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EXISTENCE AND FINITE TIME APPROXIMATION OF STRONG SOLUTIONS TO 2D g-NAVIER–STOKES EQUATIONS

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Abstract Considered here is the first initial boundary value problem for the two-dimensional g-Navier–Stokes equations in bounded domains. We first prove the existence and uniqueness of strong solutions to the problem by using the Faedo–Galerkin method. Then we study the finite time numerical approximation of the strong solutions by discretization schemes.

Keywords g-Navier–Stokes equations · Strong solution · The Faedo–Galerkin method · Finite time approximation · Discretization schemes

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. We consider the following two-dimensional (2D) non-autonomous g-Navier–Stokes equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(x,t) & \text{in } (0,T) \times \Omega, \\ \nabla \cdot (gu) = 0 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \Gamma, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where $u = u(x,t) = (u_1, u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient and u_0 is the initial velocity.

The g-Navier–Stokes equations are a variation of the standard Navier–Stokes equations. More precisely, when $g \equiv$ const we get the usual Navier–Stokes equations. The 2D g-Navier–Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [13] for the derivation of the 2D g-Navier–Stokes equations from the 3D Navier–Stokes equations and the relationships between them. As mentioned in [12], the good properties of the 2D g-Navier–Stokes equations can lead to an initial study of the Navier–Stokes equations on the thin three dimensional domain $\Omega_g = \Omega \times (0, g)$. In the last few years, the existence and long-time behavior of weak solutions to 2D g-Navier–Stokes equations have been studied extensively in both cases without and with delays (see e.g. [1, 2, 7, 8, 11–14] and the references therein). However, to the best of our knowledge, little seems to be known about strong solutions of the 2D g-Navier–Stokes equations. This is a motivation of the present paper.

The aim of this paper is to study the existence and numerical approximations of strong solutions to the two-dimensional non-autonomous g-Navier–Stokes equations. To do this, we assume that the function g satisfies the following hypothesis:

(G) $g \in W^{1,\infty}(\Omega)$ is such that

$$0 < m_0 \le g(x) \le M_0 \quad \text{for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A defined in Sect. 2).

Let us describe the contents of the paper. First, we prove the existence, uniqueness and continuous dependence on the initial data of strong solutions to problem (1.1) by using the Faedo–Galerkin method. Second, we study the convergence of a space and time discretization scheme for the 2D evolution g-Navier–Stokes equations. This scheme combines a discretization in time by an alternating direction (or decomposition) method with a discretization in space by finite elements. The convergence problem is treated by energy methods (see [15, 16] for related results on the standard Navier–Stokes equations). We also refer the reader to [4–6, 9, 10] for several other recent results on numerical approximations of the standard Navier–Stokes equations.

It is noticed that in this paper we only consider the finite time approximation of strong solutions to problem (1.1). The long-time behavior and long-time approximation of the strong solutions are important questions because the problem of numerical computation of turbulent flows is connected with the computation of the solutions for large time, and they will be the subject of a forthcoming work.

The paper is organized as follows. In Sect. 2, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the g-Navier–Stokes equations. In Sect. 3, we prove the existence and uniqueness of a strong solution to the problem by using the Faedo–Galerkin method. The finite time approximation of the strong solution is studied in the last section by using discretization schemes.

2 Preliminary results

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g \, dx, \quad u, v \in L^2(\Omega, g),$$

and

$$\big((u,v)\big)_g=\int_\Omega\sum_{j=1}^2\nabla u_j\cdot\nabla v_j g\,dx,\quad u=(u_1,u_2),\ v=(v_1,v_2)\in H_0^1(\Omega,g),$$

and norms $|u|^2=(u,u)_g$, $\|u\|^2=((u,u))_g$. Thanks to the assumption (G), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$, respectively.

Let

$$\mathcal{V}=\big\{u\in\big(C_0^\infty(\Omega)\big)^2:\nabla\cdot(gu)=0\big\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega,g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega,g)$. It follows that $V_g\subset H_g\equiv H_g'\subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle\cdot,\cdot\rangle$ for duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u,v,w)=\sum_{i,j=1}^2\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j g\,dx,$$

whenever the integrals make sense. It is easy to check that if $u,v,w\in V_g$, then

$$b(u,v,w)=-b(u,w,v).$$

Hence

$$b(u,v,v)=0\quad\forall u,v\in V_g.$$

Set A: $V_g\to V_g'$ by $\langle Au,v\rangle=((u,v))_g$, B: $V_g\times V_g\to V_g'$ by

$$\big\langle B(u,v),w\big\rangle=b(u,v,w),$$

and put $Bu=B(u,u)$. Denote $D(A)=\{u\in V_g:Au\in H_g\}$, then

$$D(A)=H^2(\Omega,g)\cap V_g$$

and $Au=-P_g\Delta u\ \forall u\in D(A)$, where P_g is the ortho-projector from $L^2(\Omega,g)$ onto H_g.

Using the Hölder inequality, the Ladyzhenskaya inequality (when $n=2$):

$$|u|_{L^4}\le c|u|^{1/2}|\nabla u|^{1/2}\quad\forall u\in H_0^1(\Omega),$$

and the interpolation inequalities, as in [15] one can prove the following result.

Lemma 2.1 *If $n=2$, then*

$$\big|b(u,v,w)\big|\le\begin{cases}c_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} & \forall u,v,w\in V_g,\\ c_2|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| & \forall u\in V_g, v\in D(A), w\in H_g,\\ c_3|u|^{1/2}|Au|^{1/2}\|v\||w| & \forall u\in D(A), v\in V_g, w\in H_g,\\ c_4|u|\|v\||w|^{1/2}|Aw|^{1/2} & \forall u\in H_g, v\in V_g, w\in D(A),\end{cases}\tag{2.1}$$

where c_i, $i=1,\ldots,4$, are appropriate constants.

