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Random fixed points of completely random operators

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Abstract. The purpose of this paper is to examine the notion of completely random operators and to prove some random fixed point theorems for such operators. Unlike many random fixed point theorems for random operators, it seems to be impossible to obtain them from deterministic fixed point theorems.

Keywords. Random operator, completely random operator, Lipschitz random operator, random fixed point.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let X, Y be separable metric spaces and let $f : \Omega \times X \to Y$ be a random operator in the sense that for each fixed x in X, the mapping $\omega \mapsto f(\omega, x)$ is measurable. An X-valued random variable ξ is said to be a random fixed point of the random operator $f : \Omega \times X \to X$ if

$$f(\omega, \xi(\omega)) = \xi(\omega) \quad \text{a.s.}$$

Random fixed point theory is an important topic of the stochastic analysis. In recent years, many random fixed point theorems have been proved (see, e.g., [1–4]). Some authors (see, e.g., [2,6–8]) have shown that under some assumptions the random operator $f : \Omega \times X \to X$ has a random fixed point if and only if for almost all ω the deterministic mapping $f_\omega : x \mapsto f(\omega, x)$ has a fixed point. Therefore, the existence of a random fixed point follows immediately from the existence of corresponding deterministic fixed point.

A random operator $f : \Omega \times X \to Y$ may be considered as an action which transforms each deterministic input x in X into a random output $f(\omega, x)$ with values in Y. Taking into account many circumstances in which the inputs are also subject to influence of a random environment, an action which transforms each

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random input with values in X into random output with values in Y is called a completely random operator from X into Y.

In Section 2, we give some definitions concerning completely random operators and some properties of such operators. Section 3 involves some results about random fixed points of weakly contractive and (f, q)-contractive completely random operators. It should be noted that the existence of random fixed point of a completely random operator does not follow from the existence of the corresponding deterministic fixed point theorem as in the case of random operator. In the last section, some applications to random equations are presented.

2 Some properties of completely random operators

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and B be a separable Banach space. A mapping $\xi : \Omega \to B$ is called a B-valued random variable if ξ is $(\mathcal{F}, \mathcal{B})$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra of B. The set of all (equivalent classes) B-valued random variables is denoted by $L_0^B(\Omega)$ and it is equipped with the topology of convergence in probability. For each $p > 0$, the set of B-valued random variables ξ such that $E\|\xi\|^p < \infty$ is denoted by $L_p^B(\Omega)$.

At first, recall that (see, e.g., [9]):

Definition 2.1. Let X, Y be two separable Banach spaces.

(1) A mapping $f : \Omega \times X \to Y$ is said to be a random operator if for each fixed x in X, the mapping $\omega \mapsto f(\omega, x)$ is measurable.

(2) The random operator $f : \Omega \times X \to Y$ is said to be continuous if for each ω in Ω the mapping $x \mapsto f(\omega, x)$ is continuous.

(3) Let $f, g : \Omega \times X \to Y$ be two random operators. Then g is said to be a modification of f if for each x in X, we have $f(\omega, x) = g(\omega, x)$ a.s. Note that the exceptional set can depend on x.

The following is the notion of the completely random operator.

Definition 2.2. Let X, Y be two separable Banach spaces.

(1) A mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is called a completely random operator.

(2) The completely random operator Φ is said to be continuous if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ a.s., we have $\lim \Phi u_n = \Phi u$ a.s.

(3) The completely random operator Φ is said to be continuous in probability if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ in probability, we have $\lim \Phi u_n = \Phi u$ in probability.

(4) The completely random operator Φ is said to be an extension of a random operator $f : \Omega \times X \to Y$ if for each x in X

$$\Phi x(\omega) = f(\omega, x) \quad \text{a.s.},$$

where for each x in X, x denotes the random variable u in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

Theorem 2.3. *Let $f : \Omega \times X \to Y$ be a random operator admitting a continuous modification. Then, there exists a continuous completely random operator*

$$\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$$

such that Φ is an extension of f.

