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Research Article

Spectral Galerkin Method in Space and Time for the 2D g-Navier-Stokes Equations

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We prove the H^2-stability and L^2-error analysis of the spectral Galerkin method in space and time with the implicit/explicit Euler scheme for the 2D g-Navier-Stokes equations in bounded domains when the initial data belong to H^1.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ. In this paper, we study the spectral Galerkin method in space and time for the following 2D g-Navier-Stokes equations:

$$\begin{gathered}\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)\, u + \nabla p = f\,(x,t) \quad \text{in } (0,+\infty)\times\Omega,\\ \nabla\cdot(gu) = 0 \quad \text{in } (0,+\infty)\times\Omega,\\ u = 0 \quad \text{on } (0,+\infty)\times\Gamma,\\ u\,(0,x) = u_0\,(x), \quad x\in\Omega,\end{gathered} \tag{1}$$

where $u = u(x,t) = (u_1,u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, and u_0 is the initial velocity.

The g-Navier-Stokes equations are a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv$ const we get the usual Navier-Stokes equations. The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [1] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [1], good properties of the 2D g-Navier-Stokes equations can lead to an initiate of the study of the Navier-Stokes equations on the thin three-dimensional domain $\Omega_g = \Omega\times(0,g)$. In the last few years, the existence and long-time behavior of both weak and strong solutions to the 2D g-Navier-Stokes equations have been studied extensively (cf. [2–9]). In this paper, we aim to study numerical approximation of the strong solutions to problem (1). To do this, we assume that

(G) $g \in W^{1,\infty}(\Omega)$ such that

$$\begin{gathered}0 < m_0 \le g\,(x) \le M_0\\ \forall x = (x_1,x_2)\in\Omega,\\ \left|\nabla g\right|_\infty < m_0\lambda_1^{1/2},\end{gathered} \tag{2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A defined in Section 2.1 below);

(F) $f \in W^{1,\infty}(\mathbb{R}^+;H_g)$; that is, $f, f_t \in L^\infty(\mathbb{R}^+;H_g)$.

In this paper, in order to study the numerical approximation of strong solutions to the 2D g-Navier-Stokes equations we will use the spectral Galerkin method in space and time, which is based on the eigen-subspaces of the g-Stokes operator. As mentioned in [10] for the Navier-Stokes equations, this method enables us to avoid solving the fully nonlinear g-Navier-Stokes equations on the low-frequency subspace, whereas to obtain the low-frequency component of the numerical solution, the usual multilevel spectral methods and the postprocessing Galerkin methods need to solve the fully nonlinear g-Navier-Stokes equations on the low-frequency subspace. In what follows, we will explain the spectral Galerkin method used in the paper. For the related function spaces, we refer the reader to Section 2.1.

Let $w_1, w_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ be the eigenvectors and eigenvalues of the g-Stokes operator. For a fixed integer m, let P_m be the orthogonal projection of H_g onto $H_m = \text{span}\{w_1, \ldots, w_m\}$. Then, the spectral Galerkin method in space is defined as follows: find $u_m(t) \in H_m$ such that

$$u_{mt} + \nu A u_m + \nu C u_m + P_m B(u_m, u_m) = P_m f, \quad t > 0, \ u_m(0) = P_m u_0. \tag{3}$$

In order to simplify the implementation of the scheme, we restrict ourselves to the semi-implicit Euler scheme applied to the spectral Galerkin method in space. We consider a spectral Galerkin method in space and time with the implicit/explicit Euler scheme: find u_m^{n+1} $(n \geq 0)$ such that

$$\frac{1}{\Delta t}\left(u_m^{n+1} - u_m^n\right) + \nu A u_m^{n+1} + \nu C u_m^{n+1} + P_m B\left(u_m^n, u_m^n\right) = P_m f^{n+1}, \quad t > 0, \ u_m^0 = P_m u_0, \tag{4}$$

where $\Delta t > 0$ is the time step size and

$$f^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(t)\, dt, \quad t_n = n\Delta t. \tag{5}$$

Here, the linear term is treated implicitly to avoid serve time step limitations, whereas the nonlinear term is kept explicitly so that the corresponding discrete system is easily invertible. It is well known that this type of scheme is only stable under some restriction on the time step size. We will obtain H^2-stability uniform in time stated in Theorem 13, provided that the following condition holds

$$C\Delta t \ln \frac{\lambda_m}{\lambda_1} \leq 1 \tag{6}$$

for some positive constant C depending on the data (u_0, ν, f, Ω). As mentioned in [11] for the case of 2D Navier-Stokes equations, the stability condition (6) is a significant improvement compared with the results provided by the nonlinear Galerkin method [12] and the multilevel method [13, 14].

We also derive an L^2-error estimate of the numerical solution u_m^n under the stability condition (6):

$$\left|u(t_n) - u_m^n\right|^2 \leq \mathscr{K}\tau^{-1}(t_n)\, e^{(16/\nu^3\gamma_0^3\lambda_1)c_0^2 C_f^2 t_n}\left(\lambda_{m+1}^{-2} + \Delta t^2\right) \quad \forall n \geq 1, \tag{7}$$

where $C_f = \sup_{t\geq 0}|f(t)|$, $\tau(t) = \min\{1, t\}$ and $\mathscr{K}$ denotes a general positive constant depending only on the data $(\nu, \Omega, |\nabla g|_\infty, \lambda_1, C_f, \|u_0\|)$. Noting that $\tau^{-1}(t_n)$ is a singular factor near $t = 0$.

Compared to He's works [11] on the spectral method of the 2D Navier-Stokes equations, here we have to address some additional difficulties. Firstly, to treat the more general condition $\nabla \cdot (gu) = 0$, instead of the usual function spaces used for the Navier-Stokes equations, we use the function spaces H_g, V_g which are defined suitably for the g-Navier-Stokes equations (see Section 2.1 for details). Secondly, we have to deal with the term Cu in the equation, which only appears for the g-Navier-Stokes equations. It is worthy noticing that when $g \equiv 1$, we of course recover the results for the Navier-Stokes equations in [11].

The paper is organized as follows. In the next section, we recall some results on function spaces and inequalities for the nonlinear terms related to the g-Navier-Stokes equations, and some discrete Gronwall inequalities are frequently used later. In Section 3, we prove several estimates for the strong solution and the Galerkin approximate solutions of problem (1). In Section 4, we study the error analysis of the spectral Galerkin method in space. Stability and error analysis of the spectral Galerkin method in space and time are discussed in the last section.

2. Preliminaries

2.1. Function Spaces and Operators. Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g\, dx, \quad u, v \in L^2(\Omega, g),$$
$$((u, v))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g\, dx, \quad u = (u_1, u_2), \quad v = (v_1, v_2) \in H_0^1(\Omega, g), \tag{8}$$

and norms $|u|^2 = (u, u)_g$, $\|u\|^2 = ((u, u))_g$. Thanks to assumption (**G**), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathscr{V} = \left\{u \in \left(C_0^\infty(\Omega)\right)^2 : \nabla \cdot (gu) = 0\right\}. \tag{9}$$

Denote by H_g the closure of $\mathscr{V}$ in $L^2(\Omega, g)$, and denote by V_g the closure of $\mathscr{V}$ in $H_0^1(\Omega, g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle\cdot,\cdot\rangle$ for duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j g dx, \quad (10)$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u,v,w) = -b(u,w,v). \quad (11)$$

Hence

$$b(u,v,v) = 0, \quad \forall u, v \in V_g. \quad (12)$$

Set $A : V_g \to V_g'$ by $\langle Au, v\rangle = ((u,v))_g$, $B : V_g \times V_g \to V_g'$ by $\langle B(u,v), w\rangle = b(u,v,w)$. Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $D(A) = H^2(\Omega, g) \cap V_g$ and $Au = -P_g \Delta u$, for all $u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g. Consequently, there exists an orthogonal basis of H_g consisting of the eigenvectors w_j of A:

$$\begin{gathered} Aw_j = \lambda_j w_j, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots, \\ \lambda_j \longrightarrow \infty \text{ as } j \longrightarrow \infty. \end{gathered} \quad (13)$$

Furthermore, we can also define the sth power A^s of A for all $s \in \mathbb{R}$. The space $D(A^s)$ is the Hilbert space when equipped with the scalar product (A^s, A^s) and norm $|A^{s\cdot}|$, where $(\cdot,\cdot)$ and $|\cdot|$ denote the scalar product and norm in H_g. In particular, $D(A^0) = H_g$ and $D(A^{1/2}) = V_g$.

Let $H_m = \operatorname{span}\{w_1, \ldots, w_m\}$. Then, the following estimates hold:

$$\lambda_1 |v|^2 \le \|v\|^2 \quad \forall v \in V_g, \qquad \|v_m\|^2 \le \lambda_m |v_m|^2 \quad \forall v_m \in H_m, \quad (14)$$

$$\lambda_{m+1} |w_m|^2 \le \|w_m\|^2 \quad \forall w_m \in V_g \setminus H_m. \quad (15)$$

Using the Hölder inequality, the Ladyzhenskaya inequality (when $n = 2$):

$$|u|_{L^4} \le c|u|^{1/2} |\nabla u|^{1/2}, \quad \forall u \in H_0^1(\Omega), \quad (16)$$

and the interpolation inequalities, as in [15, 16], one can prove the following.

Lemma 1. *If $n = 2$, then*

$$|b(u,v,w)| \le \begin{cases} c_0 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2}, & \forall u, v, w \in V_g, \\ c_0 |u|^{1/2} \|u\|^{1/2} \|v\| |Aw|^{1/2} |w|^{1/2}, & \forall u \in V_g,\ v \in D(A),\ w \in H_g, \\ c_0 |u|^{1/2} |Au|^{1/2} \|v\| |w|, & \forall u \in D(A),\ v \in V_g,\ w \in H_g, \\ c_0 |u| \|v\| |w|^{1/2} |Aw|^{1/2}, & \forall u \in H_g v \in V_g,\ w \in D(A), \end{cases} \quad (17)$$

$$|b(u,v,w)| + |b(w,v,u)| \le c_0 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w| \quad \forall u \in V_g,\ v \in D(A),\ w \in H_g, \quad (18)$$

$$|b(u,v,w)| + |b(w,v,u)| \le c_0 \left(1 + \ln \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} \|u\| \|v\| |w| \quad \forall u \in D(A),\ v \in V_g,\ w \in H_g, \quad (19)$$

where c_0 are appropriate constants depending only on Ω.

Lemma 2 (see [3]). *Let $u \in L^2(0,T; D(A)) \cap L^\infty(0,T; V_g)$, then the function Bu defined by*

$$(Bu(t), v)_g = b(u(t), u(t), v), \quad \forall v \in H_g, \text{ a.e. } t \in [0,T], \quad (20)$$

belongs to $L^4(0,T; H_g)$, and therefore also belongs to $L^2(0,T; H_g)$.

Lemma 3 (see [4]). *Let $u \in L^2(0,T; V_g)$, then the function Cu defined by*

$$\begin{aligned} (Cu(t), v)_g &= \left(\left(\frac{\nabla g}{g} \cdot \nabla\right) u, v\right)_g \\ &= b\left(\frac{\nabla g}{g}, u, v\right), \quad \forall v \in V_g, \end{aligned} \quad (21)$$

belongs to $L^2(0,T; H_g)$, and hence also belongs to $L^2(0,T; V_g')$. Moreover,

$$\begin{aligned} |Cu(t)| &\le \frac{|\nabla g|_\infty}{m_0} \cdot \|u(t)\|, \quad \text{for a.e. } t \in (0,T), \\ \|Cu(t)\|_* &\le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \cdot \|u(t)\|, \quad \text{for a.e. } t \in (0,T). \end{aligned} \quad (22)$$

Since

$$-\frac{1}{g}(\nabla \cdot g \nabla) u = -\Delta u - \left(\frac{\nabla g}{g} \cdot \nabla\right) u, \quad (23)$$

we have

$$(-\Delta u, v)_g = ((u,v))_g + \left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u, v\right)_g = (Au, v)_g + \left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u, v\right)_g, \quad \forall u, v \in V_g. \tag{24}$$

2.2. Discrete Gronwall Inequalities. Hereafter, we will frequently use the following modified discrete Gronwall lemmas.

Lemma 4 (see [12]). *Let $0 < \Delta t < 1, a_k, g_k, h_k$ for integers $k \geq 0$ be nonnegative numbers such that*

$$\frac{1}{\Delta t}(a_{k+1} - a_k) \leq g_k a_k + h_k \quad \forall 0 \leq k \leq J. \tag{25}$$

If

$$\Delta t \sum_{k=0}^{J} g_k \leq \alpha_1, \qquad \Delta t \sum_{k=0}^{J} h_k \leq \alpha_2, \tag{26}$$

for $J \leq n_0 - 1$ and

$$\Delta t \sum_{k=k_0}^{k_0+n_0-1} g_k \leq \alpha_1, \qquad \Delta t \sum_{k=k_0}^{k_0+n_0-1} h_k \leq \alpha_2, \qquad \Delta t \sum_{k=k_0}^{k_0+n_0-1} a_k \leq a_0 + \alpha_3, \tag{27}$$

for $J \geq n_0$, $0 \leq k_0 \leq J + 1 - n_0$, then

$$a_{J+1} \leq e^{\alpha_1}(a_0 + \alpha_2 + \alpha_3). \tag{28}$$

Lemma 5 (see [13]). *Let α and $a_k, b_k, h_k, k \geq 0$ be nonnegative numbers such that*

$$(1 + \gamma\Delta t) a_{k+1} - a_k + b_{k+1}\Delta t \leq h_k \Delta t \quad \forall 0 \leq k \leq J. \tag{29}$$