Lemma 2.2 *Let $u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V_g)$, then the function Bu defined by*

$$\big(Bu(t), v\big)_g = b\big(u(t), u(t), v\big) \quad \forall v \in H_g,\ a.e.\ t \in [0,T],$$

belongs to $L^4(0,T;H_g)$, therefore also belongs to $L^2(0,T;H_g)$.

Proof By Lemma 2.1, for almost every $t \in [0,T]$, we have

$$\big|Bu(t)\big| \le c_3 \big|u(t)\big|^{1/2} \big|Au(t)\big|^{1/2} \big\|u(t)\big\| \le c_3' \big\|u(t)\big\|^{3/2} \big|Au(t)\big|^{1/2}.$$

Then

$$\int_0^T \big|Bu(t)\big|^4 dt \le c_3' \int_0^T \big\|u(t)\big\|^6 \big|Au(t)\big|^2 dt \le c\|u\|^6_{L^\infty(0,T;V_g)} \int_0^T \big|Au(t)\big|^2 dt < +\infty.$$

This completes the proof. □

Lemma 2.3 [3] *Let $u \in L^2(0,T;V_g)$. Then the function Cu defined by*

$$\big(Cu(t), v\big)_g = \left(\left(\frac{\nabla g}{g} \cdot \nabla\right) u, v\right)_g = b\left(\frac{\nabla g}{g}, u, v\right) \quad \forall v \in V_g,$$

belongs to $L^2(0,T;H_g)$ and therefore also belongs to $L^2(0,T;V_g')$. Moreover,

$$\big|Cu(t)\big| \le \frac{|\nabla g|_\infty}{m_0} \cdot \big\|u(t)\big\| \quad \text{for a.e. } t \in (0,T)$$

and

$$\big\|Cu(t)\big\|_* \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \cdot \big\|u(t)\big\| \quad \text{for a.e. } t \in (0,T).$$

Since

$$-\frac{1}{g}(\nabla \cdot g \nabla) u = -\Delta u - \left(\frac{\nabla g}{g} \cdot \nabla\right) u,$$

we have

$$(-\Delta u, v)_g = \big((u,v)\big)_g + \left(\left(\frac{\nabla g}{g} \cdot \nabla\right) u, v\right)_g = (Au, v)_g + \left(\left(\frac{\nabla g}{g} \cdot \nabla\right) u, v\right)_g \quad \forall u, v \in V_g.$$

3 Existence and uniqueness of strong solutions

Definition 3.1 Given $f \in L^2(0,T;H_g)$ and $u_0 \in V_g$, a strong solution on the interval $(0,T)$ of (1.1) is a function $u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V_g)$ with $u(0) = u_0$ and such that

$$\frac{d}{dt}\big(u(t), v\big)_g + \nu\big(\big(u(t), v\big)\big)_g + \nu\big(Cu(t), v\big)_g + b\big(u(t), u(t), v\big) = \big(f(t), v\big)_g \tag{3.1}$$

for all $v \in V_g$ and for a.e. $t \in (0,T)$.

Remark 3.1 Due to Lemmas 2.2 and 2.3 from the above definition we see that any strong solution u must have the properties $u \in L^2(0,T;D(A))$ and $\frac{du}{dt} = f - \nu Au - Bu - Cu \in L^2(0,T;H_g)$. By Lemma 1.2 in [15], we have $u \in C([0,T];V_g)$, which makes the initial condition $u(0) = u_0$ meaningful. It is also noticed that if u is a strong solution of (1.1), then u satisfies the equation

$$\frac{du}{dt}(t) + \nu Au(t) + Bu(t) + Cu(t) = f(t) \quad \text{in } H_g \quad \text{for a.e. } t \in (0,T)$$

and satisfies the following energy equality for all $0 \le s < t \le T$:

$$\begin{aligned}&|u(t)|^2 + 2\nu\int_s^t \|u(r)\|^2\,dr + 2\nu\int_s^t b\left(\frac{\nabla g}{g}, u(r), u(r)\right)dr\\ &\quad= |u(s)|^2 + 2\int_s^t \big(f(r), u(r)\big)_g\,dr.\end{aligned}$$

We now prove some a priori estimates for the (sufficiently regular) strong solutions to (1.1).

Lemma 3.1 *If u is a strong solution of* (1.1) *on* $(0,T)$, *then we have*

$$\int_0^T \|u(t)\|^2\,dt \le K_1, \quad K_1 = K_1\big(|u_0|, \|f\|_{L^2(0,T;H_g)}, \nu, T, \lambda_1\big), \tag{3.2}$$

$$\sup_{s\in[0,T]} |u(s)|^2 \le K_2, \quad K_2 = K_2\big(|u_0|, \|f\|_{L^2(0,T;H_g)}, \nu, T, \lambda_1\big). \tag{3.3}$$

Proof Replacing v by $u(t)$ in (3.1) we get

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 2\big(f(t), u(t)\big)_g - 2\nu b\left(\frac{\nabla g}{g}, u(t), u(t)\right), \tag{3.4}$$

where we have used the facts that $b(u(t),u(t),u(t)) = 0$ and $(Cu(t),u(t))_g = b(\frac{\nabla g}{g}, u(t), u(t))$. Using Lemma 2.3 and the Cauchy inequality, we have

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \le 2\epsilon\nu\|u(t)\|^2 + \frac{|f(t)|^2}{2\epsilon\nu\lambda_1} + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|u(t)|^2.$$