Proof. Let g be a continuous modification of f. Define $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ by

$$\Phi u(\omega) = g(\omega, u(\omega)) \tag{2.1}$$

for each random variable u in $L_0^X(\Omega)$. This definition is well-defined. Indeed, by [5, Theorem 6.1], $g : \Omega \times X \to Y$ is measurable, hence $\omega \mapsto g(\omega, u(\omega))$ is measurable. Next, we have to show that if h is another continuous modification of f, then

$$g(\omega, u(\omega)) = h(\omega, u(\omega)) \quad \text{a.s.}$$

By the separability of X, there exists a sequence (x_n) dense in X. For each x_n, there exists a set Ω_n of probability one such that $g(\omega, x_n) = h(\omega, x_n)$ for all ω in Ω_n. Let $\Omega_0 = \bigcap_{n=1}^{\infty} \Omega_n$. Clearly, Ω_0 has probability one and we have

$$g(\omega, x_n) = h(\omega, x_n) \quad \text{for all } \omega \in \Omega_0, \text{ for all } n. \tag{2.2}$$

Fix ω in Ω_0. By the density of (x_n) in X, there exists a subsequence (x_{n_k}) converging to $u(\omega)$. By the continuity of the mapping $x \mapsto g(\omega, x)$ and the mapping $x \mapsto h(\omega, x)$, we have

$$\begin{aligned} \lim_{k\to\infty} g(\omega, x_{n_k}) &= g(\omega, u(\omega)), \\ \lim_{k\to\infty} h(\omega, x_{n_k}) &= h(\omega, u(\omega)). \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we conclude that

$$h(\omega, \xi(\omega)) = g(\omega, \xi(\omega)) \quad \text{for all } \omega \in \Omega_0$$

as claimed.

From (2.1), it is easy to show that the completely random operator Φ is continuous and is an extension of f. □

Proposition 2.4. *Let $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ be a completely random operator. Then, the continuity of Φ implies the continuity in probability of Φ.*

Proof. Let (u_n) be a sequence in $L_0^X(\Omega)$ such that $p\text{-}\lim u_n = u$. We have to show that $p\text{-}\lim \Phi u_n = \Phi u$. On the contrary, suppose that Φu_n does not converge to Φu in probability. Then, there exist $t > 0, \varepsilon > 0$ and a subsequence (u_{n_k}) such that for all u_{n_k}

$$\mathbb{P}(\|\Phi u_{n_k} - \Phi u\| > t) \geq \varepsilon.$$

Since $p\text{-}\lim u_{n_k} = u$, there is a subsequence (u'_{n_k}) converging a.s. to u. By the continuity of Φ, $(\Phi u'_{n_k})$ converges a.s. to Φu, so converges to Φu in probability. Hence,

$$0 = \lim_k \mathbb{P}(\|\Phi u'_{n_k} - \Phi u\| > t) \geq \varepsilon.$$

We get a contradiction. □

Definition 2.5. Let $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ be a completely random operator.

(1) The operator Φ is said to be $k(\omega)$-Lipschitz if there is a positive random variable $k(\omega)$ such that for each pair u, v in $L_0^X(\Omega)$

$$\|\Phi u(\omega) - \Phi v(\omega)\| \leq k(\omega)\|u(\omega) - v(\omega)\| \quad \text{a.s.}$$

Note that the exceptional set depends on u, v in general.

(2) The operator Φ is said to be probabilistic $k(\omega)$-Lipschitz if there is a real-valued random variable $k(\omega)$ such that for each pair u, v in $L_0^X(\Omega)$ and $t > 0$

$$\mathbb{P}(\|\Phi u(\omega) - \Phi v(\omega)\| > t) \leq \mathbb{P}(k(\omega)\|u(\omega) - v(\omega)\| > t).$$

(3) The operator Φ is said to be a (probabilistic) $k(\omega)$-contraction if Φ is (probabilistic) $k(\omega)$-Lipschitz with $k(\omega) < 1$ for all ω.

