aligned} \tag{23}$$

1. Suppose that Φ, Ψ have a random coincidence point u^*. Then taking $u_0 = u^*$, we obtain $M = 0$.
2. From (18) we have

$$P\big(g\big(\|\Phi u_0 - \Psi u_0\|\big) > s\big) = P\big(g\big(\|\Psi u_1 - \Psi u_0\|\big) > s\big) \le \frac{M}{s^p}. \tag{24}$$

From (23) and (24) we get

$$P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > t\big) \le \frac{Mq^{np}}{t^p}. \tag{25}$$

Taking $t = c^n$, from (25) we get

$$P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > c^n\big) \le M\frac{q^{np}}{c^{np}}, \tag{26}$$

i.e.,

$$P\big(\|\xi_{n+1} - \xi_n\| > f\big(c^n\big)\big) \le M\frac{q^{np}}{c^{np}}. \tag{27}$$

Since

$$\sum_{n=1}^{\infty} P\big(\|\xi_{n+1} - \xi_n\| > f\big(c^n\big)\big) \le M\sum_{n=1}^{\infty}\frac{q^{np}}{c^{np}} < \infty,$$

by the Borel–Cantelli lemma there is a set D with probability one such that for each ω in D, there is $N(\omega)$ satisfying

$$\big\|\xi_{n+1}(\omega) - \xi_n(\omega)\big\| \le f\big(c^n\big) \quad \forall n > N(\omega).$$

By (19) we conclude that $\sum_{n=1}^{\infty} \|\xi_{n+1}(\omega) - \xi_n(\omega)\| < \infty$ for all ω in D, which implies that there exists $\lim \xi_n(\omega)$ for all ω in D. Consequently, the sequence (ξ_n) converges a.s. to ξ in $L_0^X(\Omega)$. By assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $\Psi u^* = \xi$. So we have

$$\begin{aligned}
P\big(\big\|\Psi u_{n+1} - \Phi u^*\big\| > f(t)\big) &= P\big(\big\|\Phi u_n - \Phi u^*\big\| > f(t)\big) \\
&\le P\big(\big\|\Psi u_n - \Psi u^*\big\| > f(t/q)\big) \\
&= P\big(\|\xi_n - \xi\| > f(t/q)\big).
\end{aligned}$$

Letting $n \to \infty$, we get $P(\|\xi - \Phi u^*\| > f(t)) = 0$ for all $t > 0$; hence, $\Phi u^* = \xi = \Psi u^*$ a.s. Thus, u^* is a random coincidence point of Φ, Ψ.
3. It is easy to see that, for any $t, s > 0$,

$$g(s+t) \le g(t) + g(s).$$

Hence, for $a = \sum_{i=1}^{m} s_i$, we have

$$\begin{aligned}P\big(g\big(\|\xi_{n+m}-\xi_n\|\big)>a\big)&\le P\left(g\left(\sum_{i=1}^m\|\xi_{n+i}-\xi_{n+i-1}\|\right)>a\right)\\&\le P\left(\sum_{i=1}^m g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>a\right)\\&\le\sum_{i=1}^m P\big(g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>s_i\big).\end{aligned}$$

From (25) it follows that

$$P\big(g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>s_i\big)\le\frac{Mq^{(n+i-1)p}}{s_i^p}. \tag{28}$$

Put $r=\frac{x}{q}$, where $q<x<1$ and $s_i=s(r-1)/r^i$. An argument similar to that in the proof of Theorem 3 yields

$$\lim_{n\to\infty}P\big(g\big(\|\xi_{n+m}-\xi_n\|\big)>s\big)=0\quad\forall s>0,$$

so

$$\lim_{n\to\infty}P\big(\|\xi_{n+m}-\xi_n\|>f(s)\big)=0\quad\forall s>0.$$

Thus, we obtain

$$\lim_{n\to\infty}P\big(\|\xi_{n+m}-\xi_n\|>t\big)=0\quad\forall t>0.$$

Consequently, the sequence (ξ_n) converges in probability to ξ in $L_0^X(\Omega)$. By assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $\Psi u^*=\xi$. So, we have

$$\begin{aligned}P\big(\big\|\Psi u_{n+1}-\Phi u^*\big\|>f(t)\big)&=P\big(\big\|\Phi u_n-\Phi u^*\big\|>f(t)\big)\\&\le P\big(\big\|\Psi u_n-\Psi u^*\big\|>f(t/q)\big)\\&=P\big(\|\xi_n-\xi\|>f(t/q)\big).\end{aligned}$$