Then,

$$a_{J+1} + \Delta t \sum_{k=1}^{J+1} (1+\gamma\Delta t)^{-(J+2-k)} b_k \leq (1+\gamma\Delta t)^{-(J+1)} a_0 + \Delta t \sum_{k=0}^{J} (1+\gamma\Delta t)^{-(J+1-k)} h_k. \tag{30}$$

Lemma 6 (see [11]). *Let β and a_k, b_k, g_k, h_k for integers $k \geq 0$ be nonnegative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \Delta t \sum_{k=0}^{n} g_k a_k + \Delta t \sum_{k=0}^{n} h_k + \beta \quad \forall n \geq 0. \tag{31}$$

Suppose that $g_k \Delta t < 1$, for all $k \geq 0$, and set $\sigma_k = (1 - g_k\Delta t)^{-1}$. Then,

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \exp\left\{\Delta t \sum_{k=0}^{n} \sigma_k g_k\right\}\left(\Delta t \sum_{k=0}^{n} h_k + \beta\right) \quad \forall n \geq 0. \tag{32}$$

Moreover, if

$$a_r + \Delta t \sum_{k=r}^{n} b_k \leq \Delta t \sum_{k=r}^{n} g_k a_k + \Delta t \sum_{k=r}^{n} h_k + \beta \quad \forall 0 \leq r \leq n, \tag{33}$$

and $g_k \Delta t < 1$ for all $0 \leq k \leq n$, then

$$a_r + \Delta t \sum_{k=r}^{n} b_k \leq \exp\left\{\Delta t \sum_{k=0}^{n} \sigma_k g_k\right\}\left(\Delta t \sum_{k=0}^{n} h_k + \beta\right); \quad \forall 0 \leq r \leq n. \tag{34}$$

3. Existence and Some Estimates of Strong Solutions

In this section, we will prove some estimates for the strong solution u and the Galerkin approximate solutions u_m of problem (1). First, with the operators defined in Section 2.1, one can write this problem as follows:

$$\frac{du}{dt} + \nu Au + \nu Cu + B(u,u) = f(t), \quad t > 0, \; u(0) = u_0. \tag{35}$$

Definition 7. For $u_0 \in V_g$ given, a strong solution of problem (1) is a function $u \in L^2(0,T; D(A)) \cap C([0,T]; V_g)$ for all $T > 0$ such that $u(0) = u_0$, and u satisfies (35) in H_g for a.e. $t > 0$.

Theorem 8. *Suppose that $f, f_t \in L^\infty(\mathbb{R}^+; H_g)$ and $u_0 \in V_g$. Then, problem (1) has a unique strong solution u satisfying the following estimates for all $t > 0$,*

$$|u(t)|^2 \leq e^{-\nu\gamma_0\lambda_1 t}|u_0|^2 + \frac{1}{\nu^2\gamma_0^2\lambda_1^2} C_f^2, \qquad \int_0^t e^{\nu\gamma_0\lambda_1 s}\|u\|^2 ds \leq \frac{2}{\nu\gamma_0}|u_0|^2 + \frac{4}{\nu^3\gamma_0^3\lambda_1^2} e^{\nu\gamma_0\lambda_1 t} C_f^2, \tag{36}$$

$$\int_0^t \|u(s)\|^2 ds \leq \frac{1}{\nu\gamma_0}|u_0|^2 + \frac{tC_f^2}{\nu^2\gamma_0^2\lambda_1}, \tag{37}$$

$$\|u(t)\|^2 \leq C_{u_0,f}, \tag{38}$$

$$\tau(t)\left(|u_t(t)|^2 + |Au(t)|^2\right) + \tau^2(t)\|u_t(t)\|^2 \leq \mathcal{K}, \tag{39}$$

$$\int_0^t e^{-\nu\gamma_0\lambda_1(t-s)}\left(|Au|^2 + |u_t|^2 + \tau(s)\|u_t\|^2 + \tau^2(s)|u_{tt}|^2 + \tau^2(s)|Au_t|^2\right) ds \leq \mathcal{K}, \tag{40}$$

where $\gamma_0 = 1 - (|\nabla g|_\infty / m_0\lambda_1^{1/2}) > 0$, $\tau(t) = \min\{1, t\}$, $C_f = \sup_{t\geq 0}|f|$, $C_{u_0,f} = C(\|u_0\|, C_f)$, and $\mathcal{K}$ is a generic positive constant depending only on the data $(\nu, \Omega, |\nabla g|_\infty, \lambda_1, C_f, \|u_0\|)$. Moreover, all above estimates are also valid for the Galerkin approximate solutions u_m of problem (35).

Proof. We refer to [3] for the proof of existence and uniqueness of the strong solution u and estimates (36)–(38). We now prove (39)–(40).

First, we take the scalar product of (35) with $e^{\nu\gamma_0\lambda_1 t}Au$ and $\nu^{-1}e^{\nu\gamma_0\lambda_1 t}u_t$, respectively, and add the resulting relations to obtain

$$\begin{aligned}\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}\|u\|^2\right) &+ e^{\nu\gamma_0\lambda_1 t}\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)\\ &+ e^{\nu\gamma_0\lambda_1 t}b\left(u,u,Au+\nu^{-1}u_t\right)\\ &+ \nu e^{\nu\gamma_0\lambda_1 t}\left(Cu, Au+\nu^{-1}u_t\right)_g\\ &= \nu\gamma_0\lambda_1 e^{\nu\lambda_1 t}\|u\|^2 + e^{\nu\gamma_0\lambda_1 t}\left(f, Au+\nu^{-1}u_t\right).\end{aligned} \tag{41}$$

Using Lemmas 1 and 3 and Cauchy's inequality, we have

$$\begin{aligned}\left|b\left(u,u,Au+\nu^{-1}u_t\right)\right| &\le \frac{1}{4}\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)+c|u|^2\|u\|^2,\\ \left|\nu\left(Cu, Au+\nu^{-1}u_t\right)_g\right| &\le \frac{1}{8}\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)+c\|u\|^2,\\ \left|\left(f, Au+\nu^{-1}u_t\right)\right| &\le \frac{1}{8}\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)+c|f|^2.\end{aligned} \tag{42}$$

By combining these inequalities with (41), we get

$$\begin{aligned}\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}\|u\|^2\right) &+ \frac{1}{2}e^{\nu\gamma_0\lambda_1 t}\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)\\ &\le e^{\nu\gamma_0\lambda_1 t}\left(\nu\lambda_1+c|u|^2\|u\|^2+c\right)\|u\|^2+ce^{\nu\gamma_0\lambda_1 t}|f|^2.\end{aligned} \tag{43}$$

Integrating (43) from 0 to t and using (36)–(38), we obtain, after multiplying by $e^{-\nu\gamma_0\lambda_1 t}$, that

$$e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}\tau(s)\left(\nu|Au|^2+\nu^{-1}|u_t|^2\right)ds \le \mathcal{K} \quad \forall t\ge 0. \tag{44}$$

In view of (44), there exists a sequence $\epsilon_k \to 0$ such that

$$\tau^2(\epsilon_k)\left(\nu|Au(\epsilon_k)|^2+\nu^{-1}|u_t(\epsilon_k)|^2\right)\longrightarrow 0. \tag{45}$$

Now, differentiating (35) with respect to t yields

$$u_{tt}+\nu Au_t+\nu Cu_t+B(u_t,u)+B(u,u_t)=f_t. \tag{46}$$

We take the scalar product (46) with $2u_t$ to obtain

$$\begin{aligned}\frac{d}{dt}|u_t|^2 &+ 2\nu\|u_t\|^2+2\nu(Cu_t,u_t)_g\\ &+ 2b(u_t,u,u_t)+2b(u,u_t,u_t)=2(f_t,u_t)_g.\end{aligned} \tag{47}$$

By Lemma 3, we have

$$\begin{aligned}\frac{d}{dt}|u_t|^2 &+ 2\nu\|u_t\|^2\\ &\le 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u_t\|^2+2\left|b(u_t,u,u_t)\right|\\ &+ 2\left|b(u,u_t,u_t)\right|+2(f_t,u_t)_g.\end{aligned} \tag{48}$$

Using Lemma 1 and Cauchy's inequality, we get

$$\frac{d}{dt}|u_t|^2+\nu\gamma_0\|u_t\|^2 \le c|u_t|^2\|u\|^2+c|f_t|^2. \tag{49}$$

Multiplying the last inequality by $\tau(t)e^{\nu\gamma_0\lambda_1 t}$, we have

$$\begin{aligned}\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}\tau(t)|u_t|^2\right) &+ \nu\gamma_0 e^{\nu\gamma_0\lambda_1 t}\tau(t)\|u_t\|^2\\ &\le e^{\nu\gamma_0\lambda_1 t}\left(1+\nu\gamma_0\lambda_1+c\|u\|^2\right)|u_t|^2+ce^{\nu\gamma_0\lambda_1 t}|f_t|^2.\end{aligned} \tag{50}$$

Therefore, integrating (50) from ϵ_k to t, letting $\epsilon_k \to 0$, and using (44) and (45), one finds, after multiplying by $e^{-\nu\gamma_0\lambda_1 t}$, that

$$\tau(t)|u_t(t)|^2+\nu e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}\tau(s)\|u_t\|^2ds \le \mathcal{K} \quad \forall t\ge 0. \tag{51}$$

Moreover, in view of (35), (36)–(38), and (51), we see that

$$\begin{aligned}\tau(t)|Au(t)|^2 &\le c|f_t(t)|^2+c\tau(t)|u_t(t)|^2\\ &+ c|u(t)|^2\|u(t)\|^4 \le \mathcal{K} \quad \forall t\ge 0.\end{aligned} \tag{52}$$

Also, in view of (51), there exists a sequence $\epsilon_k \to 0$ such that

$$\tau^2(\epsilon_k)\|u_t(\epsilon_k)\|^2\longrightarrow 0. \tag{53}$$

We again take the scalar product (46) with $2Au_t$ to obtain

$$\begin{aligned}\frac{d}{dt}\|u_t\|^2 &+ 2\nu|Au_t|^2+2\nu(Cu_t,Au_t)_g\\ &+ 2b(u_t,u,Au_t)+2b(u,u_t,Au_t)=2(f_t,Au_t)_g.\end{aligned} \tag{54}$$

By Lemma 3 and Cauchy's inequality, we have

$$\begin{aligned}\frac{d}{dt}\|u_t\|^2 &+ 2\nu|Au_t|^2\\ &\le 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|Au_t|^2+\nu\frac{|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\|u_t\|^2\\ &+ 2b\left|(u_t,u,Au_t)\right|+2\left|b(u,u_t,Au_t)\right|+2(f_t,Au_t)_g.\end{aligned} \tag{55}$$

Using Lemma 1, Cauchy's inequality, and Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\|u_t\|^2 &+ \nu\gamma_0|Au_t|^2\\ &\le c\|u\|^4\|u_t\|^2+c|f_t|^2+\nu\frac{|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\|u_t\|^2.\end{aligned} \tag{56}$$

Multiplying the last inequality by $\tau^2(t)e^{\nu\gamma_0\lambda_1 t}$, we have

$$\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}\tau^2(t)\|u_t\|^2\right)+\nu\gamma_0\tau^2(t)e^{\nu\gamma_0\lambda_1 t}|Au_t|^2$$
$$\le ce^{\nu\gamma_0\lambda_1 t}|f_t|^2$$
$$+e^{\nu\gamma_0\lambda_1 t}\left(2+\nu\gamma_0\lambda_1+c\|u\|^4+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\right)\tau(t)\|u_t\|^2. \tag{57}$$

Integrating (57) from ϵ_k to t, letting $\epsilon_k \to 0$, and using (36)–(38), (51), and (53), we obtain, after a final multiplication by $e^{-\nu\gamma_0\lambda_1 t}$,

$$\tau^2(t)\|u_t(t)\|^2+\nu\gamma_0 e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}\tau^2(s)|Au_t|^2 ds\le\mathcal{K}\quad \forall t\ge 0. \tag{58}$$

Using Lemmas 1 and 3 and (46), we deduce that

$$e^{-\nu\gamma_0\lambda_1 t}\int_0^t\tau^2(t)e^{\nu\gamma_0\lambda_1 s}|u_{tt}|^2ds$$
$$\le ce^{-\nu\gamma_0\lambda_1 t}$$
$$\times\int_0^t\tau^2(t)e^{\nu\gamma_0\lambda_1 s}\left(|Au_t|^2+\|u\|^2|Au_t|^2+|f_t|^2\right)ds \quad \forall t\ge 0. \tag{59}$$

Combining (44), (51), (52), (58), and (59) yields (39) and (40).

Finally, we observe that the problem for the approximate solution u_m is similar to problem (35), and

$$|u_m(0)|\le|u_0|,\qquad \|u_m(0)\|\le\|u_0\|, \tag{60}$$

so u_m satisfies the same estimates as those for the strong solution u of problem (35). □

4. Spectral Galerkin Method in Space

For a fixed integer m, let P_m be the orthogonal projection of H_g onto $H_m = \text{span}\{w_1,\dots,w_m\}$ and $Q_m = I - P_m$. Then, every solution u of problem (1) can be decomposed uniquely into

$$u=p+q,\quad \text{where } p=P_m u,\ q=Q_m u. \tag{61}$$

Now, we apply P_m and Q_m to (35) to obtain

$$p_t+\nu Ap+\nu Cp+P_mB(u,u)=P_mf\quad\forall t>0, \tag{62}$$

$$q_t+\nu Aq+\nu Cq+Q_mB(u,u)=Q_mf\quad\forall t>0, \tag{63}$$

and the initial conditions $p(0)=P_mu_0$, $q(0)=Q_mu_0$.