(4) The operator Φ is said to be a (probabilistic) non-expansive completely random operator if Φ is (probabilistic) 1-Lipschitz.

Clearly, if Φ is $k(\omega)$-Lipschitz, then Φ is probabilistic $k(\omega)$-Lipschitz.

Proposition 2.6. *The following hold:*

(1) *If $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is a $k(\omega)$-Lipschitz completely random operator, then Φ is continuous.*

(2) *If $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is a probabilistic $k(\omega)$-Lipschitz completely random operator, then Φ is continuous in probability. In particular, a probabilistic non-expansive completely random operator is continuous in probability.*

Proof. The first assertion is easy to prove. We prove the second assertion. For each u, v in $L_0^X(\Omega)$, we have

$$\begin{aligned}\mathbb{P}(\|\Phi u - \Phi v\| > t) &\leq \mathbb{P}(k(\omega)\|u - v\| > t)\\ &= \mathbb{P}(k(\omega)\|u - v\| > t, \|u - v\| \leqslant r) + \mathbb{P}(\|u - v\| > r)\\ &\leq \mathbb{P}(rk(\omega) > t) + \mathbb{P}(\|u - v\| > r)\\ &= \mathbb{P}(k(\omega) > t/r) + \mathbb{P}(\|u - v\| > r).\end{aligned}$$

Suppose that $p\text{-}\lim u_n = u$. Then, we have

$$\mathbb{P}(\|\Phi u_n - \Phi u\| > t) \leq \mathbb{P}(k(\omega) > t/r) + \mathbb{P}(\|u_n - u\| > r).$$

So, for each $r > 0$

$$\limsup_n \mathbb{P}(\|\Phi u_n - \Phi u\| > t) \leq \mathbb{P}(k(\omega) > t/r).$$

Leting $r \to 0$, we get

$$\limsup_n \mathbb{P}(\|\Phi u_n - \Phi u\| > t) = 0. \qquad \square$$

3 Random fixed points of some completely random operators

Let $f : \Omega \times X \to X$ denote a random operator. Recall that (see, e.g., [1, 2, 4]) an X-valued random variable ξ is said to be a random fixed point of the random operator f if

$$f(\omega, \xi(\omega)) = \xi(\omega) \quad \text{a.s.}$$

Assume that the operator f is continuous. Then, by Theorem 2.3, the mapping $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ defined by

$$\Phi u(\omega) = f(\omega, u(\omega))$$

is a completely random operator extending f. For each random fixed point ξ of f, we get

$$\Phi\xi(\omega) = \xi(\omega) \quad \text{a.s.}$$

This leads us to the following definition:

Definition 3.1. Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An X-valued random variable ξ in $L_0^X(\Omega)$ is called a random fixed point of Φ if

$$\Phi\xi = \xi \quad \text{a.s.}$$

Definition 3.2. Let $f : \Omega \times [0,\infty) \to [0,\infty)$ be a mapping such that for each ω in Ω, $f(\omega,t) = 0$ if and only if $t = 0$.

(1) The completely random operator

$$\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$$

is said to be $f(\omega,t)$-weakly contractive if for each pair u, v in $L_0^X(\Omega)$

$$\|\Phi u(\omega) - \Phi v(\omega)\| \leqslant \|u(\omega) - v(\omega)\| - f(\omega, \|u(\omega) - v(\omega)\|) \quad \text{a.s.} \quad (3.1)$$

(2) The completely random operator

$$\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$$

is said to be probabilistic $f(\omega,t)$-weakly contractive if for each pair u, v in $L_0^X(\Omega)$, and $t > 0$

$$\begin{aligned} &\mathbb{P}(\|\Phi u(\omega) - \Phi v(\omega)\| > t) \\ &\qquad \leqslant \mathbb{P}(\|u(\omega) - v(\omega)\| - f(\omega, \|u(\omega) - v(\omega)\|) > t). \end{aligned} \quad (3.2)$$