Letting $n\to\infty$, we get $P(\|\xi-\Phi u^*\|>f(t))=0$ for all $t>0$ implying $\Phi u^*=\xi=\Psi u^*$ a.s. Hence, u^* is a random coincidence point of Φ,Ψ. □

Theorem 6 *Let Φ,Ψ,Θ: $L_0^X(\Omega)\to L_0^X(\Omega)$ be completely random operators, f: $[0,\infty)\to[0,\infty)$ a continuous increasing function such that $f(0)=0$, $\lim_{t\to\infty}f(t)=\infty$, and q be a positive number in $(0,1)$. Assume that*

(a) *$\Theta(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega))\subset\Theta(L_0^X(\Omega))$, $\Psi(L_0^X(\Omega))\subset\Theta(L_0^X(\Omega))$;*
(c) *for any random variables u,v in $L_0^X(\Omega)$ and $t>0$, we have*

$$P\big(\|\Phi u-\Psi v\|>f(t)\big)\le P\big(\|\Theta u-\Theta v\|>f(t/q)\big). \tag{29}$$

Then

1. *If Φ, Ψ, and Θ have a random coincidence point, then there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$M = \sup_{t>0} t^p P\big(\|\Phi u_0 - \Theta u_0\| > f(t)\big) < \infty. \tag{30}$$

2. *Assume that there exists a number c in $(q, 1)$ such that*

$$\sum_{n=1}^{\infty} f\big(c^n\big) < \infty. \tag{31}$$

Then condition (30) is sufficient for Φ, Ψ, and Θ to have a random coincidence point.

3. *Assume that for any $t, s > 0$*

$$f(t+s) \geq f(t) + f(s). \tag{32}$$

Then condition (30) is also sufficient for Φ, Ψ, and Θ to have a random coincidence point.

Proof Let $g = f^{-1}$ be the inverse function of f. Then, g: $[0, \infty) \to [0, \infty)$ is increasing with $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$. Condition (29) is equivalent to

$$P\big(g(\|\Phi u - \Psi v\|) > t\big) \leq P\big(g(\|\Theta u - \Theta v\|) > t/q\big). \tag{33}$$

Let u_0 be a random variable in $L_0^X(\Omega)$ such that (30) holds. By assumption (b) there exists a random variable u_1 in $L_0^X(\Omega)$ such that $\Theta u_1 = \Phi u_0$. Again, there exists a random variable u_2 in $L_0^X(\Omega)$ such that $\Theta u_2 = \Psi u_1$. By induction there exists a sequence (u_n) in $L_0^X(\Omega)$ with

$$\begin{aligned} &\Theta u_1 = \Phi u_0, \qquad \Theta u_2 = \Psi u_1, \ldots, \qquad \Theta u_{2n+1} = \Phi u_{2n}, \\ &\Theta u_{2n+2} = \Psi u_{2n+1}, \quad n = 1, 2, \ldots. \end{aligned} \tag{34}$$

Put $\xi_n = \Theta u_{n-1},\ n = 2, 3, \ldots$. From (33), for each n, we obtain

$$\begin{aligned} P\big(g\big(\|\xi_{2n+2} - \xi_{2n+1}\|\big) > t\big) &= P\big(g\big(\|\Phi u_{2n} - \Psi u_{2n-1}\|\big) > t\big) \\ &\leq P\big(g\big(\|\Theta u_{2n} - \Theta u_{2n-1}\|\big) > t/q\big) \\ &= P\big(g\big(\|\xi_{2n+1} - \xi_{2n}\|\big) > t/q\big) \end{aligned}$$

and

$$\begin{aligned} P\big(g\big(\|\xi_{2n+1} - \xi_{2n}\|\big) > t\big) &= P\big(g\big(\|\Phi u_{2n-2} - \Psi u_{2n-1}\|\big) > t\big) \\ &\leq P\big(g\big(\|\Theta u_{2n-2} - \Theta u_{2n-1}\|\big) > t/q\big) \\ &= P\big(g\big(\|\xi_{2n} - \xi_{2n-1}\|\big) > t/q\big). \end{aligned}$$