Using Theorem 8 and the property of P_m, we arrive at the following estimates of $q(t)=Q_mu(t)$:

$$|q(t)|^2\le\mathcal{K}\lambda_{m+1}^{-1},$$
$$\tau(t)\left(|q(t)|^2+\lambda_{m+1}^{-1}\|q(t)\|^2\right)\le 2\mathcal{K}\lambda_{m+1}^{-2}\quad\forall t\ge0, \tag{64}$$

$$e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}\left(|q|^2+\lambda_{m+1}^{-1}\|q\|^2\right)ds\le 2\mathcal{K}\lambda_{m+1}^{-2}\quad\forall t\ge0. \tag{65}$$

We now define the spectral Galerkin method as follows: find $u_m(t)\in H_m$ such that

$$u_{mt}+\nu Au_m+\nu Cu_m+P_mB(u_m,u_m)=P_mf\quad\forall t>0, \tag{66}$$

with the initial condition $u_m(0)=P_mu_0$.

In order to give an analysis of the error $u-u_m$ in the L^2-norm, we begin with a technical result concerning a dual linearized g-Navier-Stokes problem which is a similar problem to that used in [17]. We consider, for any given $t>0$ and $\xi\in L^2(0,t;H_g)$, the dual problem: find $\Phi(s)\in H_m$ such that

$$(v,\Phi_s)_g-a(v,\Phi)-\nu(Cv,\Phi)_g-b(v,u_m,\Phi)-b(u_m,v,\Phi)$$
$$=\left(v,e^{\nu\gamma_0\lambda_1 s}\xi\right)_g,\quad 0\le s<t, \tag{67}$$

for all $v\in H_m$ with $\Phi(t)=0$. It is easy to see that (67) is a well-posed problem and has a unique solution $\Phi\in L^\infty(0,t;V_g)\cap L^2(0,t;D(A))$.

Next, we prove a regularity result of problem (67).

Lemma 9. *If $u_0\in V_g$, then the solution $\Phi(s)$ of problem (67) satisfies*

$$e^{-\nu\gamma_0\lambda_1 s}\|\Phi(s)\|^2+\int_s^t e^{-\nu\gamma_0\lambda_1 r}\left(|A\Phi|^2+|\Phi_s|^2\right)dr$$
$$\le\mathcal{K}\exp\left\{\frac{2}{\nu\gamma_0}c_0^2\int_0^t\|u_m\|^2dr\right\}\int_0^t e^{\nu\gamma_0\lambda_1 r}|\xi|^2dr. \tag{68}$$

Proof. Taking $v=-2\Phi$ in (67), we obtain

$$-\frac{d}{ds}|\Phi|^2+2\nu\|\Phi\|^2+2\nu(C\Phi,\Phi)_g+2b(\Phi,u_m,\Phi)$$
$$=-2\left(\Phi,e^{\nu\gamma_0\lambda_1 s}\xi\right)_g\quad\forall 0\le s<t. \tag{69}$$

Using Lemma 3, we have

$$-\frac{d}{ds}|\Phi|^2+2\nu\gamma_0\|\Phi\|^2\le 2\left|\left(\Phi,e^{\nu\gamma_0\lambda_1 s}\xi\right)_g\right|+2\left|b(\Phi,u_m,\Phi)\right|\quad\forall 0\le s<t. \tag{70}$$

Using Lemma 1 and Cauchy's inequality, we get

$$2\left|b(\Phi,u_m,\Phi)\right|\le\frac{\nu\gamma_0}{2}\|\Phi\|^2+\frac{2}{\nu\gamma_0}c_0^2\|u_m\|^2|\Phi|^2,$$
$$2\left|\left(\Phi,e^{\nu\gamma_0\lambda_1 s}\xi\right)_g\right|\le\frac{\nu\gamma_0}{4}\|\Phi\|^2+\frac{4}{\nu\gamma_0\lambda_1}e^{2\nu\gamma_0\lambda_1 s}|\xi|^2. \tag{71}$$

Then, we have

$$-\frac{d}{ds}|\Phi|^2+\frac{5}{4}\nu\gamma_0\|\Phi\|^2 \leq \frac{2}{\nu\gamma_0}c_0^2\|u_m\|^2|\Phi|^2+\frac{4}{\nu\gamma_0\lambda_1}e^{2\nu\gamma_0\lambda_1 s}|\xi|^2 \quad \forall 0\leq s<t. \tag{72}$$

Multiplying this inequality by $e^{-\nu\gamma_0\lambda_1 s}$ and using (14), we obtain

$$-\frac{d}{ds}\left(e^{-\nu\gamma_0\lambda_1 s}|\Phi|^2\right)+\frac{\nu\gamma_0}{4}e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2 \leq \frac{2}{\nu\gamma_0}c_0^2e^{-\nu\gamma_0\lambda_1 s}\|u_m\|^2|\Phi|^2+\frac{4}{\nu\gamma_0\lambda_1}e^{\nu\gamma_0\lambda_1 s}|\xi|^2. \tag{73}$$

Multiplying this inequality by $e^{(2/\nu\gamma_0)c_0^2\int_0^s\|u_m\|^2dr}$ yields

$$-\frac{d}{ds}\left(e^{(2/\nu\gamma_0)c_0^2\int_0^s\|u_m\|^2dr}e^{-\nu\gamma_0\lambda_1 s}|\Phi|^2\right)+\frac{\nu\gamma_0}{4}e^{(2/\nu\gamma_0)c_0^2\int_0^s\|u_m\|^2dr}e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2 \leq \frac{4}{\nu\gamma_0\lambda_1}e^{(2/\nu\gamma_0)c_0^2\int_0^s\|u_m\|^2dr}e^{\nu\gamma_0\lambda_1 s}|\xi|^2. \tag{74}$$

Integrating (74) from s to t and noting that $\Phi(t)=0$, we obtain, after multiplying by $e^{-(2/\nu\gamma_0)c_0^2\int_0^s\|u_m\|^2dr}$, that

$$e^{-\nu\gamma_0\lambda_1 s}|\Phi(s)|^2+\frac{\nu\gamma_0}{4}\int_s^t e^{-\nu\gamma_0\lambda_1 r}\|\Phi\|^2dr \leq \frac{4}{\nu\gamma_0\lambda_1}e^{(2/\nu\gamma_0)c_0^2\int_0^t\|u_m\|^2dr}\int_0^t e^{\nu\gamma_0\lambda_1 r}|\xi|^2dr \quad \forall 0\leq s\leq t. \tag{75}$$

Moreover, inserting $v=-2A\Phi$ into (67), we get

$$-\frac{d}{dt}\|\Phi\|^2+2\nu|A\Phi|^2+2\nu(CA\Phi,\Phi)_g-2\{b(A\Phi,\Phi,u_m)+b(u_m,\Phi,A\Phi)\} = -2\left(A\Phi,e^{\nu\gamma_0\lambda_1 s}\xi^k\right)_g. \tag{76}$$

Using Lemma 3, (14), and Cauchy's inequality, we have

$$2\nu(CA\Phi,\Phi)_g\leq 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|A\Phi|^2+\frac{\nu|\nabla g|_\infty\lambda_m}{2m_0\lambda_1^{1/2}}\|\Phi\|^2. \tag{77}$$

Therefore,

$$-\frac{d}{dt}\|\Phi\|^2+2\nu\gamma_0|A\Phi|^2 \leq 2\left|\left(A\Phi,e^{\nu\gamma_0\lambda_1 s}\xi^k\right)_g\right|+2\left|b(A\Phi,\Phi,u_m)+b(u_m,\Phi,A\Phi)\right| +\frac{\nu|\nabla g|_\infty\lambda_m}{2m_0\lambda_1^{1/2}}\|\Phi\|^2. \tag{78}$$

Multiplying this inequality, by $e^{-\nu\gamma_0\lambda_1 s}$ and using (14), we obtain

$$-\frac{d}{dt}\left(e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2\right)+\nu\gamma_0e^{-\nu\gamma_0\lambda_1 s}|A\Phi|^2 \leq 2\left|\left(A\Phi,\xi^k\right)_g\right| +2e^{-\nu\gamma_0\lambda_1 s}\left|b(A\Phi,\Phi,u_m)+b(u_m,\Phi,A\Phi)\right| +\frac{\nu|\nabla g|_\infty\lambda_m}{2m_0\lambda_1^{1/2}}e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2. \tag{79}$$

Using Lemma 1, Cauchy's inequality and Young's inequality, we have

$$\begin{aligned} 2\left|b(A\Phi,\Phi,u_m)\right|+2\left|b(u_m,\Phi,A\Phi)\right| &\leq c_0|A\Phi|^{3/2}\|\Phi\|^{1/2}\|u_m\|^{1/2}|u_m|^{1/2} \\ &\leq \frac{\nu\gamma_0}{2}|A\Phi|^2+c\|\Phi\|^2\|u_m\|^2|u_m|^2, \\ 2\left|\left(A\Phi,\xi^k\right)_g\right| &\leq \frac{\nu\gamma_0}{4}e^{-\nu\gamma_0\lambda_1 s}|A\Phi|^2+\frac{4}{\nu\gamma_0}e^{\nu\gamma_0\lambda_1 s}\left|\xi^k\right|^2. \end{aligned} \tag{80}$$

Combining the above estimates with (79) yields

$$-\frac{d}{dt}\left(e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2\right)+\frac{\nu\gamma_0}{4}e^{-\nu\gamma_0\lambda_1 s}|A\Phi|^2 \leq ce^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2\|u_m\|^2|u_m|^2 +\frac{\nu|\nabla g|_\infty\lambda_m}{2m_0\lambda_1^{1/2}}e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2+\frac{4}{\nu\gamma_0}e^{\nu\gamma_0\lambda_1 s}|\xi|^2. \tag{81}$$

Integrating this inequality from s to t and using (75), we have

$$e^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2+\frac{\nu\gamma_0}{4}\int_s^t e^{-\nu\gamma_0\lambda_1 r}|A\Phi|^2dr \leq c\left(1+\frac{\nu|\nabla g|_\infty\lambda_m}{2m_0\lambda_1^{1/2}}+\sup_{t\geq 0}|u_m(t)|^2\|u_m(t)\|^2\right) \times e^{(2/\nu\gamma_0)c_0^2\int_0^t\|u_m\|^2dr}\int_0^t e^{\nu\gamma_0\lambda_1 r}|\xi|^2dr. \tag{82}$$

Taking $v=\Phi_s$ in (67), then using (14), Lemmas 1 and 3, we obtain

$$\left|\Phi_s\right|\leq \nu|A\Phi|+\nu\frac{|\nabla g|_\infty\lambda_m}{m_0\lambda_1^{1/2}}\|\Phi\|+c\|u_m\||A\Phi|+e^{\nu\gamma_0\lambda_1 s}|\xi|. \tag{83}$$

Hence,

$$e^{-\nu\gamma_0\lambda_1 s}\left|\Phi_s\right|^2\leq c\left(1+\|u_m\|^2\right)e^{-\nu\gamma_0\lambda_1 s}|A\Phi|^2 +ce^{-\nu\gamma_0\lambda_1 s}\|\Phi\|^2+ce^{\nu\gamma_0\lambda_1 s}|\xi|^2. \tag{84}$$

Integrating (84) from s to t and using (82), (75), and Theorem 8, we complete the proof. □

Lemma 10. *If $u_0 \in V_g$, then the error $P_m u(t) - u_m(t)$ satisfies*

$$\left|P_m u(t) - u_m(t)\right|^2 + \frac{\nu\gamma_0}{4} e^{-\nu\gamma_0\lambda_1 t} \int_0^t e^{\nu\gamma_0\lambda_1 s} \left\|P_m u - u_m\right\|^2 ds \leq \mathscr{K} e^{(2/\nu\gamma_0)c_0^2 \int_0^t \|u(r)\|^2 dr} \lambda_{m+1}^{-1} \quad \forall t \geq 0. \tag{85}$$

Proof. We set $e(t) = P_m u(t) - u_m(t)$ and subtract (66) from (62) to obtain

$$e_t + \nu A e + \nu C e + P_m B\left(Q_m u + e, u\right) + P_m B\left(u_m, Q u_m + e\right) = 0, \tag{86}$$

with $e(0) = 0$. Taking the scalar product of (86) with $2e$, we obtain

$$\frac{d}{dt}|e|^2 + 2\nu\|e\|^2 + 2(Ce, e)_g + 2\left(b(e, u, e) + b\left(u_m, Q_m u, e\right) + b\left(Q_m u, u, e\right)\right) = 0. \tag{87}$$

Using Lemma 3, we get

$$\frac{d}{dt}|e|^2 + 2\nu\gamma_0\|e\|^2 \leq 2\left|b(e, u, e) + b\left(u_m, Q_m u, e\right) + b\left(Q_m u, u, e\right)\right|. \tag{88}$$