Hence

$$\frac{d}{dt}|u(t)|^2 + 2\nu(\gamma_0 - \epsilon)\|u(t)\|^2 \le \frac{|f(t)|^2}{2\epsilon\nu\lambda_1}, \tag{3.5}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$ and $\epsilon > 0$ is chosen so that $\gamma_0 - \epsilon > 0$. By integrating in t from 0 to T, after dropping the unnecessary term, we obtain (3.2). Then by integrating in t of (3.5) from 0 to s, $0 < s < T$, we obtain

$$|u(s)|^2 \le |u_0|^2 + \frac{1}{2\epsilon\nu\lambda_1}\int_0^T |f(t)|^2\,dt.$$

Hence we get (3.3). □

Lemma 3.2 *If u is a sufficiently regular solution of* (1.1) *on* $(0,T)$, *then*

$$\sup_{t\in[0,T]} \|u(t)\|^2 \le K_3, \quad K_3 = K_3(K_1, K_2), \tag{3.6}$$

$$\int_0^T |Au(t)|^2\,dt \le K_4, \quad K_4 = K_4(K_1, K_2). \tag{3.7}$$

Proof Thanks to (3.1), replacing v by $Au(t)$, we get

$$\frac{d}{dt}\big(u(t), Au(t)\big)_g + \nu\big(\big(u(t), Au(t)\big)\big)_g + \nu\big(Cu(t), Au(t)\big)_g + b\big(u(t), u(t), Au(t)\big)$$
$$= \big(f(t), Au(t)\big)_g. \tag{3.8}$$

Since $((\phi,\psi))_g = \langle A\phi, \psi\rangle\ \forall\phi,\psi \in V_g$, this relation can be rewritten as follows:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 + \nu\big(Cu(t), Au(t)\big)_g + b\big(u(t), u(t), Au(t)\big)$$
$$= \big(f(t), Au(t)\big)_g \le \frac{\nu}{4}|Au(t)|^2 + \frac{1}{\nu}|f(t)|^2. \tag{3.9}$$

By Lemmas 2.1 and 2.3, (3.9) implies that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 \le \frac{\nu}{4}|Au(t)|^2 + \frac{1}{\nu}|f(t)|^2 + c_3|u(t)|^{1/2}|Au(t)|^{3/2}\|u(t)\|$$
$$+ \frac{\nu|\nabla g|_\infty}{m_0}\|u(t)\|\,|Au(t)|. \tag{3.10}$$

Using the Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 \le \frac{\nu}{4}|Au(t)|^2 + \frac{1}{\nu}|f(t)|^2$$
$$+ \frac{\nu}{4}|Au(t)|^2 + c_3'|u(t)|^2\|u(t)\|^4$$
$$+ \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|Au(t)|^2 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{4m_0}\|u(t)\|^2. \tag{3.11}$$

Then,

$$\frac{d}{dt}\|u(t)\|^2 + \nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)|Au(t)|^2$$
$$\le \frac{2}{\nu}|f(t)|^2 + 2c_3'|u(t)|^2\|u(t)\|^4 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\|u(t)\|^2. \tag{3.12}$$

Dropping the term $\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})|Au(t)|^2$, we obtain the differential inequality

$$y' \le a + \theta y,$$

where

$$y(t)=\|u(t)\|^2,\qquad a(t)=\frac{2}{\nu}|f(t)|^2,\qquad \theta(t)=c_3'|u(t)|^2\|u(t)\|^2+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0},$$

from which, by applying the Gronwall inequality, we obtain

$$y(t)\le y(0)\exp\left(\int_0^t\theta(\tau)\,d\tau\right)+\int_0^t a(s)\exp\left(\int_0^t\theta(\tau)\,d\tau\right)ds,$$

or

$$\begin{aligned}\|u(t)\|^2\le{}&\|u_0\|^2\exp\left(\int_0^t\left(c_3'|u(\tau)|^2\|u(\tau)\|^2+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\right)d\tau\right)\\&+\frac{2}{\nu}\int_0^t|f(s)|^2\exp\left(\int_0^t\left(c_3'|u(\tau)|^2\|u(\tau)\|^2+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\right)d\tau\right)ds.\end{aligned}\qquad(3.13)$$

By Lemma 3.1, we get (3.6). Integrating (3.12) from 0 to T, we obtain (3.7). □

Theorem 3.1 *Suppose that $f\in L^2(0,T;H_g)$ and $u_0\in V_g$ are given. Then there exists a unique strong solution u of (1.1) on $(0,T)$. Moreover, the map $u_0\mapsto u(t)$ is continuous on V_g for all $t\in[0,T]$, that is, the strong solution depends continuously on the initial data.*

Proof (i) *Uniqueness and continuous dependence.* Assume that u and v are two strong solutions of (1.1) with initial data u_0, v_0. Setting $w=u-v$, we see that

$$w\in L^2\big(0,T;D(A)\big)\cap L^\infty(0,T;V_g)$$

and

$$\frac{d}{dt}w+\nu Aw+\nu Cw=Bv-Bu,$$
$$w(0)=u_0-v_0.$$

Taking the inner product with Aw, we have

$$\frac{d}{dt}(w,Aw)_g+\nu(Aw,Aw)_g+\nu(Cw,Aw)_g=b(v,v,Aw)-b(u,u,Aw).$$

By the equality $(A\phi,\psi)_g=((\phi,\psi))_g$ and the trilinearness of $b(\cdot,\cdot,\cdot)$ we have

$$\frac{d}{dt}\|w\|^2+2\nu|Aw|^2=-2\nu(Cw,Aw)_g-2b(u,w,Aw)-2b(w,v,Aw).$$

Hence, by Lemmas 2.1 and 2.3,

$$\begin{aligned}\frac{d}{dt}\|w\|^2+2\nu|Aw|^2\le{}&2\nu\frac{|\nabla g|_\infty}{m_0}\|w\||Aw|\\&+2c_3|u|^{1/2}|Au|^{1/2}\|w\||Aw|+2c_3|w|^{1/2}|Aw|^{1/2}\|v\||Aw|.\end{aligned}$$

Since $\|w\|^2 \le \lambda_1 |Aw|^2$, by Cauchy's inequality and Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\|w\|^2 + 2\nu|Aw|^2 \le \nu|Aw|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}\|w\|^2 \\ + \frac{\nu}{2}|Aw|^2 + \frac{2c_3^2}{\nu}|u||Au|\|w\|^2 \\ + \frac{\nu}{2}|Aw|^2 + \frac{6c_3^4}{\nu\lambda_1}\|v\|^4\|w\|^2.\end{aligned}$$