By induction we obtain that, for each n,

$$\begin{aligned} P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > t\big) &\leq P\big(g\big(\|\Theta u_1 - \Theta u_0\|\big) > t/q^n\big) \\ &= P\big(g\big(\|\Phi u_0 - \Theta u_0\|\big) > t/q^n\big). \end{aligned} \tag{35}$$

1. Suppose that Φ, Ψ, and Θ have a random coincidence point u^*. Then taking $u_0 = u^*$, we obtain $M = 0$.
2. From (30) we have

$$P\big(g\big(\|\Phi u_0 - \Theta u_0\|\big) > s\big) = P\big(\|\Phi u_0 - \Theta u_0\| > f(s)\big) \le \frac{M}{s^p}. \tag{36}$$

From (35) and (36) we get

$$P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > t\big) \le \frac{M q^{np}}{t^p}. \tag{37}$$

Taking $t = c^n$, from (37) we get

$$P\big(g\big(\|\xi_{n+1} - \xi_n\|\big) > c^n\big) \le M \frac{q^{np}}{c^{np}}, \tag{38}$$

i.e.,

$$P\big(\|\xi_{n+1} - \xi_n\| > f\big(c^n\big)\big) \le M \frac{q^{np}}{c^{np}}. \tag{39}$$

Since

$$\sum_{n=1}^{\infty} P\big(\|\xi_{n+1} - \xi_n\| > f\big(c^n\big)\big) \le M \sum_{n=1}^{\infty} \frac{q^{np}}{c^{np}} < \infty,$$

by the Borel–Cantelli lemma, there is a set D with probability one such that for each ω in D, there is $N(\omega)$ with

$$\big\|\xi_{n+1}(\omega) - \xi_n(\omega)\big\| \le f\big(c^n\big) \quad \forall n > N(\omega).$$

By (31) we conclude that $\sum_{n=1}^{\infty} \|\xi_{n+1}(\omega) - \xi_n(\omega)\| < \infty$ for all ω in D, which implies that there exists $\lim \xi_n(\omega)$ for all ω in D. Consequently, the sequence (ξ_n) converges a.s. to ξ in $L_0^X(\Omega)$. By assumption (a) there exists u^* in $L_0^X(\Omega)$ with $\Theta u^* = \xi$. So we have

$$\begin{aligned} P\big(\big\|\xi_{2n+2} - \Psi u^*\big\| > f(t)\big) &= P\big(\big\|\Phi u_{2n} - \Psi u^*\big\| > f(t)\big) \\ &\le P\big(\big\|\Theta u_{2n} - \Theta u^*\big\| > f(t/q)\big) \\ &= P\big(\|\xi_{2n+1} - \xi\| > f(t/q)\big). \end{aligned}$$

Letting $n \to \infty$, we get $P(\|\xi - \Psi u^*\| > f(t)) = 0$ for all $t > 0$ implying $\Psi u^* = \xi = \Theta u^*$ a.s. By the same proof we have $\Phi u^* = \xi$ a.s. Hence, u^* is a random coincidence point of Φ, Ψ, and Θ.
3. It is easy to see that, for any $t, s > 0$,

$$g(s + t) \le g(t) + g(s).$$

Hence, for $a = \sum_{i=1}^{m} s_i$, we have

$$
\begin{aligned}
P\big(g\big(\|\xi_{n+m}-\xi_n\|\big)>a\big) &\le P\left(g\left(\sum_{i=1}^{m}\|\xi_{n+i}-\xi_{n+i-1}\|\right)>a\right)\\
&\le P\left(\sum_{i=1}^{m} g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>a\right)\\
&\le \sum_{i=1}^{m} P\big(g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>s_i\big).
\end{aligned}
$$