Multiplying the last inequality by $e^{\nu\gamma_0\lambda_1 t}$ and using (14), we obtain

$$\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}|e|^2\right) + \nu\gamma_0 e^{\nu\gamma_0\lambda_1 t}\|e\|^2 \leq 2e^{\nu\gamma_0\lambda_1 t}\left|b(e, u, e) + b\left(u_m, Q_m u, e\right) + b\left(Q_m u, u, e\right)\right|. \tag{89}$$

Due to Lemma 1 and Cauchy's inequality, we have

$$\begin{aligned} 2\left|b(e, u, e)\right| &\leq 2c_0 |e| \|e\| \|u\| \leq \frac{\nu\gamma_0}{2}\|e\|^2 + \frac{2}{\nu\gamma_0} c_0^2 \|u\|^2 |e|^2, \\ 2\left|b\left(u_m, Q_m u, e\right)\right| + 2\left|b\left(Q_m u, u, e\right)\right| &\leq 2c_0 \lambda_1^{-1/2}\left|Q_m u\right|\left(|Au| + \left|Au_m\right|\right)\|e\| \\ &\leq \frac{\nu\gamma_0}{4}\|e\|^2 + c\left(|Au|^2 + \left|Au_m\right|^2\right)\left|Q_m u\right|^2. \end{aligned} \tag{90}$$

Combining (89) with the above estimate yields

$$\frac{d}{dt}\left(e^{\nu\gamma_0\lambda_1 t}|e|^2\right) + \frac{\nu\gamma_0}{4} e^{\nu\gamma_0\lambda_1 t}\|e\|^2 \leq \frac{2}{\nu\gamma_0} c_0^2 e^{\nu\gamma_0\lambda_1 t}\|u\|^2|e|^2 + c e^{\nu\gamma_0\lambda_1 t}\left(|Au|^2 + \left|Au_m\right|^2\right)\left|Q_m u\right|^2. \tag{91}$$

Multiplying (91) by $e^{-(2/\nu\gamma_0)c_0^2 \int_0^s \|u\|^2 dr}$ yields

$$\begin{aligned} &\frac{d}{dt}\left(e^{-(2/\nu\gamma_0)c_0^2 \int_0^s \|u\|^2 dr} e^{\nu\gamma_0\lambda_1 t}|e|^2\right) + \frac{\nu\gamma_0}{4} e^{-(2/\nu\gamma_0)c_0^2 \int_0^s \|u\|^2 dr} e^{\nu\gamma_0\lambda_1 t}\|e\|^2 \\ &\quad \leq c e^{-(2/\nu\gamma_0)c_0^2 \int_0^s \|u\|^2 dr} e^{\nu\gamma_0\lambda_1 t}\left(|Au|^2 + \left|Au_m\right|^2\right)\left|Q_m u\right|^2 \\ &\quad \leq c e^{\nu\gamma_0\lambda_1 t}\left(|Au|^2 + \left|Au_m\right|^2\right)\left|Q_m u\right|^2. \end{aligned} \tag{92}$$

Integrating this inequality from 0 to t and using (64), Theorem 8, we obtain, after a final multiplication by $e^{(2/\nu\gamma_0)c_0^2 \int_0^s \|u\|^2 dr} e^{-\nu\gamma_0\lambda_1 t}$, that

$$|e|^2 + \frac{\nu\gamma_0}{4} e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}\|e\|^2 ds \leq \mathscr{K} e^{(2/\nu\gamma_0)c_0^2 \int_0^t \|u\|^2 dr} \lambda_{m+1}^{-1} \quad \forall t \geq 0, \tag{93}$$

which is (85). □

Lemma 11. *If $u_0 \in V_g$, then the error $P_m u(t) - u_m(t)$ satisfies the following bound:*

$$\tau(t)\left|P_m u(t) - u_m(t)\right|^2 \leq \mathscr{K} e^{(4/\nu\gamma_0)c_0^2 \int_0^t (\|u\|^2 + \|u_m\|^2) dr} \lambda_{m+1}^{-2} \quad \forall t \geq 0. \tag{94}$$

Proof. Take $v = e(s)$ and $\xi = e(s)$ in (67) to obtain

$$\left(e, \Phi_s\right)_g - a(e, \Phi) - \nu(Ce, \Phi)_g - b\left(e, u_m, \Phi\right) - b\left(u_m, e, \phi\right) = e^{\nu\gamma_0\lambda_1 s}|e|^2. \tag{95}$$

Multiplying (86) by Φ, we have

$$\left(e_s, \Phi\right)_g + a(e, \Phi) + \nu(Ce, \Phi)_g + b\left(u_m, e + Q_m u, \Phi\right) + b\left(e + Q_m u, u_m, \Phi\right) + b\left(u - u_m, u - u_m, \Phi\right) = 0. \tag{96}$$

Adding (96) and (95), we get

$$e^{\nu\gamma_0\lambda_1 s}|e|^2 = \frac{d}{ds}(e, \Phi)_g + b\left(u_m, Q_m u, \Phi\right) + b\left(Q_m u, u_m, \Phi\right) + b\left(u - u_m, u - u_m, \Phi\right). \tag{97}$$

Using Lemma 1, we have

$$\begin{aligned} \left|b\left(u_m, Q_m u, \Phi\right)\right| + \left|b\left(Q_m u, u_m, \Phi\right)\right| &\leq c\left\|u_m\right\|\left|Q_m u\right||A\Phi|, \\ \left|b\left(u - u_m, u - u_m, \Phi\right)\right| &\leq c\left|u - u_m\right|\left|u - u_m\right|\|A\Phi\|. \end{aligned} \tag{98}$$

Hence, we deduce from (97) that

$$e^{\nu\gamma_0\lambda_1 s}|e|^2 \leq \frac{d}{ds}(e, \Phi)_g + c\left(\left\|u_m\right\|\left|Q_m u\right| + \left|u - u_m\right|\left\|u - u_m\right\|\right)|A\Phi|. \tag{99}$$

Integrating (99) from 0 to t and noting that $e(0) = \Phi(t) = 0$, we obtain

$$\int_0^t e^{\nu\gamma_0\lambda_1 s}|e|^2 ds \le c\left(\int_0^t e^{\nu\gamma_0\lambda_1 s}\left(\|u_m\|^2|Q_m u|^2 + |u-u_m|^2\|u-u_m\|^2\right)ds\right)^{1/2} \times\left(\int_0^t e^{-\nu\gamma_0\lambda_1 s}|A\Phi|^2 ds\right)^{1/2}. \tag{100}$$

Using (64), (65), Theorem 8, and Lemmas 9 and 10, we deduce from (100) that

$$e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 s}|e|^2 ds \le \mathscr{K} e^{(4/\nu\gamma_0)c_0^2\int_0^t(\|u\|^2+\|u_m\|^2)dr}\lambda_{m+1}^{-2}. \tag{101}$$

Now, multiplying (91) by $\tau(t)$, we have

$$\frac{d}{dt}\left(\tau(t)e^{\nu\gamma_0\lambda_1 t}|e|^2\right) + \frac{\nu\gamma_0}{4}\tau(t)e^{\nu\gamma_0\lambda_1 t}\|e\|^2 \le \left(1+\frac{2}{\nu\gamma_0}c_0^2\|u\|^2\right)e^{\nu\gamma_0\lambda_1 t}|e|^2 + ce^{\nu\gamma_0\lambda_1 t}\tau(t)\left(|Au|^2+|Au_m|^2\right)|Q_m u|^2. \tag{102}$$

Integrating (102) from 0 to t and using Theorem 8, we obtain, after a final multiplication by $e^{-\nu\gamma_0\lambda_1 t}$,

$$\tau(t)|e(t)|^2 + \frac{\nu\gamma_0}{4}e^{-\nu\gamma_0\lambda_1 t}\int_0^t \tau(s)e^{\nu\gamma_0\lambda_1 s}\|e\|^2 ds \le \mathscr{K}e^{-\nu\gamma_0\lambda_1 t}\int_0^t e^{\nu\gamma_0\lambda_1 t}\left(|e|^2+|Q_m u|^2\right)ds. \tag{103}$$

Using (65) and (101) in (103) yields

$$\tau(t)|e(t)|^2 + \frac{\nu\gamma_0}{4}e^{-\nu\gamma_0\lambda_1 t}\int_0^t \tau(s)e^{\nu\gamma_0\lambda_1 s}\|e\|^2 ds \le \mathscr{K}e^{(4/\nu\gamma_0)c_0^2\int_0^t(\|u\|^2+\|u_m\|^2)dr}\lambda_{m+1}^{-2}, \tag{104}$$

which is (94). □

Finally, by combining Lemma 11 with (64) and using Theorem 8, we get the following error estimate.

Theorem 12. *If $u \in V_g$, then the error $u(t) - u_m(t)$ satisfies the following bound:*

$$\tau(t)\left|u(t)-u_m(t)\right|^2 \le \mathscr{K}e^{(8/\nu^3\gamma_0^3\lambda_1)c_0^2C_f^2 t}\lambda_{m+1}^{-2} \quad \forall t \ge 0. \tag{105}$$

5. Spectral Galerkin Method in Space and Time

5.1. Stability Analysis. In this subsection, we consider the semi-implicit Euler scheme applied to the spatially discrete spectral Galerkin approximation, show the stability of this scheme, and establish some preliminaries related to the error analysis uniform in time.

We consider the semi-implicit Euler scheme and define recursively a solution $\{u_m^n\} \subset H_m$ such that

$$d_t u_m^{n+1} + \nu A u_m^{n+1} + \nu C u_m^{n+1} + P_m B\left(u_m^n, u_m^n\right) = P_m f^{n+1}, \tag{106}$$

for $n \ge 0$ with the initial condition $u_m^0 = P_m u_0$, where $\Delta t > 0$ is a time step such that $n_0\Delta t = 1$ for some integer n_0 and

$$d_t u_m^{n+1} = \frac{1}{\Delta t}\left(u_m^{n+1} - u_m^n\right), \tag{107}$$

$$f^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(t)\,dt, \quad t_n = n\Delta t. \tag{108}$$

In order to derive the L^2-bound on the error $u_m(t_n) - u_m^n$, we will begin with a time discrete duality argument which is similar to the one used in [11, 17]. We consider the dual scheme correponding to scheme (106): for any fixed $n \ge 1$ and $u_m^{k-1}, \xi^k \in H_m$, $1 \le k \le n$, find $\Phi^{k-1} \in H_m$, $1 \le k \le n$ such that

$$\left(v, d_t\Phi^k\right)_g - a\left(v,\Phi^{k-1}\right) - \nu\left(Cv,\Phi^{k-1}\right)_g - b\left(v,u_m^{k-1},\Phi^{k-1}\right) - b\left(u_m^{k-1},v,\Phi^{k-1}\right) = \left(1+\nu\gamma_0\lambda_1\Delta t\right)^k\left(v,\xi^k\right)_g \quad \forall v \in H_m, \tag{109}$$

with an initial condition $\Phi^n = 0$.

The following theorem provides the H^2-stability of scheme (106).

Theorem 13. *Under the assumptions of Theorem 8, if Δt and m satisfy the following condition:*

$$\frac{8}{\nu\gamma_0}c_0^2\ln\frac{\lambda_m}{\lambda_1}\Delta t C_{u_0,f} \le 1, \tag{110}$$

then the semi-implicit Euler scheme is the H^2-stability; that is,

$$\Delta t\sum_{k=1}^n\left\|u_m^k\right\|^2 \le \frac{1}{\nu\gamma_0}|u_0|^2 + \frac{1}{\nu^2\gamma_0^2\lambda_1}C_f^2 t_n \quad \forall n \ge 0, \tag{111}$$

$$\Delta t\sum_{k=k_0+1}^n\left\|u_m^k\right\|^2 \le \frac{1}{\nu\gamma_0}|u_0|^2 + \frac{2}{\nu^3\gamma_0^3\lambda_1^2}C_f^2 + \frac{2}{\nu^2\gamma_0^2\lambda_1}C_f^2\left(t_n - t_{k_0}\right) \quad \forall n \ge k_0 \ge 0, \tag{112}$$

$$\left|u_m^n\right|^2 \le \left(1+\nu\gamma_0\lambda_1\Delta t\right)^{-n}\left|u_0\right|^2 + \frac{1}{\nu^2\gamma_0^2\lambda_1^2}C_f^2 \quad \forall n \ge 0, \tag{113}$$

$$\left\|u_m^n\right\|^2 \le C_{u_0,f} \quad \forall n \ge 0, \tag{114}$$

$$\begin{aligned}&\Delta t\sum_{k=1}^{n}\left(1+\nu\gamma_0\lambda_1\Delta t\right)^{-(n-1-k)} \\ &\quad\times\left(\left|Au_m^k\right|^2+\left|d_tu_m^k\right|^2+\tau\left(t_k\right)\left\|d_tu_m^k\right\|^2\right) \\ &\quad+\tau\left(t_n\right)\left|Au_m^n\right|^2+\tau\left(t_n\right)\left|d_tu_m^n\right|^2 \le \mathscr{K} \quad \forall n \ge 0.\end{aligned} \tag{115}$$

Proof. Clearly, scheme (106) defines a unique sequence $\{u_m^n\} \subset H_m$. Now, we will prove (111)–(115).