Clearly,

- If the operator Φ is probabilistic $f(\omega,t)$-weakly contractive, then it is probabilistic non-expansive, so it is continuous in probability.
- If the operator Φ is a (probabilistic) $k(\omega)$-contraction, then Φ is a (probabilistic) $f(\omega,t)$-weakly contractive completely random operator where

$$f(\omega,t) = (1 - k(\omega))t.$$

Theorem 3.3. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be an $f(\omega,t)$-weakly contractive completely random operator where for each ω in Ω, the function $t \mapsto f(\omega,t)$ is non-decreasing. Then, Φ has a unique random fixed point.*

Proof. Let u_0 be an arbitrary X-valued random variable. We define the sequence $(u_n) \subset L_0^X(\Omega)$ by

$$u_{n+1} = \Phi u_n, \quad n = 0, 1, \ldots. \quad (3.3)$$

By (3.1), for each pair (i,j), we have

$$\|\Phi u_i - \Phi u_j\| \le \|u_i - u_j\| - f(\omega, \|u_i - u_j\|) \quad \text{a.s.}$$

Hence, there is a set D of probability one such that for each ω in D and for all pairs (i,j)

$$\|\Phi u_i(\omega) - \Phi u_j(\omega)\| \le \|u_i(\omega) - u_j(\omega)\| - f(\omega, \|u_i(\omega) - u_j(\omega)\|). \quad (3.4)$$

In particular, for each ω in D and for all pairs (i, j)

$$\|\Phi u_i(\omega) - \Phi u_j(\omega)\| \le \|u_i(\omega) - u_j(\omega)\|. \tag{3.5}$$

Claim 1. *For each ω in D, we have*

$$\lim \|u_{i+1}(\omega) - u_i(\omega)\| = 0.$$

By (3.5), we receive

$$\begin{aligned}\|u_{i+1}(\omega) - u_i(\omega)\| &= \|\Phi u_i(\omega) - \Phi u_{i-1}(\omega)\| \\ &\le \|u_i(\omega) - u_{i-1}(\omega)\|.\end{aligned}$$

This implies that

$$\lim \|u_{i+1}(\omega) - u_i(\omega)\| = L(\omega) \ge 0 \quad \text{for all } \omega \in D.$$

We have to show that $L(\omega) = 0$ for all ω in D. On the contrary, suppose that there exists ω in D such that $L(\omega) > 0$. Then, $\|u_{i+1}(\omega) - u_i(\omega)\| \ge L(\omega)$ and $f(\omega, \|u_{i+1}(\omega) - u_i(\omega)\|) \ge f(\omega, L(\omega)) = L > 0$. Hence, for each i

$$\begin{aligned}\|u_{i+2}(\omega) - u_{i+1}(\omega)\| &= \|\Phi u_{i+1}(\omega) - \Phi u_i(\omega)\| \\ &\le \|u_{i+1}(\omega) - u_i(\omega)\| - f(\omega, \|u_{i+1}(\omega) - u_i(\omega)\|) \\ &\le \|u_{i+1}(\omega) - u_i(\omega)\| - f(\omega, L(\omega)) \\ &= \|u_{i+1}(\omega) - u_i(\omega)\| - L.\end{aligned}$$

Adding the above inequalities for $i = 0, 1, \ldots, n-1$, we get for all n

$$\|u_{n+1}(\omega) - u_n(\omega)\| \le \|u_1(\omega) - u_0(\omega)\| - nL$$

which is a contradiction.