From (30) we have

$$
P\big(g\big(\|\xi_{n+i}-\xi_{n+i-1}\|\big)>s_i\big) \le \frac{Mq^{(n+i-1)p}}{s_i^p}. \tag{40}
$$

Put $r=\frac{x}{q}$, where $q<x<1$ and $s_i=s(r-1)/r^i$. An argument similar to that in the proof of Theorem 3 yields

$$
\lim_{n\to\infty} P\big(g\big(\|\xi_{n+m}-\xi_n\|\big)>s\big)=0 \quad \forall s>0,
$$

so

$$
\lim_{n\to\infty} P\big(\|\xi_{n+m}-\xi_n\|>f(s)\big)=0 \quad \forall s>0.
$$

Thus, we obtain

$$
\lim_{n\to\infty} P\big(\|\xi_{n+m}-\xi_n\|>t\big)=0 \quad \forall t>0.
$$

Consequently, the sequence (ξ_n) converges in probability to ξ in $L_0^X(\Omega)$. By assumption (a) there exists u^* in $L_0^X(\Omega)$ satisfying $\Theta u^*=\xi$. So, we have

$$
\begin{aligned}
P\big(\big\|\xi_{2n+2}-\Psi u^*\big\|>f(t)\big) &= P\big(\big\|\Phi u_{2n}-\Psi u^*\big\|>f(t)\big)\\
&\le P\big(\big\|\Theta u_{2n}-\Theta u^*\big\|>f(t/q)\big)\\
&= P\big(\|\xi_{2n+1}-\xi\|>f(t/q)\big).
\end{aligned}
$$

Letting $n\to\infty$, we get $P(\|\xi-\Psi u^*\|>f(t))=0$ for all $t>0$, implying $\Psi u^*=\xi=\Theta u^*$ a.s. By the same proof we have $\Phi u^*=\xi$ a.s. Hence, u^* is a random coincidence point of Φ, Ψ, and Θ.

□

4 Applications to random fixed point theorems and random equations

In this section, we present some applications of the obtained results to random fixed point theorems and random equations.

Theorem 7 *Let Φ: $L_0^X(\Omega)\to L_0^X(\Omega)$ be a completely random operator, f: $[0,\infty)\to[0,\infty)$ be a continuous, increasing function such that $f(0)=0$, $\lim_{t\to\infty}f(t)=\infty$, and q be a positive number. Assume that*

$$
P\big(\|\Phi u-\Phi v\|>f(t)\big)\le P\big(\|u-v\|>f(t/q)\big) \tag{41}
$$

for each pair u, v in $L_0^X(\Omega)$. Then

1. *If Φ has a random fixed point, then the random fixed point is unique. Moreover, there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$M = \sup_{t>0} t^p P\big(\|\Phi u_0 - u_0\| > f(t)\big) < \infty. \tag{42}$$

2. *Assume that there exists a number c in $(q, 1)$ satisfying*

$$\sum_{n=1}^{\infty} f\big(c^n\big) < \infty. \tag{43}$$

   Then condition (42) is sufficient for Φ to have a unique random fixed point.
3. *Assume that, for any $t, s > 0$,*

$$f(t+s) \geq f(t) + f(s). \tag{44}$$

   Then condition (42) is also sufficient for Φ to have a unique random fixed point.

Proof Let ξ, η be two random fixed points of Φ. Then, for each $t > 0$, we have

$$P\big(\|\xi - \eta\| > f(t)\big) = P\big(\|\Phi\xi - \Phi\eta\| > f(t)\big) \leq P\big(\|\xi - \eta\| > f(t/q)\big).$$

By induction it follows that

$$P\big(\|\xi - \eta\| > f(t)\big) \leq P\big(\|\xi - \eta\| > f\big(t/q^n\big)\big) \quad \forall n.$$

Since $\lim_{n\to\infty} f(t/q^n) = +\infty$, we conclude that $P(\|\xi - \eta\| > f(t)) = 0$ for each $t > 0$. Hence, $g(\|\xi - \eta\|) = 0$ a.s., with g being the inverse function of f. So we have $\xi = \eta$ a.s. as claimed.

Suppose that Φ has a random fixed point ξ. Then, taking $u_0 = \xi$, we obtain $M = 0$.