Taking the scalar product of (106) with $2u_m^{n+1}\Delta t$, we obtain

$$\begin{aligned}&\left|u_m^{n+1}\right|^2-\left|u_m^n\right|^2+\left|u_m^{n+1}-u_m^n\right|^2+2\nu\left\|u_m^{n+1}\right\|^2\Delta t \\ &\quad+2\nu\Delta t\left(Cu_m^{n+1},u_m^{n+1}\right)_g+2b\left(u_m^n,u_m^n-u_m^{n+1},u_m^{n+1}\right)\Delta t \\ &\quad=2\left(f^{n+1},u_m^{n+1}\right)_g\Delta t.\end{aligned} \tag{116}$$

Using Lemma 3 and Cauchy's inequality, we have

$$\begin{aligned}&\left|u_m^{n+1}\right|^2-\left|u_m^n\right|^2+\left|u_m^{n+1}-u_m^n\right|^2+2\nu\left\|u_m^{n+1}\right\|^2\Delta t \\ &\quad\le 2\nu\frac{\left|\nabla g\right|_\infty}{m_0\lambda_1^{1/2}}\left\|u_m^{n+1}\right\|^2\Delta t+2\left(f^{n+1},u_m^{n+1}\right)_g\Delta t \\ &\quad-2b\left(u_m^n,u_m^n-u_m^{n+1},u_m^{n+1}\right)\Delta t.\end{aligned} \tag{117}$$

Hence,

$$\begin{aligned}&\left|u_m^{n+1}\right|^2-\left|u_m^n\right|^2+\left|u_m^{n+1}-u_m^n\right|^2+2\nu\gamma_0\left\|u_m^{n+1}\right\|^2\Delta t \\ &\quad\le 2\left(f^{n+1},u_m^{n+1}\right)_g\Delta t-2b\left(u_m^n,u_m^n-u_m^{n+1},u_m^{n+1}\right)\Delta t.\end{aligned} \tag{118}$$

By (19), we have

$$\begin{aligned}&2\left(f^{n+1},u_m^{n+1}\right)_g \le \frac{\nu\gamma_0}{2}\left\|u_m^{n+1}\right\|^2+\frac{2}{\nu\gamma_0\lambda_1}\left|f^{n+1}\right|^2\Delta t, \\ &2\left|b\left(u_m^n,u_m^n-u_m^{n+1},u_m^{n+1}\right)\right| \\ &\quad\le \frac{\nu\gamma_0}{2}\left\|u_m^{n+1}\right\|^2+L\left(u_m^n\right)\left|u_m^n-u_m^{n+1}\right|^2,\end{aligned} \tag{119}$$

where

$$L\left(u_m^n\right)=\frac{8}{\nu\gamma_0}c_0^2\left(1+\ln\frac{\left|Au_m^n\right|^2}{\lambda_1\left\|u_m^n\right\|^2}\right)\left\|u_m^n\right\|^2. \tag{120}$$

Substituting those into (118), we get

$$\begin{aligned}&\left|u_m^{n+1}\right|^2-\left|u_m^n\right|^2+\left(1-L\left(u_m^n\right)\Delta t\right)\left|u_m^{n+1}-u_m^n\right|^2 \\ &\quad+\nu\gamma_0\left\|u_m^{n+1}\right\|^2\Delta t \le \frac{2}{\nu\gamma_0\lambda_1}\left|f^{n+1}\right|^2\Delta t.\end{aligned} \tag{121}$$

Next, by taking the scalar product of (106) with $2Au_m^{n+1}\Delta t$, we obtain

$$\begin{aligned}&\left\|u_m^{n+1}\right\|^2-\left\|u_m^n\right\|^2+\left\|u_m^{n+1}-u_m^n\right\|^2+2\nu\left|Au_m^{n+1}\right|^2\Delta t \\ &\quad+2\nu\Delta t\left(Cu_m^{n+1},Au_m^{n+1}\right)_g+2b\left(u_m^n,u_m^n,Au_m^{n+1}\right)\Delta t \\ &\quad=2\left(f^{n+1},Au_m^{n+1}\right)_g\Delta t.\end{aligned} \tag{122}$$

Using Lemma 3 and Cauchy's inequality, we have

$$\begin{aligned}&2\nu\left(Cu_m^{n+1},Au_m^{n+1}\right)_g \\ &\quad\le 2\nu\frac{\left|\nabla g\right|_\infty}{m_0\lambda_1^{1/2}}\left|Au_m^{n+1}\right|^2+\frac{\nu\left|\nabla g\right|_\infty\lambda_1^{1/2}}{2m_0}\left\|u_m^{n+1}\right\|^2.\end{aligned} \tag{123}$$

Substituting into (122), we get

$$\begin{aligned}&\left\|u_m^{n+1}\right\|^2-\left\|u_m^n\right\|^2+\left\|u_m^{n+1}-u_m^n\right\|^2+2\nu\gamma_0\left|Au_m^{n+1}\right|^2\Delta t \\ &\quad\le 2\left(f^{n+1},Au_m^{n+1}\right)_g\Delta t-2b\left(u_m^n,u_m^n,Au_m^{n+1}\right)\Delta t \\ &\quad+\frac{\nu\left|\nabla g\right|_\infty\lambda_1^{1/2}}{2m_0}\left\|u_m^{n+1}\right\|^2\Delta t.\end{aligned} \tag{124}$$

Due to (17)–(19), Cauchy's inequality, and Young's inequality, we have

$$\begin{aligned}&2\left|b\left(u_m^n,u_m^{n+1}-u_m^n,Au_m^{n+1}\right)\right| \\ &\quad\le \frac{\nu\gamma_0}{4}\left|Au_m^{n+1}\right|^2+L\left(u_m^n\right)\left\|u_m^{n+1}-u_m^n\right\|^2, \\ &2\left|b\left(u_m^n,u_m^{n+1},Au_m^{n+1}\right)\right| \\ &\quad\le \nu\gamma_0\left|Au_m^{n+1}\right|^2+2\left(\nu\gamma_0\right)^{-3}c_0^4\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2, \\ &2\left(f^{n+1},Au_m^{n+1}\right)_g \le \frac{\nu\gamma_0}{2}\left|Au_m^{n+1}\right|^2+\frac{2}{\nu\gamma_0}C_f^2.\end{aligned} \tag{125}$$

By combining (124) with the above estimates, we obtain

$$\begin{aligned}&\left\|u_m^{n+1}\right\|^2-\left\|u_m^n\right\|^2+\left(1-L\left(u_m^n\right)\Delta t\right)\left\|u_m^{n+1}-u_m^n\right\|^2 \\ &\quad\le 2\left(\nu\gamma_0\right)^{-3}c_0^4\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2\Delta t+\frac{2}{\nu\gamma_0}C_f^2\Delta t \\ &\quad+\frac{\nu\left|\nabla g\right|_\infty\lambda_1^{1/2}}{2m_0}\left\|u_m^{n+1}\right\|^2\Delta t.\end{aligned} \tag{126}$$

On the other hand, from (17)–(19), Cauchy's inequality, and Young's inequality, we have

$$\begin{aligned} 2\left|b\left(u_m^n, u_m^{n+1}-u_m^n, Au_m^{n+1}\right)\right| &\leq \frac{\nu\gamma_0}{4}\left|Au_m^{n+1}\right|^2 + L\left(u_m^n\right)\left\|u_m^{n+1}-u_m^n\right\|^2, \\ 2\left|b\left(u_m^n, u_m^{n+1}, Au_m^{n+1}\right)\right| &\leq \frac{\nu\gamma_0}{8}\left|Au_m^{n+1}\right|^2 + c\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2, \\ 2\left(f^{n+1}, Au_m^{n+1}\right)_g &\leq \frac{\nu\gamma_0}{8}\left|Au_m^{n+1}\right|^2 + cC_f^2. \end{aligned} \tag{127}$$

Combining (124) with those estimates, we obtain

$$\begin{aligned} &\left(1+\nu\gamma_0\lambda_1\Delta t\right)\left\|u_m^{n+1}\right\|^2 - \left\|u_m^n\right\|^2 \\ &\quad + \left(1-L\left(u_m^n\right)\Delta t\right)\left\|u_m^{n+1}-u_m^n\right\|^2 + \frac{\nu\gamma_0}{2}\left|Au_m^{n+1}\right|^2\Delta t \\ &\leq c\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2\Delta t + cC_f^2\Delta t + c\left\|u_m^{n+1}\right\|^2\Delta t. \end{aligned} \tag{128}$$

Next, from (106) we have

$$\begin{aligned} &\left|d_t u_m^{n+1}\right|^2 + \nu\left(Au_m^{n+1}, d_t u_m^{n+1}\right) + \nu\left(Cu_m^{n+1}, d_t u_m^{n+1}\right)_g \\ &\quad + b\left(u^n, u^n, d_t u_m^{n+1}\right) = \left(f^{n+1}, d_t u_m^{n+1}\right)_g. \end{aligned} \tag{129}$$

Using (11), Lemmas 1 and 3, and Cauchy's inequality, we get

$$\begin{aligned} \left|b\left(u^n, u^n, d_t u_m^{n+1}\right)\right| &\leq c\left|u_m^n\right|^2\left\|u_m^n\right\|\left\|u_m^{n+1}\right\|^2, \\ \nu\left|\left(Au_m^{n+1}, d_t u_m^{n+1}\right)_g\right| &\leq \frac{1}{4}\left|d_t u_m^{n+1}\right|^2 + \nu^2\left|Au_m^{n+1}\right|^2, \\ \nu\left|\left(Cu_m^{n+1}, d_t u_m^{n+1}\right)_g\right| &\leq \frac{1}{4}\left|d_t u_m^{n+1}\right|^2 + c\left\|u_m^{n+1}\right\|^2, \\ \left|\left(f^{n+1}, d_t u_m^{n+1}\right)_g\right| &\leq \frac{1}{4}\left|d_t u_m^{n+1}\right|^2 + cC_f^2. \end{aligned} \tag{130}$$

Therefore, we have

$$\begin{aligned} &\frac{\gamma_0}{16\nu}\left|d_t u_m^{n+1}\right|^2 - \frac{\nu\gamma_0}{4}\left|Au_m^{n+1}\right|^2 \\ &\quad \leq c\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2 + cC_f^2 + c\left\|u_m^{n+1}\right\|^2. \end{aligned} \tag{131}$$

Combining this inequality with (128) yields

$$\begin{aligned} &\left(1+\nu\gamma_0\lambda_1\Delta t\right)\left\|u_m^{n+1}\right\|^2 \\ &\quad - \left\|u_m^n\right\|^2 + \left(1-L\left(u_m^n\right)\Delta t\right)\left\|u_m^{n+1}-u_m^n\right\|^2 \\ &\quad + \frac{\gamma_0}{16\nu}\left|d_t u_m^{n+1}\right|^2\Delta t + \frac{\nu\gamma_0}{4}\left|Au_m^{n+1}\right|^2\Delta t \\ &\leq c\left|u_m^n\right|^2\left\|u_m^n\right\|^2\left\|u_m^{n+1}\right\|^2\Delta t + cC_f^2\Delta t + c\left\|u_m^{n+1}\right\|^2\Delta t. \end{aligned} \tag{132}$$

Moreover, we deduce from (106) that

$$\begin{aligned} &d_{tt}u_m^{n+1} + \nu A d_t u_m^{n+1} + \nu C d_t u_m^{n+1} + P_m B\left(d_t u_m^n, u_m^n\right) \\ &\quad + P_m B\left(u_m^n, d_t u_m^n\right) = P_m d_t f^{n+1}, \end{aligned} \tag{133}$$

where $d_t u_m^0$ is defined by

$$d_t u_m^0 + \nu A u_m^0 + \nu C u_m^0 + P_m B\left(u_m^0, u_m^0\right) = P_m f(0). \tag{134}$$

Taking the scalar product of (133) with $2d_t u_m^{n+1}\Delta t$, we obtain

$$\begin{aligned} &\left|d_t u_m^{n+1}\right|^2 - \left|d_t u_m^n\right|^2 + \left|d_t u_m^{n+1} - d_t u_m^n\right|^2 \\ &\quad + 2\nu\left\|d_t u_m^{n+1}\right\|^2\Delta t + 2\nu\left(C d_t u_m^{n+1}, d_t u_m^{n+1}\right)\Delta t \\ &\quad + 2b\left(d_t u_m^n - d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\Delta t \\ &\quad + 2b\left(d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\Delta t \\ &\quad + 2b\left(d_t u_m^{n+1}, d_t u_m^n - d_t u_m^{n+1}, d_t u_m^{n+1}\right)\Delta t \\ &= 2\left(d_t f^{n+1}, d_t u_m^{n+1}\right)\Delta t. \end{aligned} \tag{135}$$

By Lemma 3, we get

$$\begin{aligned} &\left|d_t u_m^{n+1}\right|^2 - \left|d_t u_m^n\right|^2 + \left|d_t u_m^{n+1} - d_t u_m^n\right|^2 + 2\nu\gamma_0\left\|d_t u_m^{n+1}\right\|^2\Delta t \\ &\leq 2\left(d_t f^{n+1}, d_t u_m^{n+1}\right)\Delta t \\ &\quad - 2b\left(d_t u_m^n - d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\Delta t \\ &\quad - 2b\left(d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\Delta t \\ &\quad - 2b\left(d_t u_m^{n+1}, d_t u_m^n - d_t u_m^{n+1}, d_t u_m^{n+1}\right)\Delta t. \end{aligned} \tag{136}$$