Claim 2. *There exists ξ in $L_0^X(\Omega)$ such that*

$$\lim u_n = \xi \quad a.s.$$

Fix ω in D. Given $\varepsilon > 0$. By Claim 1, there exists N such that

$$\|u_{N+1}(\omega) - u_N(\omega)\| < \min\{\varepsilon, f(\omega, \varepsilon)\}.$$

We shall show that for each $n > N$,

$$\|u_n(\omega) - u_N(\omega)\| \le 2\varepsilon. \tag{3.6}$$

Prove by induction. For $n = N + 1$, inequality (3.6) holds. Suppose that (3.6) holds for $n > N$. If $\|u_n(\omega) - u_N(\omega)\| \le \varepsilon$, then by (3.5)

$$\begin{aligned}\|u_{n+1}(\omega) - u_N(\omega)\| &\le \|\Phi u_n(\omega) - \Phi u_N(\omega)\| + \|\Phi u_N(\omega) - u_N(\omega)\| \\ &\le \|u_n(\omega) - u_N(\omega)\| + \|u_{N+1}(\omega) - u_N(\omega)\| \le 2\varepsilon.\end{aligned}$$

If $\varepsilon \le \|u_n(\omega) - u_N(\omega)\| \le 2\varepsilon$, then

$$\begin{aligned}\|u_{n+1}(\omega) - u_N(\omega)\| &\le \|\Phi u_n(\omega) - \Phi u_N(\omega)\| + \|\Phi u_N(\omega) - u_N(\omega)\| \\ &\le \|u_n(\omega) - u_N(\omega)\| - f(\omega, \|u_n(\omega) - u_N(\omega)\|) \\ &\qquad + \|u_{N+1}(\omega) - u_N(\omega)\| \\ &\le \|u_n(\omega) - u_N(\omega)\| - f(\omega, \varepsilon) + \|u_{N+1}(\omega) - u_N(\omega)\| \\ &\le \|u_n(\omega) - u_N(\omega)\| \le 2\varepsilon,\end{aligned}$$

i.e. (3.6) holds for $n + 1$. Hence (3.6) is proved. From this, $(u_n(\omega))$ is a Cauchy sequence for each ω in D, which implies Claim 2.

Since Φ is continuous in probability, from (3.3), let $n \to \infty$ we get $\xi = \Phi\xi$ a.s. Therefore, ξ is a random fixed point of Φ.

Suppose that η is another random fixed point of Φ. There is a set D' of probability one such that for all ω in D'

$$\|\xi(\omega) - \eta(\omega)\| = \|\Phi\xi(\omega) - \Phi\eta(\omega)\| \le \|\xi(\omega) - \eta(\omega)\| - f(\omega, \|\xi(\omega) - \eta(\omega)\|).$$

Hence, $f(\omega, \|\xi(\omega)-\eta(\omega)\|) = 0$ which implies $\|\xi(\omega)-\eta(\omega)\| = 0$ for all ω in D', i.e. $\xi = \eta$ a.s. □

As a consequence of Theorem 3.3, we get

Theorem 3.4. *If the completely random operator Φ is a $k(\omega)$-contraction, then Φ has a unique random fixed point.*

Theorem 3.5. *Let Φ be a probabilistic $f(t)$-weakly contractive completely random operator where the function $f(\omega, t) = f(t) < t$ for all $t > 0$. For each $t > 0$, define*

$$h(t) = \inf_{s \ge t} \frac{f(s)}{s}.$$

Assume that $h(t) > 0$ for all $t > 0$. Then,

(1) *Φ has a random fixed point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$E\,\|\Phi u_0 - u_0\|^p < \infty. \tag{3.7}$$

In this case, Φ has a unique random fixed point.

(2) *Let* (u_n) *in* $L_0^X(\Omega)$ *be a sequence given by*

$$u_{n+1} = \Phi u_n, \quad n = 0, 1, \ldots, \tag{3.8}$$

and ξ *be the random fixed point of* Φ. *Then, we have the following estimation:*

$$\mathbb{P}(\|u_n - \xi\| > t) \leq \frac{M}{(1 - q^{\frac{p}{1+p}})^{1+p}} \frac{(q^p)^n}{t^p},$$

where $M = E\,\|\Phi u_0 - u_0\|^p$, $q = 1 - h(t)$.