Hence

$$\frac{d}{dt}\|w\|^2 \le \left(\frac{\nu|\nabla g|_\infty^2}{m_0^2} + \frac{2c_3^2}{\nu}|u||Au| + \frac{6c_3^4}{\nu\lambda_1}\|v\|^4\right)\|w\|^2.$$

Thus, one has

$$\|w(t)\|^2 \le \|w(0)\|^2 \exp\left(\int_0^t \left(\frac{\nu|\nabla g|_\infty^2}{m_0^2} + \frac{2c_3^2}{\nu}|u(s)||Au(s)| + \frac{6c_3^4}{\nu\lambda_1}\|v(s)\|^4\right) ds\right),$$

or

$$\|u(t) - v(t)\|^2 \le \|u_0 - v_0\|^2 \exp\left(\int_0^t \left(\frac{\nu|\nabla g|_\infty^2}{m_0^2} + \frac{2c_3^2}{\nu}|u(s)||Au(s)| + \frac{6c_3^4}{\nu\lambda_1}\|v(s)\|^4\right) ds\right).$$

This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of the strong solution on the initial data.

(ii) *Existence.* We split the proof of the existence into several steps.

Step 1: A Galerkin scheme. Let $v_1, v_2, \dots$ be a basis of V_g consisting of eigenfunctions of the operator A, which is orthonormal in H_g. Denote $V_m = \operatorname{span}\{v_1, \dots, v_m\}$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j)v_j$. Define also

$$u^m(t) = \sum_{j=1}^m \alpha_{m,j}(t)v_j,$$

where the coefficients $\alpha_{m,j}$ are required to satisfy the following system:

$$\begin{aligned}&\frac{d}{dt}\big(u^m(t), v_j\big)_g + \nu\big(Au^m(t), v_j\big)_g + \nu\big(Cu^m(t), v_j\big)_g + b\big(u^m(t), u^m(t), v_j\big) \\ &\quad = \big(f(t), v_j\big)_g \quad \forall j = 1, \dots, m,\end{aligned} \tag{3.14}$$

and the initial condition $u^m(0) = P_m u_0$. This system of ordinary differential equations in the unknown $(\alpha_{m,1}(t), \dots, \alpha_{m,m}(t))$ fulfills the conditions of the Peano theorem, so the approximate solutions u_m exist.

Step 2: A priori estimates. Using (3.14) and replacing v_j by $Au^m(t)$, we get

$$\begin{aligned}&\frac{d}{dt}\big(u^m(t), Au^m(t)\big)_g + \nu\big(\big(u^m(t), Au^m(t)\big)\big)_g + \nu\big(Cu^m(t), Au^m(t)\big)_g \\ &\quad + b\big(u^m(t), u^m(t), Au^m(t)\big) = \big(f(t), Au^m(t)\big)_g.\end{aligned} \tag{3.15}$$

This relation is similar to (3.8). Doing exactly as in Lemma 3.2 for u^m, with u_0 replaced by u_0^m, and noticing that

$$\|u_0^m\| = \|P_m u_0\| \le \|u_0\|,$$

we conclude that

$$\{u^m\} \text{ is bounded in } L^2\big(0,T;D(A)\big) \cap L^\infty(0,T;V_g). \tag{3.16}$$

Now, observe that (3.14) is equivalent to

$$\frac{du^m}{dt} = -\nu Au^m - \nu Cu^m - P_m Bu^m + P_m f.$$

Hence, by Lemma 2.2 we have

$$\big\{(u^m)'\big\} \text{ is bounded in } L^2(0,T;H_g).$$

Step 3: Passage to the limit. From the above estimates we conclude that there exist $u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V_g)$ with $u' \in L^2(0,T;H_g)$, and a subsequence of $\{u^m\}$, relabeled the same, such that

$$\begin{aligned}
&\{u^m\} \text{ converges weakly-star to } u \text{ in } L^\infty(0,T;V_g),\\
&\{u^m\} \text{ converges weakly to } u \text{ in } L^2\big(0,T;D(A)\big) \quad \text{and}\\
&\big\{(u^m)'\big\} \text{ converges weakly to } u' \text{ in } L^2(0,T;H_g).
\end{aligned} \tag{3.17}$$

Since Ω is bounded, we can use the Compactness Lemma (see e.g. [15, Chap. III, Theorem 2.1]) to deduce the existence of a subsequence (still denoted by) u^m which converges strongly to u in $L^2(0,T;V_g)$.

Then we can pass to the limit in the nonlinearity b thanks to the following lemma whose proof is exactly the proof of Lemma 3.2 in [15, Chap. III].

Lemma 3.3 *If u_m converges to u in $L^2(0,T;V_g)$ strongly then, for any vector function w with components belonging to $C^1([0,T]\times\overline{\Omega})$, we have*

$$\int_0^T b\big(u_m(t),u_m(t),w(t)\big)\,dt \to \int_0^T b\big(u(t),u(t),w(t)\big)\,dt.$$

Finally, we prove that $u(0)=u_0$. Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T)=0$. Multiplying (3.14) by $\psi(t)$ and integrating by parts the first term, we get

$$\begin{aligned}
&-\int_0^T \big(u^m(t), v_j\psi'(t)\big)_g\,dt + \nu\int_0^T \big(Au^m(t), v_j\psi(t)\big)_g\,dt\\
&\quad + \nu\int_0^T \big(Cu^m(t), v_j\psi(t)\big)_g\,dt + \int_0^T b\big(u^m(t),u^m(t),v_j\psi(t)\big)\,dt\\
&= \big(u^m(0), v_j\big)_g\psi(0) + \int_0^T \big(f(t), v_j\psi(t)\big)_g\,dt.
\end{aligned}$$