Using Lemma 1 and (19), we have

$$\begin{aligned} &2\left|b\left(d_t u_m^n - d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\right| \\ &\quad + 2\left|b\left(d_t u_m^{n+1}, d_t u_m^n - d_t u_m^{n+1}, d_t u_m^{n+1}\right)\right| \\ &\quad \leq \frac{\nu\gamma_0}{2}\left\|d_t u_m^{n+1}\right\|^2 + \frac{1}{2}\left(L\left(u_m^n\right) + L\left(u_m^{n+1}\right)\right)\left|d_t u_m^{n+1} - d_t u_m^n\right|^2, \\ &2\left|b\left(d_t u_m^{n+1}, u_m^n, d_t u_m^{n+1}\right)\right| \leq \frac{\nu\gamma_0}{8}\left\|d_t u_m^{n+1}\right\|^2 \\ &\qquad + \frac{8}{\nu\gamma_0}c_0^2\left\|u_m^n\right\|^2\left|d_t u_m^{n+1}\right|^2, \\ &2\left(d_t f^{n+1}, d_t u_m^{n+1}\right) \leq \frac{\nu\gamma_0}{8}\left\|d_t u_m^{n+1}\right\|^2 + \frac{8}{\nu\gamma_0\lambda_1}\sup_{t\geq 0}\left|f_t(t)\right|^2. \end{aligned} \tag{137}$$

By combining (136) with the above estimates, we arrive at

$$\begin{aligned}(1+\nu\gamma_0\lambda_1\Delta t)\left|d_tu_m^{n+1}\right|^2-\left|d_tu_m^n\right|^2 &+\left(1-\frac{1}{2}\left(L\left(u_m^n\right)+L\left(u_m^{n+1}\right)\right)\Delta t\right)\left|d_tu_m^{n+1}-d_tu_m^n\right|^2\\ &+\frac{\nu\gamma_0}{4}\left\|d_tu_m^{n+1}\right\|^2\Delta t\le\frac{8}{\nu\gamma_0}c_0^2\left\|u_m^n\right\|^2\left|d_tu_m^{n+1}\right|^2\Delta t\\ &+\frac{8}{\nu\gamma_0\lambda_1}\sup_{t\ge0}\left|f_t(t)\right|^2.\end{aligned}\tag{138}$$

Multiplying (138) by $\tau(t_{n+1})$ and noting that $\tau(t_{n+1})\le\tau(t_n)+\Delta t$, we obtain

$$\begin{aligned}(1+\nu\gamma_0\lambda_1\Delta t)\,\tau(t_{n+1})\left|d_tu_m^{n+1}\right|^2-\tau(t_n)\left|d_tu_m^n\right|^2 &+\left(1-\frac{1}{2}\left(L\left(u_m^n\right)+L\left(u_m^{n+1}\right)\right)\Delta t\right)\\ &\times\tau(t_{n+1})\left|d_tu_m^{n+1}-d_tu_m^n\right|^2\\ &+\frac{\nu\gamma_0}{4}\tau(t_{n+1})\left\|d_tu_m^{n+1}\right\|^2\Delta t\\ \le\left|d_tu_m^n\right|^2\Delta t+\frac{8}{\nu\gamma_0}c_0^2\left\|u_m^n\right\|^2\left|d_tu_m^{n+1}\right|^2\Delta t &+\frac{8}{\nu\gamma_0\lambda_1}\sup_{t\ge0}\left|f_t(t)\right|^2\Delta t.\end{aligned}\tag{139}$$

Now, we will prove (111)–(115) by induction. Obviously, (111)–(115) are true for $n=0$. Assuming that (111)–(115) hold for $n=0,1,\ldots,J$, we need to prove (111)–(115) for $n=J+1$. In view of (110) and the inductive assumption, we obtain

$$1-L\left(u_m^n\right)\Delta t\ge0\quad\forall 0\le n\le J.\tag{140}$$

Summing (121) from 0 to J and k_0+1 to J, respectively, and using (140), we obtain (111) and (112) with $n=J+1$ after a final multiplication by $(\nu\gamma_0)^{-1}$. Noting that $\lambda_1|v|^2\le\|v\|^2$, using (140) in (121), and then applying Lemma 5 with

$$\begin{aligned}a_k&=\left|u_m^k\right|^2, & b_k&=0,\\ h_k&=\frac{2}{\nu\gamma_0\lambda_1}C_f^2, & \gamma&=\nu\gamma_0\lambda_1,\end{aligned}\tag{141}$$

we obtain (113) for $n=J+1$.

Furthermore, setting

$$\begin{aligned}a_k&=\left\|u_m^k\right\|^2, \qquad g_k=2(\nu\gamma_0)^{-3}c_0^4\left|u_m^k\right|^2\left\|u_m^{k+1}\right\|^2,\\ h_k&=\frac{2}{\nu\gamma_0}C_f^2+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}\left\|u_m^{k+1}\right\|^2,\end{aligned}\tag{142}$$

in (126), using (111)–(113) and (140), we obtain

$$\frac{1}{\Delta t}\left(a_{k+1}-a_k\right)\le g_ka_k+h_k\quad\forall 0\le k\le J,\tag{143}$$

with

$$\Delta t\sum_{k=0}^{J}g_k\le\alpha_1,\qquad \Delta t\sum_{k=0}^{J}h_k\le\alpha_1,\qquad a_0\le\left\|u_0\right\|^2,\tag{144}$$

for all $J+1\le n_0-1$ and

$$\begin{aligned}&\Delta t\sum_{k=k_0}^{k_0+n_0-1}g_k\le\alpha_1,\qquad \Delta t\sum_{k=k_0}^{k_0+n_0-1}h_k\le\alpha_2,\\ &\Delta t\sum_{k=k_0}^{k_0+n_0-1}a_k\le a_0+\alpha_3,\quad 0\le k_0\le J+1-n_0,\end{aligned}\tag{145}$$

for all $J+1\ge n_0$. Applying Lemma 4 to (143), we obtain (114) for $n=J+1$.

Applying again Lemma 4 to (132) with

$$\begin{aligned}a_k&=\left\|u_m^k\right\|^2,\\ b_k&=\frac{1}{16\nu\gamma_0}\left|d_tu_m^k\right|^2+\frac{\nu\gamma_0}{4}\left|Au_m^k\right|^2,\quad \gamma=\nu\gamma_0\lambda_1,\\ h_k&=c\left|u_m^k\right|^2\left\|u_m^k\right\|^2\left\|u_m^{k+1}\right\|^2+cC_f^2+c\left\|u_m^{k+1}\right\|^2,\end{aligned}\tag{146}$$

and using (111)–(114) and (140), we obtain

$$\Delta t\sum_{k=1}^{J+1}(1+\nu\gamma_0\lambda_1\Delta t)^{-(J+2-k)}\left(\left|Au_m^k\right|+\left|d_tu_m^k\right|^2\right)\le\mathcal{K}.\tag{147}$$

Applying Lemma 5 to (139) with

$$\begin{aligned}a_k&=\tau(t_k)\left|d_tu_m^k\right|^2,\qquad b_k=\frac{\nu\gamma_0}{4}\tau(t_k)\left\|d_tu_m^k\right\|^2,\qquad \gamma=\nu\gamma_0\lambda_1,\\ h_k&=\left|d_tu_m^k\right|^2\Delta t+\frac{8}{\nu\gamma_0}c_0^2\left\|u_m^k\right\|^2\left|d_tu_m^{k+1}\right|^2+\frac{8}{\nu\gamma_0\lambda_1}\sup_{t\ge0}\left|f_t(t)\right|^2,\end{aligned}\tag{148}$$

and using (114), (140), and (147) yields

$$\begin{aligned}&\tau(t_{J+1})\left|d_tu_m^{J+1}\right|^2\\ &+\frac{\nu\gamma_0}{4}\Delta t\sum_{k=1}^{J+1}(1+\nu\gamma_0\lambda_1\Delta t)^{-(J+2-k)}\tau(t_k)\left\|d_tu_m^k\right\|^2\le\mathcal{K}.\end{aligned}\tag{149}$$

Finally, due to (17)–(19), we have

$$\begin{aligned}\left|b\left(u_m^J,u^J-u_m^{J+1},Au_m^{J+1}\right)\right|&\le\frac{\nu\gamma_0}{8}\left|Au_m^{J+1}\right|^2\\ &\quad+cL\left(u_m^J\right)\left\|d_tu_m^{J+1}\right\|^2\Delta t^2,\\ \left|b\left(u_m^J,u_m^{J+1},Au_m^{J+1}\right)\right|&\le\frac{\nu\gamma_0}{8}\left|Au_m^{J+1}\right|^2\\ &\quad+c\left|u_m^J\right|^2\left\|u_m^J\right\|^2\left\|u_m^{J+1}\right\|^2.\end{aligned}\tag{150}$$

Combining these inequalities with (106) and (140) yields

$$\tau t_{J+1}\left|Au_m^{J+1}\right|^2 \le c\tau\left(t_{J+1}\right)\left|d_t u_m^{J+1}\right|^2 + c\left|u_m^J\right|^2\left\|u_m^J\right\|^2\left\|u_m^{J+1}\right\|^2 + cC_f^2 + c\tau\left(t_{J+1}\right)\left\|d_t u_m^{J+1}\right\|^2\Delta t. \tag{151}$$

Using (113), (114) and (149) in this inequality yields

$$\tau\left(t_{J+1}\right)\left|Au_m^{J+1}\right|^2 \le \mathcal{K}. \tag{152}$$

Combining this inequality with (149) and (147) implies (115) for $n = J + 1$. □

The following lemma provides the stability of scheme (109).

Lemma 14. *If* $u_0 \in V_g$, Δt *and* m *satisfy* (110), *then problem* (109) *admits a unique solution* $\{\Phi^k\}_0^n \subset H_m$ *satisfying*

$$(1+\nu\gamma_0\lambda_1\Delta t)^{-r}\left\|\Phi^r\right\|^2 + \Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(k-1)}\left|A\Phi^k\right|^2 \le \mathcal{K}\exp\left(\frac{4}{\nu\gamma_0}c_0^2\Delta t\sum_{k=1}^{n}\left\|u_m^k\right\|^2\right)\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k\left|\xi^k\right|^2. \tag{153}$$

Proof. In view of (17), (110), and (114), we can prove that the following bilinear form

$$\frac{1}{\Delta t}(v,\Phi) + a(v,\Phi) + \nu(Cv,\Phi) + b\left(v,u_m^{k-1},\Phi\right) + b\left(u_m^{k-1},v,\Phi\right) \quad \forall v,\Phi \in H_m \tag{154}$$

is elliptic. Hence, problem (109) has a unique solution Φ^{k-1} for $1 \le k \le n$. Next, by taking $v = -2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\Phi^{k-1}\Delta t$ in (109) and using (11), we have

$$\begin{aligned}
&-2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left(\Phi^{k-1},\Phi^k\right)_g \\
&\quad + 2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|\Phi^{k-1}\right|^2 \\
&\quad + 2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|\Phi^{k-1}\right\|^2\Delta t \\
&\quad + 2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left(C\Phi^{k-1},\Phi^{k-1}\right)_g\Delta t \\
&\quad + 2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}b\left(\Phi^{k-1},u_m^{k-1},\Phi^{k-1}\right)\Delta t \\
&= -2\left(\Phi^{k-1},\xi^k\right)_g\Delta t.
\end{aligned} \tag{155}$$

Using Lemma 3, we get

$$\begin{aligned}
&2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|\Phi^{k-1}\right|^2 \\
&\quad + 2\nu\gamma_0(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|\Phi^{k-1}\right\|^2\Delta t \\
&\le 2\left|\left(\Phi^{k-1},\xi^k\right)_g\right|\Delta t + 2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|\left(\Phi^{k-1},\Phi^k\right)_g\right| \\
&\quad + 2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|b\left(\Phi^{k-1},u_m^{k-1},\Phi^{k-1}\right)\right|\Delta t.
\end{aligned} \tag{156}$$

Using (17) and Cauchy's inequality, we have

$$\begin{aligned}
2\left|\left(\Phi^{k-1},\Phi^k\right)_g\right| &\le \left|\Phi^{k-1}\right|^2 + \left|\Phi^k\right|^2, \\
2\left|\left(\Phi^{k-1},\xi^k\right)_g\right| &\le \frac{\nu\gamma_0}{4}(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|\Phi^{k-1}\right\|^2 + \frac{4}{\nu\gamma_0\lambda_1}(1+\nu\gamma_0\lambda_1\Delta t)^k\left|\xi^k\right|^2, \\
2\left|b\left(\Phi^{k-1},u_m^{k-1},\Phi^{k-1}\right)\right| &\le \frac{\nu\gamma_0}{2}\left\|\Phi^{k-1}\right\|^2 + \frac{2}{\nu\gamma_0}c_0^2\left|\Phi^{k-1}\right|^2\left\|u^{k-1}\right\|^2.
\end{aligned} \tag{157}$$

Combining above inequalities and using (14), we obtain

$$\begin{aligned}
&(1+\nu\gamma_0\lambda_1\Delta t)^{-(k-1)}\left|\Phi^{k-1}\right|^2 - (1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|\Phi^k\right|^2 \\
&\quad + \frac{\nu\gamma_0}{4}(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|\Phi^{k-1}\right\|^2\Delta t \\
&\le \frac{2}{\nu\gamma_0}c_0^2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|u_m^{k-1}\right\|^2\left|\Phi^{k-1}\right|^2\Delta t \\
&\quad + \frac{4}{\nu\gamma_0\lambda_1}(1+\nu\gamma_0\lambda_1\Delta t)^k\left|\xi^k\right|^2\Delta t.
\end{aligned} \tag{158}$$