Proof. (1) If Φ has a random fixed point ξ, then (3.7) holds with $u_0 = \xi$ for any $p > 0$.

Conversely, suppose that $E\,\|\Phi u_0 - u_0\|^p < \infty$ for some $u_0 \in L_0^X(\Omega)$, $p > 0$. We will show that (u_n) given by $u_{n+1} = \Phi u_n$ $(n = 0, 1, \ldots)$ is a Cauchy sequence in $L_0^X(\Omega)$. Define the function $g(t)$, $t > 0$, by

$$g(t) = 1 - \frac{f(t)}{t}.$$

So, we have

$$f(t) = (1 - g(t))t.$$

Since $f(t) > 0$ for all $t > 0$, we get $g(t) < 1$ for all $t > 0$. For any u, v in $L_0^X(\Omega)$, we have

$$\mathbb{P}(\|\Phi u(\omega) - \Phi v(\omega)\| > t) \leqslant \mathbb{P}(\|u(\omega) - v(\omega)\| - f(\|u(\omega) - v(\omega)\|) > t).$$

Equivalently,

$$\mathbb{P}(\|\Phi u(\omega) - \Phi v(\omega)\| > t) \leqslant \mathbb{P}(g(\|u(\omega) - v(\omega)\|)\|u(\omega) - v(\omega)\| > t). \tag{3.9}$$

Fix $t > 0$. For each $s \geq t > 0$, we have

$$g(s) = 1 - \frac{f(s)}{s} \leq 1 - h(t) = q(t).$$

Since $g(t) < 1$, we get

$$\{g(\|u - v\|)\|u - v\| > t\} \subset \{\|u - v\| > t\}.$$

Hence,

$$\begin{aligned}
\mathbb{P}(\|\Phi u - \Phi v\| > t) &\leq \mathbb{P}(g(\|u - v\|)\|u - v\| > t) \\
&= \mathbb{P}(g(\|u - v\|)\|u - v\| > t, \|u - v\| > t) \\
&\leq \mathbb{P}(q(t)\|u - v\| > t, \|u - v\| > t) \\
&\leq \mathbb{P}(q(t)\|u - v\| > t) \\
&= \mathbb{P}(\|u - v\| > t/q(t)) = \mathbb{P}(\|u - v\| > t/q)
\end{aligned}$$

where $q = q(t)$. Note that $q < 1$ since $h(t) > 0$.

From this for each n, we obtain

$$\mathbb{P}(\|u_{n+1} - u_n\| > t) = \mathbb{P}(\|\Phi u_n - \Phi u_{n-1}\| > t) \le \mathbb{P}(\|u_n - u_{n-1}\| > t/q).$$

By induction and Chebyshev's inequality, we get

$$\begin{aligned}\mathbb{P}(\|u_{n+1} - u_n\| > t) &\le \mathbb{P}(\|u_n - u_{n-1}\| > t/q)\\ &\le \cdots \le \mathbb{P}(\|u_1 - u_0\| > t/q^n)\\ &= \mathbb{P}(\|\Phi u_0 - u_0\| > t/q^n)\\ &\le E\,\|\Phi u_0 - u_0\|^p \frac{(q^n)^p}{t^p} = M\frac{(q^n)^p}{t^p}.\end{aligned}$$

Let $r = \frac{x}{q}$ where $q < x < 1$. Then, we have $r > 1$ and

$$(r-1)\left(\frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^m}\right) + \frac{1}{r^m} = 1 \quad \text{for all } m \ge 1.$$