Conversely, consider the completely random operator Ψ given by $\Psi u = u$. According to Theorem 5, Φ and Ψ have a random coincidence point ξ, which is exactly the random fixed point of Φ. □

Theorem 8 *Let $\Phi, \Psi : L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators, and $f : [0, \infty) \to [0, \infty)$ be a mapping such that $f(t) = 0$ if and only if $t = 0$ and $f(t) < t$ for all $t > 0$. For each $t > 0$, define*

$$h(t) = \inf_{s\geq t} \frac{f(s)}{s}. \tag{45}$$

Assume that $h(t) > 0$ for all $t > 0$ and

(a) *$\Psi(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset \Psi(L_0^X(\Omega))$;*
(c) *for each pair u, v in $L_0^X(\Omega)$ and all $t > 0$, we have*

$$P\big(\|\Phi u - \Phi v\| > t\big) \leq P\big(\|\Psi u - \Psi v\| - f\big(\|\Psi u - \Psi v\|\big) > t\big). \tag{46}$$

(d) *Φ, Ψ commute, i.e., $\Phi\Psi u = \Psi\Phi u$ for any random variable u in $L_0^X(\Omega)$.*

Then Φ and Ψ have a unique common random fixed point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that

$$M = E\|\Phi u_0 - \Psi u_0\|^p < \infty. \tag{47}$$

Proof If Φ and Ψ have a common random fixed point ξ, then (47) holds with $u_0 = \xi$ and any $p > 0$. Conversely, suppose that (47) holds. By Theorem 3 there exists u^* such that $\Phi u^* = \Psi u^* = \xi$. For $t > 0$, we have

$$\begin{aligned} P\big(\|\Phi\xi - \xi\| > t\big) &= P\big(\|\Phi\xi - \Phi u^*\| > t\big) \le P\big(\|\Psi\xi - \Psi u^*\| > t/q\big) \\ &= P\big(\|\Psi\Phi u^* - \xi\| > t/q\big) = P\big(\|\Phi\Psi u^* - \xi\| > t/q\big) \\ &= P\big(\|\Phi\xi - \xi\| > t/q\big). \end{aligned}$$

By induction it follows that $P(\|\Phi\xi - \xi\| > t) \le P(\|\Phi\xi - \xi\| > t/q^n)$ for any $n \in N$. Letting $n \to \infty$, we have $P(\|\Phi\xi - \xi\| > t) = 0$ for any $t > 0$. Thus, $\Phi\xi = \xi$, i.e., ξ is a random fixed point of Φ. We have $\Psi\xi = \Psi\Phi u^* = \Phi\Psi u^* = \Phi\xi = \xi$. Hence, ξ is also a random fixed point of Ψ.

Let ξ_1 and ξ_2 be two common random fixed points of Φ and Ψ. For each $t > 0$, we have

$$\begin{aligned} P\big(\|\xi_1 - \xi_2\| > t\big) &= P\big(\|\Phi\xi_1 - \Phi\xi_2\| > t\big) \le P\big(\|\Psi\xi_1 - \Psi\xi_2\| > t/q\big) \\ &= P\big(\|\xi_1 - \xi_2\| > t/q\big) \le \cdots \le P\big(\|\xi_1 - \xi_2\| > t/q^n\big). \end{aligned}$$

Letting $n \to \infty$, we have $P(\|\xi_1 - \xi_2\| > t) = 0$ for all $t > 0$. Hence, $\xi_1 = \xi_2$. □

Corollary 2 *Let Φ: $L_0^X(\Omega) \to L_0^X(\Omega)$ be a probabilistic q-contraction completely random operator in the sense that there exists a number q in $(0, 1)$ such that*

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(\|u - v\| > t/q\big)$$

for all random variables u, v in $L_0^X(\Omega)$ and $t > 0$. Then Φ has a unique random fixed point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that

$$E\|\Phi u_0 - u_0\|^p < \infty.$$

Proof Consider the operator Ψ: $L_0^X(\Omega) \to L_0^X(\Omega)$ given by $\Psi u = u$, the functions $f(t) = (1-q)t$ and $h(t) = 1 - q > 0$. It is clear that Φ, Ψ, and $f(t)$ satisfy the conditions stated in Theorem 8 and Φ, Ψ commute. Thus, Φ and Ψ have a common random fixed point ξ, i.e., Φ has a random fixed point ξ. □