Summing (158) from $r+1$ to n, noting that $\Phi^n = 0$, and using Theorem 13, we arrive at

$$\begin{aligned}
&(1+\nu\gamma_0\lambda_1\Delta t)^{-r}\left|\Phi^r\right|^2 \\
&\quad + \frac{\nu\gamma_0}{4}\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(k+1)}\left\|\Phi^k\right\|^2 \\
&\le \frac{2}{\nu\gamma_0}c_0^2\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left\|u_m^k\right\|^2\left|\Phi^k\right|^2 \\
&\quad + \frac{4}{\nu\gamma_0\lambda_1}\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k\left|\xi^k\right|^2
\end{aligned} \tag{159}$$

for all $0 \le r \le n$. Let

$$a_r = (1+\nu\gamma_0\lambda_1\Delta t)^{-r}|\Phi^r|^2,$$
$$b_r = \frac{\nu\gamma_0}{4}(1+\nu\gamma_0\lambda_1\Delta t)^{-(r+1)}\|\Phi^r\|^2,$$
$$h_r = \frac{4}{\nu\gamma_0\lambda_1}(1+\nu\gamma_0\lambda_1\Delta t)^{r}|\xi^k|^2, \quad (160)$$
$$g_r = \frac{2}{\nu\gamma_0}c_0^2\|u_m^k\|^2, \qquad \beta = 0,$$

in (159) to obtain

$$a_r + \Delta t\sum_{k=r}^{n} b_k \le \Delta t\sum_{k=r}^{n} g_k a_k + \Delta t\sum_{k=r}^{n} h_k \quad \forall 0 \le r \le n, \quad (161)$$

with $\sigma_r = (1-g_r\Delta t)^{-1} \le 2$. Applying Lemma 6 to this inequality yields

$$(1+\nu\gamma_0\lambda_1\Delta t)^{-r}|\Phi^r|^2 + \frac{\nu\gamma_0}{2}\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(k+1)}\|\Phi^k\|^2$$
$$\le \mathscr{K}\exp\left(\frac{4}{\nu\gamma_0}c_0^2\Delta t\sum_{k=1}^{n}\|u_m^k\|^2\right)\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k}|\xi^k|^2. \quad (162)$$

Moreover, by taking $v = -2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}A\Phi^{k-1}\Delta t$ in (109), we have

$$\begin{aligned} &-2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}(A\Phi^{k-1},\Phi^k)_g \\ &+2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\|\Phi^{k-1}\|^2 \\ &+2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}|A\Phi^{k-1}|^2\Delta t \\ &+2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}(CA\Phi^{k-1},\Phi^{k-1})_g\Delta t \\ &+2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}b(A\Phi^{k-1},\Phi^{k-1},u_m^{k-1})\Delta t \\ &+2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}b(u_m^{k-1},\Phi^{k-1},A\Phi^{k-1})\Delta t \\ &=2(A\Phi^{k-1},\xi^k)_g\Delta t. \end{aligned} \quad (163)$$

Using Lemma 3 and Cauchy's inequality, we deduce that

$$\begin{aligned} &2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|(CA\Phi^{k-1},\Phi^{k-1})_g\right| \\ &\le 2\nu(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|A\Phi^{k-1}|^2 \\ &+c(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\|\Phi^{k-1}\|^2. \end{aligned} \quad (164)$$

Therefore,

$$\begin{aligned} &2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\|\Phi^{k-1}\|^2 \\ &+2\nu\gamma_0(1+\nu\gamma_0\lambda_1\Delta t)^{-k}|A\Phi^{k-1}|^2\Delta t \\ &\le 2(A\Phi^{k-1},\xi^k)_g\Delta t + 2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}(A\Phi^{k-1},\Phi^k)_g \\ &+2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|b(A\Phi^{k-1},\Phi^{k-1},u_m^{k-1})\right|\Delta t \\ &+2(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left|b(u_m^{k-1},\Phi^{k-1},A\Phi^{k-1})\right|\Delta t \\ &+c(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\|\Phi^{k-1}\|^2\Delta t. \end{aligned} \quad (165)$$

Using (17)–(19) and Cauchy's inequality, we have

$$\begin{aligned} &2\left|b(A\Phi^{k-1},\Phi^{k-1},u_m^{k-1})\right| + 2\left|b(u_m^{k-1},\Phi^{k-1},A\Phi^{k-1})\right| \\ &\le \frac{\nu\gamma_0}{4}|A\Phi^{k-1}|^2 + c|u_m^{k-1}|^2\|u_m^{k-1}\|^2\|\Phi^{k-1}\|^2, \\ &2(A\Phi^{k-1},\Phi^k)_g = 2((\Phi^{k-1},\Phi^k))_g \le \|\Phi^{k-1}\|^2 + \|\Phi^k\|^2, \\ &2(A\Phi^{k-1},\xi^k)_g \le \frac{\nu\gamma_0}{4}(1+\nu\gamma_0\lambda_1\Delta t)^{-k}|A\Phi^{k-1}|^2 \\ &\qquad + \frac{4}{\nu\gamma_0}(1+\nu\gamma_0\lambda_1\Delta t)^{k}|\xi^k|^2. \end{aligned} \quad (166)$$

Combining above inequality, noting that $\lambda_1\|v\|^2 \le |Av|^2$, we obtain

$$\begin{aligned} &(1+\nu\gamma_0\lambda_1\Delta t)^{-(k-1)}\|\Phi^{k-1}\|^2 - (1+\nu\gamma_0\lambda_1\Delta t)^{-k}\|\Phi^k\|^2 \\ &+\frac{\nu\gamma_0}{2}(1+\nu\gamma_0\lambda_1\Delta t)^{-k}|A\Phi^{k-1}|^2\Delta t \\ &\le c(1+\nu\gamma_0\lambda_1\Delta t)^{-k}\left(|u_m^{k-1}|^2\|u_m^{k-1}\|^2+1\right)\|\Phi^{k-1}\|^2\Delta t \\ &+c(1+\nu\gamma_0\lambda_1\Delta t)^{k}|\xi^k|^2\Delta t \end{aligned} \quad (167)$$

for all $0 \le k \le n$. Summing (167) from $r+1$ to n, we obtain

$$\begin{aligned} &(1+\nu\gamma_0\lambda_1\Delta t)^{-r}\|\Phi^r\|^2 + \frac{\nu\gamma_0}{2}\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(k+1)}|A\Phi^k|^2 \\ &\le c\sup_{k\ge 0}\left\{\left(|u_m^k|^2\|u_m^k\|^2+1\right)\right\}\Delta t\sum_{k=r}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(k+1)}\|\Phi^k\|^2 \\ &+c\Delta t\sum_{k=r+1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k}|\xi^k|^2 \quad \forall 0 \le r \le n. \end{aligned} \quad (168)$$

Combining (168) with (162) and using Theorem 13, we complete the proof. □

5.2. Error Analysis. In this subsection, we will establish the H^1- and L^2-error estimates uniform in time for the fully discrete spectral Galerkin method with the explicit time discretization for the nonlinear term. To do this, by integrating (35) from t_n to t_{n+1}, we obtain

$$d_t u_m(t_{n+1}) + \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} A u_m(t)\,dt + \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} C u_m(t)\,dt + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} P_m B(u_m(t), u_m(t))\,dt = f^{n+1}. \tag{169}$$

Subtracting (106) from (153) and setting $e^n = u_m(t_n) - u_m^n$, we have

$$d_t e^{n+1} + \nu A e^{n+1} + \nu C e^{n+1} + P_m B(e^n, u_m^n) + P_m B(u_m(t_n), e^n) = P_m E_{n+1}, \tag{170}$$

with $e^0 = 0$ and

$$\begin{aligned} E_{n+1} &= \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} A(u_m(t_{n+1}) - u_m(t))\,dt + \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} C(u_m(t_{n+1}) - u_m(t))\,dt \\ &\quad + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (B(u_m(t_n), u_m(t_n)) - B(u_m(t), u_m(t)))\,dt \\ &= \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n) A u_{mt}\,dt + \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n) C u_{mt}\,dt \\ &\quad - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t_{n+1} - t)(B(u_{mt}, u_m) + B(u_m, u_{mt}))\,dt. \end{aligned} \tag{171}$$

To derive a bound on e^n, we need to provide the following estimates of E_n.

Lemma 15. *Under the assumptions of Theorem 13, the error E_n satisfies the following bounds:*

$$\Delta t \sum_{k=1}^{n} (1 + \nu\gamma_0\lambda_1)^{k-1} \left|A^{-1} P_m E_k\right|^2 \le \mathcal{K}(1 + \nu\gamma_0\lambda_1)^n \Delta t^2 \quad \forall n \ge 1, \tag{172}$$

$$\Delta t \sum_{k=1}^{n} (1 + \nu\gamma_0\lambda_1)^{k-1} \left|A^{-1/2} P_m E_k\right|^2 \le \mathcal{K}(1 + \nu\gamma_0\lambda_1)^n \Delta t \quad \forall n \ge 1, \tag{173}$$

$$\Delta t \sum_{k=1}^{n} (1 + \nu\gamma_0\lambda_1)^{k-1} \tau(t_k) \left|A^{-1/2} P_m E_k\right|^2 \le \mathcal{K}(1 + \nu\gamma_0\lambda_1)^n \Delta t^2 \quad \forall n \ge 1. \tag{174}$$

Proof. In view of (17)–(19), Lemma 3, and Theorem 8, we deduce from (171) that

$$\begin{aligned} &\left|A^{-1} P_m E_n\right| \\ &= \sup_{v \in H_m} \frac{|(E_n, v)|}{|Av|} \\ &\le \Delta t^{-1/2} \left(\int_{t_{n-1}}^{t_n} \left(c_1 \left((t - t_{n-1})^2 + \frac{|\nabla g|_\infty}{m_0} (t - t_{n-1})^2 \right) |u_{mt}|^2 + c_2 (t_n - t)^2 \|u_m\|^2 |u_{mt}|^2 \right) dt \right)^{1/2} \\ &\le c\Delta t^{-1/2} \left(\int_{t_{n-1}}^{t_n} \left((t - t_{n-1})^2 + (t_n - t)^2 \|u_m\|^2 \right) |u_{mt}|^2 dt \right)^{1/2}, \\ &\left|A^{-1/2} P_m E_n\right| = \sup_{v \in H_m} \frac{|(E_n, v)|}{\|v\|} \\ &\le c\Delta t^{-1/2} \times \left(\int_{t_{n-1}}^{t_n} \left((t - t_{n-1})^2 + (t_n - t)^2 \|u_m\|^2 \right) \times \|u_{mt}\|^2 dt \right)^{1/2}. \end{aligned} \tag{175}$$

By Theorem 8, (175), and the fact that $(1 + \nu\gamma_0\lambda_1)^{k-1} \le e^{\nu\gamma_0\lambda_1 t_{k-1}}$, we have

$$\begin{aligned} &(1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \left|A^{-1} P_m E_k\right|^2 \Delta t \le \mathcal{K}\Delta t^3 (1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \quad \forall k \ge n_0 + 1, \\ &(1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \left|A^{-1/2} P_m E_k\right|^2 \Delta t \le \mathcal{K}\Delta t^3 (1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \quad \forall k \ge n_0 + 1, \end{aligned} \tag{176}$$

$$(1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \left|A^{-1} P_m E_k\right|^2 \Delta t \le \mathcal{K}\Delta t^2 \int_{t_{k-1}}^{t_k} e^{\nu\gamma_0\lambda_1 t} |u_{mt}|^2 dt \quad \forall 1 \le k \le n_0, \tag{177}$$

$$(1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \left|A^{-1/2} P_m E_k\right|^2 \Delta t \le \mathcal{K}\Delta t^2 \int_{t_{k-1}}^{t_k} e^{\nu\gamma_0\lambda_1 t} \|u_{mt}\|^2 dt \quad \forall 1 \le k \le n_0. \tag{178}$$

For $n \ge n_0 + 1$, summing (176) from $n_0 + 1$ to n, we have

$$\Delta t \sum_{k=n_0+1}^{n} (1 + \nu\gamma_0\lambda_1\Delta t)^{k-1} \left(\left|A^{-1} P_m E_k\right|^2 + \left|A^{-1/2} P_m E_k\right|^2 \right) \le \mathcal{K}\Delta t^3 \sum_{k=n_0+1}^{n} (1 + \nu\gamma_0\lambda_1\Delta t)^{k-1}$$

$$\leq \mathcal{K}\Delta t^3(1+\nu\gamma_0\lambda_1\Delta t)^{n_0}\frac{1-(1+\nu\gamma_0\lambda_1\Delta t)^{n-n_0}}{-\nu\gamma_0\lambda_1\Delta t} \leq \mathcal{K}\Delta t^2(1+\nu\gamma_0\lambda_1\Delta t)^{n}. \quad (179)$$

For $n \leq n_0$, we sum (177) from 1 to n and use Theorem 8 to obtain

$$\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left|A^{-1}P_mE_k\right|^2 \leq \mathcal{K}\Delta t^2 e^{\nu\gamma_0\lambda_1} \leq \mathcal{K}\Delta t^2(1+\nu\gamma_0\lambda_1\Delta t)^{n} \quad \forall n\leq n_0, \quad (180)$$

which, together with (179), gives (172).