Passing to the limit and noting that the set $\{v_j\}_{j=1}^{\infty}$ is dense in V_g, we have

$$\begin{aligned} -\int_0^T \big(u(t), v\psi'(t)\big)_g\,dt + \nu \int_0^T \big(Au(t), v\psi(t)\big)_g\,dt \\ + \nu \int_0^T \big(Cu(t), v\psi(t)\big)_g\,dt + \int_0^T b\big(u(t), u(t), v\psi(t)\big)\,dt \\ = (u_0, v)_g \psi(0) + \int_0^T \big(f(t), v\psi(t)\big)_g\,dt \end{aligned} \tag{3.18}$$

holds for any $v \in V_g$. On the other hand, we can multiply (3.1) by $\psi(t)$, integrate on $(0, T)$ and apply the integration by part for the first term to get

$$\begin{aligned} -\int_0^T \big(u(t), v\psi'(t)\big)_g\,dt + \nu \int_0^T \big(Au(t), v\psi(t)\big)_g\,dt \\ + \nu \int_0^T \big(Cu(t), v\psi(t)\big)_g\,dt + \int_0^T b\big(u(t), u(t), v\psi(t)\big)\,dt \\ = \big(u(0), v\big)_g \psi(0) + \int_0^T \big(f(t), v\psi(t)\big)_g\,dt. \end{aligned} \tag{3.19}$$

By a comparison with (3.18), we have

$$\big(u(0) - u_0, v\big)_g \psi(0) = 0.$$

We can choose ψ with $\psi(0) \neq 0$, thus $u(0) = u_0$. This completes the proof. □

4 Finite time approximation of the strong solutions

In this section, we will study the finite time numerical approximation of the strong solutions by using a space and time discretization scheme. This scheme combines a discretization in time by an alternating direction method with a discretization in space by finite elements.

Denote by $\mathcal{H}$ a regular triangulation of Ω and by $W_h, h \in \mathcal{H}$, a family of finite dimensional subspaces of $H_0^1(\Omega, g)$ such that $\cup_{h\in\mathcal{H}} W_h$ is dense in $H_0^1(\Omega, g)$. For every h, V_h is a subspace of W_h. The family $V_h, h \in \mathcal{H}$, constitutes an external approximation of V_g. From the results in [15, Chap. I], the preceding approximation is stable and convergent.

For every h, let u_{0h} be the projection (in $H_0^1(\Omega, g)$) of u_0 on V_h, i.e.,

$$\begin{aligned} &u_{0h} \in V_h, \\ &\big((u_{0h}, v_h)\big)_g = \big((u_0, v_h)\big)_g \quad \forall v_h \in V_h. \end{aligned} \tag{4.1}$$

Let N be an integer, $k = T/N$. For every h and k, we now recursively define a family $u_h^{m+i/2}$ of elements of $V_h, m = 0, \ldots, N-1, i = 1, 2$. We start with

$$u_h^0 = u_{0h}.$$

Assuming that $u_h^m, m \geq 0$, is known, we define $u_h^{m+1/2}$ and u_h^{m+1} as follows:

$$u_h^{m+1/2} \in V_h,$$
$$\frac{1}{k}\big(u_h^{m+1/2} - u_h^m, v_h\big)_g + \frac{\nu}{2}\big(\big(u_h^{m+1/2}, v_h\big)\big)_g + \frac{\nu}{2}\big(Cu_h^{m+1/2}, v_h\big)_g = \big(f^m, v_h\big)_g \quad \forall v_h \in V_h, \tag{4.2}$$

where

$$f^m = \frac{1}{k}\int_{mk}^{(m+1)k} f(t)\,dt,$$

and

$$\begin{aligned} &u_h^{m+1} \in W_h, \\ &\frac{1}{k}\big(u_h^{m+1} - u_h^{m+1/2}, v_h\big)_g + \frac{\nu}{2}\big(\big(u_h^{m+1}, v_h\big)\big)_g + \frac{\nu}{2}\big(Cu_h^{m+1}, v_h\big)_g \\ &\quad + \widetilde{b}\big(u_h^{m+1}, u_h^{m+1}, v_h\big) = 0 \quad \forall v_h \in W_h, \end{aligned} \tag{4.3}$$

where

$$\widetilde{b}(u, v, w) = \sum_{i,j=1}^{2} \frac{1}{2}\int_\Omega \big\{u_i\big[(D_i v_j)w_j - v_j(D_i w_j)\big]\big\} g\,dx.$$

The existence and uniqueness of a solution $u_h^{m+1/2}$ of (4.2) follows from the Riesz representation theorem. The existence of a solution $u_h^{m+1} \in W_h$ of (4.3) follows from the Brouwer fixed point theorem (see [15, Chap. II, Lemma 1.4]). We associate this family of elements $u_h^{m+i/2}$ of W_h with the following functions defined on $[0, T]$:

- $u_k^{(i)}$ is the piecewise constant function which is equal to $u_h^{m+i/2}$ on $[mk, (m+1)k)$, $i = 1, 2$; $m = 0, \ldots, N-1$.
- $\widetilde{u}_k^{(i)}$ is the continuous function from $[0, T]$ into W_h, which is linear on $(mk, (m+1)k)$ and equal to $u_h^{m+i/2}$ at mk, $i = 1, 2$; $m = 0, \ldots, N-1$.