Theorem 9 *Let Φ, Ψ: $L_0^X(\Omega) \to L_0^X(\Omega)$ be probabilistic completely random operators such that*

$$P\big(\|\Phi u - \Phi v\| > f(t)\big) \le P\big(\|\Psi u - \Psi v\| > f(t/q)\big) \tag{48}$$

for all u, v in $L_0^X(\Omega)$, $t > 0$, where f: $[0, \infty) \to [0, \infty)$ is a continuous increasing function such that $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$ satisfying either (43) or (44), and q is a positive number. Consider a random equation of the form

$$\Phi u - \lambda\Psi u = \eta, \tag{49}$$

where λ *is a real number, and* η *is a random variable in* $L_0^X(\Omega)$.

Assume that

$$\Psi\big(L_0^X(\Omega)\big) \quad \text{is closed in } L_0^X(\Omega), \tag{50}$$

$$\Phi\big(L_0^X(\Omega)\big) \subset \lambda\Psi\big(L_0^X(\Omega)\big) + \eta, \tag{51}$$

$$|\lambda| \geq \sup_{t>0} \frac{f(\frac{q}{q'}t)}{f(t)}, \tag{52}$$

where q' *is in* $(0,1)$. *Then Eq.* (49) *has a unique random solution if and only if there exist a random variable* u_0 *in* $L_0^X(\Omega)$ *and a number* $p>0$ *such that*

$$M = \sup_{t>0} t^p P\big(\|\Phi u_0 - \lambda\Psi u_0 - \eta\| > |\lambda| f(t)\big) < \infty. \tag{53}$$

Proof Suppose that Eq. (49) has a solution ξ. Then condition (53) holds for $u_0 = \xi$.

Conversely, suppose that condition (53) holds. Define the completely random operator Θ by

$$\Theta u = \frac{\Phi u - \eta}{\lambda}.$$

From (51) and (53) it follows that

$$\Theta\big(L_0^X(\Omega)\big) \subset \Psi\big(L_0^X(\Omega)\big), \qquad M = \sup_{t>0} t^p P\big(\|\Theta u_0 - \Psi u_0\| > f(t)\big) < \infty.$$

Let $g = f^{-1}$ be the inverse function of f. Then g: $[0,\infty) \to [0,\infty)$ is a continuous increasing function with $g(0)=0$ and $\lim_{t\to\infty} g(t) = \infty$. For each $t>0$, there exists t' with $f(t') = |\lambda| f(t)$, i.e., $t' = g(|\lambda| f(t))$. So we have

$$\begin{aligned}
P\big(\|\Theta u - \Theta v\| > f(t)\big) &= P\big(\|\Phi u - \Phi v\| > |\lambda| f(t)\big) \\
&= P\big(\|\Phi u - \Phi v\| > f\big(t'\big)\big) \\
&\leq P\big(\|\Psi u - \Psi v\| > f\big(t'/q\big)\big) \\
&= P\left(\|\Psi u - \Psi v\| > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right).
\end{aligned}$$

From (52) we get $|\lambda| f(t) \geq f(\frac{q}{q'}t)$. Then we deduce that $g(|\lambda| f(t)) \geq \frac{q}{q'}t$. So $t' \geq \frac{q}{q'}t$ and $\frac{q't'}{qt} \geq 1$. Hence, we get

$$P\left(\|\Psi u - \Psi v\| > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right) \leq P\big(\|\Psi u - \Psi v\| > f\big(t/q'\big)\big),$$

which implies

$$P\big(\|\Theta u - \Theta v\| > f(t)\big) \leq P\big(\|\Psi u - \Psi v\| > f\big(t/q'\big)\big).$$

Consequently, Θ and Ψ satisfy conditions (a)–(c) stated in Theorem 5. Thus, Θ and Ψ has a random coincidence point ξ, i.e., Eq. (49) has a random solution ξ. □

Corollary 3 *Let $\Phi\colon L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator satisfying the condition*

$$P\big(\|\Phi u - \Phi v\| > f(t)\big) \le P\big(\|u - v\| > f(t/q)\big) \tag{54}$$

for all u, v in $L_0^X(\Omega)$, $t > 0$, where $f\colon [0,\infty) \to [0,\infty)$ is a continuous increasing function such that $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$ satisfying either (43) or (44), and q is a positive number. Consider a random equation of the form

$$\Phi u - \lambda u = \eta, \tag{55}$$

where λ is a real number, and η is a random variable in $L_0^X(\Omega)$.