Finally, multiplying (178) by $\tau^j(t_k)$, $j=0,1$; $2\leq k\leq n_0$, and noting that $\Delta t\leq \tau(t_{k-1})$, $\tau(t_k)\leq 2\tau(t_{k-1})$, we obtain

$$(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\tau^j(t_k)\left|A^{-1/2}P_mE_k\right|^2\Delta t \leq \mathcal{K}\Delta t^{1-j}\int_{t_{k-1}}^{t_k} e^{\nu\gamma_0\lambda_1 t}\tau^j(t)\left\|u_{mt}\right\|^2 dt \quad \forall 2\leq k\leq n_0. \quad (181)$$

Observing again (170) with $n=1$ and using Theorem 8 yields

$$\tau^j(t_1)\left|A^{-1/2}P_mE_k\right|^2\Delta t \leq \mathcal{K}\sup_{0\leq t\leq t_1}\left\|u_m(t)-u_m(0)\right\|^2\Delta t^{1-j}. \quad (182)$$

For $2\leq n\leq n_0$, summing (181) from 2 to n and using Theorem 12, we obtain

$$\Delta t\sum_{k=2}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\tau^j(t_k)\left|A^{-1/2}P_mE_k\right|^2 \leq \mathcal{K}\Delta t^{1-j}e^{\nu\gamma_0\lambda_1}. \quad (183)$$

Combining (182) and (183) with (179) yields (173) and (174). □

Now, we prove the following error estimate.

Lemma 16. *Under the assumptions of Theorem 13, one has*

$$\left|e^n\right|^2+\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{-(n-1-k)} \times\left(\frac{\nu\gamma_0}{8}\left\|e^k\right\|^2\Delta t+\left|e^k-e^{k-1}\right|^2\right) \leq \mathcal{K}\exp\left(\frac{8}{\nu\gamma_0}c_0^2\Delta t\sum_{k=1}^{n}\left\|u_m^k\right\|^2\right)\Delta t \quad \forall n\geq 1. \quad (184)$$

Proof. Taking the scalar product of (170) with $2e^{n+1}\Delta t$, using (11), Lemma 3 and noting that $\lambda_1|v|^2\leq\|v\|^2$, we have

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)\left|e^{n+1}\right|^2-\left|e^n\right|^2 + \left|e^{n+1}-e^n\right|^2+\nu\gamma_0\left\|e^{n+1}\right\|^2\Delta t\\&+2\left(b\left(e^{n+1},u_m^n,e^{n+1}\right)+b\left(e^n-e^{n+1},u_m^n,e^{n+1}\right) + b\left(u_m(t_n),e^n-e^{n+1},e^{n+1}\right)\right)\Delta t\\&\leq 2\left(E_{n+1},e^{n+1}\right)\Delta t.\end{aligned} \quad (185)$$

Using (17)–(19), we get

$$\begin{aligned}&2\left|b\left(e^{n+1},u_m^n,e^{n+1}\right)\right|\leq\frac{\nu\gamma_0}{4}\left\|e^{n+1}\right\|^2+\frac{4}{\nu\gamma_0}c_0^2\left\|u_m^n\right\|^2\left|e^{n+1}\right|^2,\\&2\left|b\left(e^n-e^{n+1},u_m^n,e^{n+1}\right)\right| + 2\left|b\left(u_m(t_n),e^n-e^{n+1},e^{n+1}\right)\right|\\&\leq\frac{9\nu\gamma_0}{16}\left\|e^{n+1}\right\|^2+\frac{4}{9}\left(L\left(u_m^n\right)+L\left(u_m(t_n)\right)\right)\left|e^{n+1}-e^n\right|^2,\\&2\left|\left(E_{n+1},e^{n+1}\right)\right|\leq\frac{\nu\gamma_0}{16}\left\|e^{n+1}\right\|^2+\frac{16}{\nu\gamma_0}\left|A^{-1/2}P_mE_{n+1}\right|^2.\end{aligned} \quad (186)$$

Hence, by combining the above inequalities with (185) and using Theorem 8, (110) and (114), we obtain

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^{n+1}\left|e^{n+1}\right|^2-(1+\nu\gamma_0\lambda_1\Delta t)^{n}\left|e^n\right|^2 + (1+\nu\gamma_0\lambda_1\Delta t)^{n}\left(\frac{\nu\gamma_0}{8}\left\|e^{n+1}\right\|^2\Delta t+\frac{1}{9}\left|e^{n+1}-e^n\right|^2\right)\\&\leq\frac{4}{\nu\gamma_0}c_0^2\left\|u_m^n\right\|^2(1+\nu\gamma_0\lambda_1\Delta t)^{n}\left|e^{n+1}\right|^2\Delta t + \frac{16}{\nu\gamma_0}(1+\nu\gamma_0\lambda_1\Delta t)^{n}\left|A^{-1/2}P_mE_{n+1}\right|^2\Delta t,\end{aligned} \quad (187)$$

for all $n\geq 1$. Summing (187) from 1 to $n-1$ and using Lemma 15, we obtain

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^{n}\left|e^n\right|^2 + \sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left(\frac{\nu\gamma_0}{8}\left\|e^k\right\|^2\Delta t+\frac{1}{9}\left|e^k-e^{k-1}\right|^2\right)\\&\leq\frac{4}{\nu\gamma_0}c_0^2\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left\|u_m^{k-1}\right\|^2\left|e^k\right|^2 + \frac{16}{\nu\gamma_0}\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left|A^{-1/2}P_mE_k\right|^2.\end{aligned} \quad (188)$$

We set

$$a_k = (1+\nu\gamma_0\lambda_1\Delta t)^k |e^k|^2,$$
$$b_k = (1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left(\frac{\nu\gamma_0}{8}\|e^k\|^2 + \frac{1}{9}|e^k - e^{k-1}|^2\Delta t^{-1}\right),$$
$$g_k = \frac{4}{\nu\gamma_0}c_0^2\|u_m^{k-1}\|^2,$$
$$h_k = \frac{16}{\nu\gamma_0}\Delta t(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}|A^{-1/2}P_m E_k|^2, \qquad \beta = 0 \tag{189}$$

in (188) to obtain

$$a_n + \Delta t\sum_{k=1}^{n} b_k \le \Delta t\sum_{k=1}^{n} g_k a_k + \Delta t\sum_{k=1}^{n} h_k, \tag{190}$$

with $\sigma_k = (1-g_k\Delta t)^{-1} \le 2$. Applying Lemma 6 to (190) and using Lemma 15, we deduce that

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^n|e^n|^2 \\ &\quad + \sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^{k-1}\left(\frac{\nu\gamma_0}{8}\|e^k\|^2\Delta t + \frac{1}{9}|e^k - e^{k-1}|^2\right) \\ &\quad \le (1+\nu\gamma_0\lambda_1\Delta t)^n \mathscr{K}\exp\left(\frac{8}{\nu\gamma_0}c_0^2\Delta t\sum_{k=0}^{n-1}\|u_m^k\|^2\right)\Delta t.\end{aligned} \tag{191}$$

Multiplying (191) by $(1+\nu\gamma_0\lambda_1\Delta t)^{-n}$, we get (184). □

Lemma 17. *Under the assumptions of Theorem 13, one has*

$$\begin{aligned}&\tau(t_n)|u_m(t_n) - u_m^n|^2 \\ &\quad \le \mathscr{K}\exp\left(\frac{16}{\nu\gamma_0}c_0^2\Delta t\sum_{k=0}^{n-1}\|u_m^k\|^2\right)\Delta t^2 \quad \forall n \ge 1.\end{aligned} \tag{192}$$

Proof. Replacing $n+1$ by k in (170) and taking the scalar product of (170) with Φ^{k-1}, we obtain

$$\begin{aligned}&(d_t e^k, \Phi^{k-1})_g + a(e^k, \Phi^{k-1}) + \nu(Ce^k, \Phi^{k-1})_g \\ &\quad + b(e^{k-1}, u_m^{k-1}, \Phi^{k-1}) + b(u_m^{k-1}, e^{k-1}, \Phi^{k-1}) \\ &\quad + b(e^{k-1}, e^{k-1}, \Phi^{k-1}) = (E_k, \Phi^{k-1})_g.\end{aligned} \tag{193}$$

Next, setting $v = e^k$, $\xi^k = e^k$, $1 \le k \le n$ in (109), we get

$$\begin{aligned}&(e^k, d_t\Phi^k)_g - a(e^k, \Phi^{k-1}) - \nu(Ce^k, \Phi^{k-1})_g \\ &\quad - b(e^k, u_m^{k-1}, \Phi^{k-1}) - b(u_m^{k-1}, e^k, \Phi^k) \\ &\quad = (1+\nu\gamma_0\lambda_1\Delta t)^k|\xi^k|^2.\end{aligned} \tag{194}$$

Adding (193) to (194), we arrive at

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^k|\xi^k|^2 \\ &\quad = \frac{1}{\Delta t}\left((e^k, \Phi^k)_g - (e^{k-1}, \Phi^{k-1})_g\right) \\ &\quad + b(e^{k-1} - e^k, u_m^{k-1}, \Phi^{k-1}) + b(u_m^{k-1}, e^{k-1} - e^k, \Phi^{k-1}) \\ &\quad + b(e^{k-1}, e^{k-1}, \Phi^{k-1}) - (E_k, \Phi^{k-1})_g.\end{aligned} \tag{195}$$

From (17) and (18), noting that $\lambda_1|v|^2 \le \|v\|^2$, one finds that

$$\begin{aligned}&|b(e^{k-1} - e^k, u_m^{k-1}, \Phi^{k-1}) + b(u_m^{k-1}, e^{k-1} - e^k, \Phi^{k-1})| \\ &\quad \le c|e^k - e^{k-1}|\,\|u_m^{k-1}\|\,|A\Phi^{k-1}|, \\ &|b(e^{k-1}, e^{k-1}, \Phi^{k-1})| \le c\|e^{k-1}\|\,|e^{k-1}|\,|A\Phi^{k-1}|, \\ &|(E_k, \Phi^{k-1})_g| \le |A^{-1}P_m E_k|\,|A\Phi^{k-1}|.\end{aligned} \tag{196}$$

Combining (195) with the above estimates yields

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^k|\xi^k|^2\Delta t \\ &\quad \le \left((e^k, \Phi^k)_g - (e^{k-1}, \Phi^{k-1})_g\right) + |A^{-1}P_m E_k|\,|A\Phi^{k-1}|\Delta t \\ &\quad + c\left(|e^k - e^{k-1}|\,\|u_m^{k-1}\| + \|e^{k-1}\|\,|e^{k-1}|\right)|A\Phi^{k-1}|\Delta t,\end{aligned} \tag{197}$$

with $e^0 = \Phi^n = 0$. Summing (197) from 1 to n and using Lemma 15, we obtain

$$\begin{aligned}&\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k|\xi^k|^2 \\ &\quad \le \mathscr{K}\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k \\ &\qquad \times\left(|e^k - e^{k-1}|^2\|u_m^{k-1}\|^2 + \|e^{k-1}\|^2|e^{k-1}|^2\right) \\ &\qquad + \Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k|A^{-1}P_m E_k|^2.\end{aligned} \tag{198}$$

Applying Lemmas 15 and 16 and Theorem 8 in (198), we get

$$\begin{aligned}&(1+\nu\gamma_0\lambda_1\Delta t)^{-n}\Delta t\sum_{k=1}^{n}(1+\nu\gamma_0\lambda_1\Delta t)^k|\xi^k|^2 \\ &\quad \le \mathscr{K}\exp\left(\frac{16}{\nu\gamma_0}c_0^2\Delta t\sum_{n=0}^{n-1}\|u_m^k\|^2\right)\Delta t^2 \quad \forall n \ge 1.\end{aligned} \tag{199}$$

Now, multiplying (187) by $\tau(t_{n+1})$ and noting that

$$\tau(t_{n+1}) \le \Delta t + \tau(t_n), \quad \Delta t \le \tau(t_n)\ \forall 1 \le n \le N,\ e^0 = 0, \tag{200}$$

we arrive at

$$\begin{aligned}
&\left(1+\nu\gamma_0\lambda_1\Delta t\right)^{n+1}\tau\left(t_{n+1}\right)\left|e^{n+1}\right|^2 \\
&\quad-\left(1+\nu\gamma_0\lambda_1\Delta t\right)^{n}\tau\left(t_{n}\right)\left|e^{n}\right|^2 \\
&\leq \mathscr{K}\left(1+\nu\gamma_0\lambda_1\Delta t\right)^{n}\left|e^{n}\right|^2\Delta t \\
&\quad+k\left(1+\nu\gamma_0\lambda_1\Delta t\right)^{n}\tau\left(t_{n+1}\right)\left|A^{-1/2}P_mE_{n+1}\right|^2\Delta t.
\end{aligned} \tag{201}$$

Summing (201) from 0 to $n-1$ and using (199) and (174), we obtain, after a final multiplication by $(1+\nu\gamma_0\lambda_1\Delta t)^{-n}$, the estimate (192). □

Finally, combining Theorems 12, 8, and 13 with Lemma 17, we obtain the error estimate of the numerical solution $\{u_m^n\}$.

Theorem 18. *Under the assumptions of Theorem 13, the following error estimates holds:*

$$\left|u\left(t_n\right)-u_m^n\right|^2 \leq \mathscr{K}\tau^{-1}\left(t_n\right)e^{(16/\nu^3\gamma_0^3\lambda_1)c_0^2C_f^2t_n}\left(\lambda_{m+1}^{-2}+\Delta t^2\right) \quad \forall n\geq 1. \tag{202}$$

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