By adding (4.2) and (4.3) we obtain a relation which can be reinterpreted in terms of these functions as:

$$\begin{aligned} &\left(\frac{d\widetilde{u}_k^{(2)}}{dt}(t-k), v_h\right)_g + \frac{\nu}{2}\big(\big(u_k^{(1)}(t) + u_k^{(2)}(t), v_h\big)\big)_g \\ &\quad + \frac{\nu}{2}\big(Cu_k^{(1)}(t) + Cu_k^{(2)}(t), v_h\big)_g + b\big(u_k^{(2)}(t), u_k^{(2)}(t), v_h\big) \\ &= \big(f_k(t), v_h\big)_g \quad \forall v_h \in V_h, \end{aligned} \tag{4.4}$$

where

$$f_k(t) = f^m \quad \text{for } t \in [mk, (m+1)k).$$

Similarly, by adding (4.3) to the relation (4.2) for $m+1$, we arrive at an equation which is equivalent to

$$\left(\frac{d\widetilde{u}_k^{(1)}}{dt}(t), v_h\right)_g + \frac{\nu}{2}\big(\big(u_k^{(1)}(t+k) + u_k^{(2)}(t), v_h\big)\big)_g$$

$$+\frac{\nu}{2}\big(Cu_k^{(1)}(t+k)+Cu_k^{(2)}(t),v_h\big)_g+b\big(u_k^{(2)}(t),u_k^{(2)}(t),v_h\big)$$
$$=\big(f_k(t+k),v_h\big)_g\quad\forall v_h\in V_h. \tag{4.5}$$

We now discuss the behavior of these functions $u_k^{(i)}, \widetilde{u}_k^{(i)}$, as h and k tend to 0.

Theorem 4.1 *Under the above assumptions, the functions $u_k^{(i)}, \widetilde{u}_k^{(i)}; i=1,2$, belong to a bounded set of $L^2(0,T;H_0^1(\Omega,g))\cap L^\infty(0,T;L^2(\Omega,g))$.*

As k and $h\to 0$, $u_k^{(i)}$ and $\widetilde{u}_k^{(i)}$ converge to the strong solution u of (1.1) in $L^2(0,T;H_0^1(\Omega,g))$ and $L^q(0,T;L^2(\Omega,g))$ for all $1\le q<\infty$.

Proof (i) *A priori estimates.* Setting $v_h=u_h^{m+1/2}$ in (4.2) and observing that

$$(a-b,a)=\frac{1}{2}\big(|a|^2-|b|^2+|a-b|^2\big)\quad\forall a,b\in H_g,$$

we get

$$\begin{aligned}
&\big|u_h^{m+1/2}\big|^2-\big|u_h^m\big|^2+\big|u_h^{m+1/2}-u_h^m\big|^2+\nu k\big\|u_h^{m+1/2}\big\|^2\\
&\quad=2k\big(f^m,u_h^{m+1/2}\big)_g-\nu k\big(Cu_h^{m+1/2},u_h^{m+1/2}\big)_g\\
&\quad\le\nu k\epsilon\big\|u_h^{m+1/2}\big\|^2+\frac{k}{\epsilon\nu\lambda_1}\big|f^m\big|^2+\nu k\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\big\|u_h^{m+1/2}\big\|^2,
\end{aligned}\tag{4.6}$$

whence

$$\big|u_h^{m+1/2}\big|^2-\big|u_h^m\big|^2+\big|u_h^{m+1/2}-u_h^m\big|^2+\nu k(\gamma_0-\epsilon)\big\|u_h^{m+1/2}\big\|^2\le\frac{k}{\epsilon\nu\lambda_1}\big|f^m\big|^2, \tag{4.7}$$

where $\gamma_0=1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}>0$, and ϵ is chosen such that $\gamma_0-\epsilon>0$. Similarly, taking $v_h=u_h^{m+1}$ in (4.3), we can deduce that

$$\big|u_h^{m+1}\big|^2-\big|u_h^{m+1/2}\big|^2+\big|u_h^{m+1}-u_h^{m+1/2}\big|^2+\nu k\big\|u_h^{m+1}\big\|^2-\nu k\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\big\|u_h^{m+1}\big\|^2\le 0,$$

or

$$\big|u_h^{m+1}\big|^2-\big|u_h^{m+1/2}\big|^2+\big|u_h^{m+1}-u_h^{m+1/2}\big|^2+\nu k\gamma_0\big\|u_h^{m+1}\big\|^2\le 0. \tag{4.8}$$

By adding all the relations (4.7), (4.8) for $m=0,\dots,N-1$, we obtain

$$\begin{aligned}
&\big|u_h^N\big|^2+\sum_{m=0}^{N-1}\big(\big|u_h^{m+1}-u_h^{m+1/2}\big|^2+\big|u_h^{m+1/2}-u_h^m\big|^2\big)\\
&\qquad+\nu k\sum_{m=0}^{N-1}\big((\gamma_0-\epsilon)\big\|u_h^{m+1/2}\big\|^2+\gamma_0\big\|u_h^{m+1}\big\|^2\big)\\
&\quad\le|u_{0h}|^2+\frac{k}{\epsilon\nu\lambda_1}\sum_{m=0}^{N-1}\big|f^m\big|^2
\end{aligned}$$

$$\leq |u_{0h}|^2 + \frac{k}{\epsilon\nu\lambda_1}\sum_{m=0}^{N-1}\left|\int_{mk}^{(m+1)k} f(s)\,ds\right|^2$$

$$\leq |u_{0h}|^2 + \frac{1}{\epsilon\nu\lambda_1}\int_0^T \left|f(s)\right|^2 ds \quad \text{(by Schwarz's inequality).} \tag{4.9}$$

Since $\|u_{0h}\| \leq \|u_0\|$, we then conclude that

$$\sum_{m=0}^{N-1}\left(\left|u_h^{m+1} - u_h^{m+1/2}\right|^2 + \left|u_h^{m+1/2} - u_h^m\right|^2\right) \leq L, \tag{4.10}$$

$$k\sum_{m=0}^{N-1}\left((\gamma_0 - \epsilon)\left\|u_h^{m+1/2}\right\|^2 + \gamma_0\left\|u_h^{m+1}\right\|^2\right) \leq \frac{L}{\nu}, \tag{4.11}$$

where

$$L = \frac{1}{\lambda_1}\|u_0\|^2 + \frac{1}{\epsilon\nu\lambda_1}\int_0^T \left|f(s)\right|^2 ds.$$