Assume that

$$|\lambda| \ge \sup_{t>0} \frac{f(\frac{q}{q'}t)}{f(t)}, \tag{56}$$

where q' is in $(0,1)$. Then Eq. (55) has a unique random solution if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and a number $p > 0$ such that

$$M = \sup_{t>0} t^p P\big(\|\Phi u_0 - \lambda u_0 - \eta\| > |\lambda| f(t)\big) < \infty. \tag{57}$$

Proof It suffices to apply Theorem 9 for the completely random operator Ψ given by $\Psi u = u$. □

Corollary 4 *Let $\Phi, \Psi\colon L_0^X(\Omega) \to L_0^X(\Omega)$ be two completely random operators satisfying the condition*

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(\|\Psi u - \Psi v\| > t/q\big). \tag{58}$$

Consider the random equation

$$\Phi u - \lambda \Psi u = \eta, \tag{59}$$

where λ is a real number, and η is a random variable in $L_p^X(\Omega)$, $p > 0$.

Assume that

$$\begin{aligned}
&\Psi\big(L_0^X(\Omega)\big) \quad \text{is closed in } L_0^X(\Omega),\\
&\Phi\big(L_0^X(\Omega)\big) \subset \lambda\Psi\big(L_0^X(\Omega)\big) + \eta,\\
&|\lambda| > q.
\end{aligned}$$

Then the random equation (59) has a solution if and only if there exists a random variable u_0 in $L_0^X(\Omega)$ such that

$$E\|\Phi u_0 - \lambda \Psi u_0\|^p < \infty. \tag{60}$$

Proof Suppose that Eq. (59) has a solution ξ. Then condition (60) holds for $u_0 = \xi$.

Conversely, suppose that there exists a random variable u_0 in $L_0^X(\Omega)$ such that (60) holds. So Φ and Ψ satisfy (54), where $f(t) = t$. Take $q < s < |\lambda|$ and observe that $q' = q/s < 1$,

$$|\lambda| > s = \frac{q}{q'} = \frac{f(\frac{q}{q'}t)}{f(t)}.$$

Moreover, for each $t > 0$,

$$t^p P\big(\|\Phi u_0 - \lambda \Psi u_0 - \eta\| > |\lambda| t\big) \le \frac{E\|\Phi u_0 - \lambda \Psi u_0 - \eta\|^p}{|\lambda|^p} < \infty$$

since

$$E\big(\|\Phi u_0 - \lambda \Psi u_0 - \eta\|^p\big) \le C_p E\big(\|\Phi u_0 - \lambda \Psi u_0\|^p\big) + C_p E\|\eta\|^p < \infty,$$

where C_p is a constant. Hence condition (53) is satisfied. By Theorem 9 we conclude that Eq. (59) has a random solution. □

Considering the completely random operator Ψ given by $\Psi u = u$, we obtain the following:

Corollary 5 *Let* Φ: $L_0^X(\Omega) \to L_0^X(\Omega)$ *be a completely random operator satisfying the condition*

$$P\big(\|\Phi u - \Phi v\| > t\big) \le P\big(\|u - v\| > t/q\big). \tag{61}$$

Consider the random equation

$$\Phi u - \lambda u = \eta, \tag{62}$$

where λ *is a real number satisfying* $|\lambda| > q$, *and* η *is a random variable in* $L_p^X(\Omega)$, $p > 0$. *Then the random equation (62) has a unique random solution if and only if there exists a random variable* u_0 *in* $L_0^X(\Omega)$ *such that*

$$E\|\Phi u_0 - \lambda u_0\|^p < \infty. \tag{63}$$

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