By adding the relations (4.7) and (4.9) for $m = 0, \ldots, p$, after a simplification we obtain

$$\left|u_h^{p+1}\right|^2 \leq L, \quad p = 0, \ldots, N-1. \tag{4.12}$$

Adding the relations (4.7) for $m = 0, \ldots, p$ and (4.9) for $m = 0, \ldots, p-1$, and dropping unnecessary terms, we find that

$$\left|u_h^{p+1/2}\right|^2 \leq L, \quad p = 0, \ldots, N-1. \tag{4.13}$$

The relations (4.11)–(4.13) imply that the functions $u_k^{(i)}$, $\widetilde{u}_k^{(i)}$, $i = 1, 2$, belong to a bounded set of $L^\infty(0, T; L^2(\Omega, g))$ and that $u_k^{(1)}, u_k^{(2)}$ belong to a bounded set of $L^2(0, T; H_0^1(\Omega, g))$. In order to show that $\widetilde{u}_k^{(i)}$ also belongs to a bounded set of $L^2(0, T; H_0^1(\Omega, g))$, we observe by a direct calculation that

$$\left|u_k^{(i)} - \widetilde{u}_k^{(i)}\right|^2_{L^2(0,T;L^2(\Omega,g))} = \frac{k}{3}\sum_{m=1}^{N}\left|u_h^{m+i/2} - u_h^{m-1+i/2}\right|^2$$

$$\leq \frac{kL}{3} \quad \text{(by (4.10)).} \tag{4.14}$$

(ii) *Convergence of* $u_k^{(i)}, \widetilde{u}_k^{(i)}$ *as* $k, h \to 0$. Due to the previous a priori estimates, there exist a subsequence and $u^{(1)}, u^{(2)}$ in $L^2(0, T; H_0^1(\Omega, g)) \cap L^\infty(0, T; L^2(\Omega, g))$ such that

$$\begin{aligned} &u_k^{(i)} \to u^{(i)} \text{ weakly in } L^2\big(0, T; H_0^1(\Omega, g)\big), \\ &u_k^{(i)} \to u^{(i)} \text{ weakly-star in } L^\infty(0, T; H_g), \quad i = 1, 2. \end{aligned} \tag{4.15}$$

The relations (4.10) and (4.14) imply that the same is true for $\widetilde{u}_k^{(i)}$, $i = 1, 2$. Now we infer from (4.10) that

$$\left|u_k^{(2)} - u_k^{(1)}\right|_{L^2(0,T;H_g)} \leq kL, \tag{4.16}$$

and therefore

$$u_k^{(2)} - u_k^{(1)} \to 0 \quad \text{in } L^2(0,T;H_g) \quad \text{as } k \to 0,$$

so that $u^{(2)} = u^{(1)}$. We also deduce from the properties of the $V_h's$ (which constitute an external approximation of V_g) that $u^{(1)}$ belongs in fact to $L^2(0,T;V_g) \cap L^\infty(0,T;H_g)$:

$$u^{(2)} = u^{(1)} = u_* \in L^2(0,T;V_g) \cap L^\infty(0,T;H_g). \tag{4.17}$$

To prove that u_* is a strong solution of (1.1), we need the following result.

Lemma 4.1 [16, Theorem 13.3] *Assume that X and Y are two Banach spaces with the property*

$$Y \subset X, \quad \textit{the injection being compact.}$$

Let $\mathcal{G}$ be a bounded set in $L^1(0,T;Y)$ and $L^p(0,T;X)$, $T>0$, $p>1$, such that

$$\int_0^{T-a} \left|g(a+s) - g(s)\right|_X^p ds \to 0 \quad \textit{as } a \to 0 \textit{ uniformly for } g \in \mathcal{G}. \tag{4.18}$$

Then $\mathcal{G}$ is relatively compact in $L^q(0,T;X)$ for any q, $1 \le q < p$.

By integration of (4.5),

$$\begin{aligned}\left(\widetilde{u}_k^{(1)}(t+a) - \widetilde{u}_k^{(1)}(t)\right)_g = -\int_t^{t+a} \Big\{ & \frac{\nu}{2}\left(\left(u_k^{(1)}(s+k) + u_k^{(2)}(s), v_h\right)\right)_g \\ & + \frac{\nu}{2}\left(Cu_k^{(1)}(s+k) + Cu_k^{(2)}(s), v_h\right)_g + b\left(u_k^{(2)}(s), u_k^{(2)}(s), v_h\right) \\ & - \left(f_k(s+k), v_h\right)_g \Big\} ds \quad \forall v_h \in V_h.\end{aligned}$$

We majorize the absolute value of the right-hand side of this equality as follows:

$$\begin{aligned}\left|\frac{\nu}{2}\int_t^{t+a} \left(\left(u_k^{(1)}(s+k), v_h\right)\right)_g ds\right| &\le \frac{\nu}{2} a^{1/2} \|v_h\| \left(\int_t^{t+a} \|u_k^{(1)}(s+k)\|^2 ds\right)^{1/2} \\ &\le \frac{\nu}{2} a^{1/2} \|v_h\| \left(\int_t^{T} \|u_k^{(1)}(s+k)\|^2 ds\right)^{1/2} \\ &\le c_1 a^{1/2} \|v_h\|,\end{aligned}$$

where $c_1 = c_1(u_0, f, \nu, T, \Omega)$. Similarly, we have

$$\begin{aligned}&\left|\frac{\nu}{2}\int_t^{t+a} \left(\left(u_k^{(2)}(s), v_h\right)\right)_g ds\right| \le c_2 a^{1/2} \|v_h\|, \\ &\left|\int_t^{t+a} \left(f_k(s+k), v_h\right)_g ds\right| \le c_3 a^{1/2} \|v_h\|, \\ &\left|\frac{\nu}{2}\int_t^{t+a} \left(Cu_k^{(1)}(s+k) + Cu_k^{(2)}(s), v_h\right)_g ds\right|\end{aligned}$$

$$\le \left|\frac{\nu}{2}\int_t^{t+a}\big(Cu_k^{(1)}(s+k),v_h\big)_g\,ds\right| + \left|\frac{\nu}{2}\int_t^{t+a}\big(Cu_k^{(2)}(s),v_h\big)_g\,ds\right|$$
$$\le \frac{\nu}{2}\int_t^{t+a}\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\big\|u_k^{(1)}(s+k)\big\|\|v_h\|\,ds + \frac{\nu}{2}\int_t^{t+a}\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\big\|u_k^{(2)}(s)\big\|\|v_h\|\,ds$$
$$\le c_4 a^{1/2}\|v_h\|,$$

where $c_4 = c_4(u_0, f, \nu, T, \Omega, m_0, \lambda_1, |\nabla g|_\infty)$.

For the term involving b, using Lemma 2.1, the Hölder inequality, and the fact that $u_k^{(2)}$ is bounded in $L^\infty(0,T;L^2(\Omega,g))$, we have

$$\left|\int_t^{t+a} b\big(u_k^{(2)}(s), u_k^{(2)}(s), v_h\big)\,ds\right| \le c_5 a^{1/4}\|v_h\|.$$

Finally, we obtain the majorization

$$\big|\big(\widetilde{u}_k^{(1)}(t+a)-\widetilde{u}_k^{(1)}(t), v_h\big)_g\big| \le c_6 a^{1/4}\|v_h\| \quad \forall v_h \in V_h. \tag{4.19}$$

Since $\widetilde{u}_k^{(1)}$ is a V_h-valued function, we can take $v_h = \widetilde{u}_k^{(1)}(t+a) - \widetilde{u}_k^{(1)}(t)$, and we find after integration with respect to t that

$$\int_0^{T-a}\big|\widetilde{u}_k^{(1)}(t+a)-\widetilde{u}_k^{(1)}(t)\big|^2\,dt \le c_6 a^{1/4}\int_0^{T-a}\big\|\widetilde{u}_k^{(1)}(t+a)-\widetilde{u}_k^{(1)}(t)\big\|\,dt$$
$$\le c_6 a^{1/4}(T)^{1/2}\left(\int_0^{T-a}\big\|\widetilde{u}_k^{(1)}(t+a)-\widetilde{u}_k^{(1)}(t)\big\|^2\,dt\right)^{1/2}.$$

Then

$$\int_0^{T-a}\big|\widetilde{u}_k^{(1)}(t+a)-\widetilde{u}_k^{(1)}(t)\big|^2\,dt \le c_7 a^{1/4}. \tag{4.20}$$

We apply Lemma 4.1 as follows: $\mathcal{G}$ is the family of functions $\widetilde{u}_k^{(1)}$ which is bounded in $L^2(0,T;H_0^1(\Omega,g)) \cap L^\infty(0,T;L^2(\Omega,g))$, $(Y = H_0^1(\Omega,g), X = L^2(\Omega,g), p = 2)$, so that (4.18) follows from (4.20). We conclude that $\widetilde{u}_k^{(1)}$ is relatively compact in $L^q(0,T;L^2(\Omega,g))$, $1 \le q < 2$, and since $\widetilde{u}_k^{(1)}$ converges weakly in $L^q(0,T;L^2(\Omega,g))$ to u_*, $\widetilde{u}_k^{(1)}$ converges to u_* strongly in $L^q(0,T;L^2(\Omega,g))$, $1 \le q < 2$. As the sequence $\widetilde{u}_k^{(1)}$ is bounded in $L^\infty(0,T;L^2(\Omega,g))$, we conclude from the Lebesgue dominated convergence theorem that

$$\widetilde{u}_k^{(1)} \to u_* \text{ strongly in } L^q\big(0,T;L^2(\Omega,g)\big), \quad 1 \le q < \infty. \tag{4.21}$$

Combining (4.14), (4.16) and (4.21), we have

$$\widetilde{u}_k^{(i)}, u_k^{(i)} \text{ converge to } u_* \text{ strongly in } L^q\big(0,T;L^2(\Omega,g)\big), \quad 1 \le q < \infty. \tag{4.22}$$

(iii) *Passage to the limit in* (4.5). Let ψ be any C^1 scalar function on $[0,T]$ which vanishes near T. We multiply (4.5) by $\psi(t)$, integrate in t and integrate by parts the first term to get

$$\int_0^T \Big\{-\big(\widetilde{u}_k^{(1)}(t), v_h\psi'(t)\big)_g + \frac{\nu}{2}\big(\big(u_k^{(1)}(t+k)+u_k^{(2)}(t), v_h\psi(t)\big)\big)_g$$

$$+\frac{\nu}{2}\Big(Cu_k^{(1)}(t+k)+Cu_k^{(2)}(t),v_h\psi(t)\Big)_g+b\Big(u_k^{(2)}(t),u_k^{(2)}(t),v_h\psi(t)\Big)\Bigg\}\,dt$$

$$=\int_0^T\big(f_k(t+k),v_h\psi(t)\big)_g\,dt+(u_{0h},v_h)\psi(0). \tag{4.23}$$

Let v be an arbitrary element of V_g. Choosing an approximation v_h of v, and passing to the limit in (4.23) we get

$$\int_0^T\Big\{-\big(u_*(t),v\psi'(t)\big)_g+\nu\big(\big(u_*(t),v\psi(t)\big)\big)_g$$

$$+\nu\big(Cu_*(t),v\psi(t)\big)_g+b\big(u_*(t),u_*(t),v\psi(t)\big)\Big\}\,dt$$

$$=\int_0^T\big(f_k(t),v\psi(t)\big)_g\,dt+(u_0,v)\psi(0), \tag{4.24}$$

where we have used the strong convergence (4.22) to pass to the limit in the nonlinear term. We deduce from (4.24) that u_* is a strong solution of (1.1) with the initial datum u_0, and therefore, by the uniqueness, $u_*=u$. This completes the proof. □

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