Thus, for any $t > 0$ and m, n in $\mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}(\|u_{n+m} - u_n\| > t) &\le \mathbb{P}(\|u_{n+m} - u_n\| > (1 - 1/r^m)t)\\ &\le \mathbb{P}(\|u_{n+m} - u_{n+m-1}\| > t(r-1)/r^m)\\ &\qquad + \cdots + \mathbb{P}(\|u_{n+1} - u_n\| > t(r-1)/r)\\ &\le \frac{M}{[(r-1)t]^p}\left[(r^m)^p (q^{n+m-1})^p + \cdots + r^p (q^n)^p\right]\\ &= \frac{M}{[(r-1)t]^p}(q^n)^p r^p\left[(qr)^{p(m-1)} + \cdots + (qr)^p + 1\right]\\ &= \frac{M}{[(r-1)t]^p}(q^n)^p r^p \frac{1-(qr)^{mp}}{1-(qr)^p}\\ &< \frac{Mr^p}{[(r-1)t]^p[1-(qr)^p]}(q^p)^n \end{aligned} \tag{3.10}$$

which tends to 0 as $n \to \infty$. It implies that (u_n) is a Cauchy sequence in $L_0^X(\Omega)$. Hence, there exists ξ in $L_0^X(\Omega)$ such that p-$\lim u_n = \xi$. Let $n \to \infty$ in (3.8). Since Φ is continuous in probability, we get that $\xi = \Phi\xi$ a.s.

Let η be another random fixed point of Φ. Then, for any $t > 0$, we have

$$\begin{aligned}\mathbb{P}(\|\xi - \eta\| > t) &= \mathbb{P}(\|\Phi\xi - \Phi\eta\| > t)\\ &\le \mathbb{P}(\|\xi - \eta\| > t/q) \le \cdots \le \mathbb{P}(\|\xi - \eta\| > t/q^n)\end{aligned}$$

for all $n > 0$. Let $n \to \infty$, we have $\mathbb{P}(\|\xi - \eta\| > t) = 0$ for any $t > 0$, i.e. $\xi = \eta$ a.s. Thus, Φ has a unique random fixed point.

(2) From (3.10), letting $m \to \infty$, we have

$$\begin{aligned}\mathbb{P}(\|u_n - \xi\| > t) &\le M \frac{(q^p)^n}{t^p}\left(\frac{r}{r-1}\right)^p \frac{1}{1-(qr)^p} \\ &= M \frac{(q^p)^n}{t^p}\left(\frac{x}{x-q}\right)^p \frac{1}{1-x^p}\end{aligned}$$

for any x in $(q, 1)$. Let $f(x)$ be a function defined in $(q, 1)$ by

$$f(x) = \left(\frac{x}{x-q}\right)^p \frac{1}{1-x^p}.$$

By a standard argument, we get

$$\min_{x \in (q,1)} f(x) = \frac{1}{(1-q^{\frac{p}{1+p}})^{1+p}}.$$

Hence,

$$\mathbb{P}(\|u_n - \xi\| > t) \le \frac{M}{(1-q^{\frac{p}{1+p}})^{1+p}} \frac{(q^p)^n}{t^p}. \qquad \square$$

If $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ is a probabilistic k-contraction, then Φ is a probabilistic $f(t)$-weakly contractive completely random operator where $f(t) = (1-k)t$. In this case, $0 < h(t) = 1 - k < 1$ for all $t > 0$. Hence, we get the following corollary:

Corollary 3.6. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a probabilistic k-contraction. Then, Φ has a unique random point if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$E\|\Phi u_0 - u_0\|^p < \infty.$$

Definition 3.7. Let $f : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$ and let q be a positive number.

(1) The completely random operator $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ is said to be probabilistic (f, q)-Lipschitz if for each pair u, v in $L_0^X(\Omega)$

$$\mathbb{P}(\|\Phi u(\omega) - \Phi v(\omega)\| > f(t)) \leqslant \mathbb{P}(\|u(\omega) - v(\omega)\| > f(t/q)). \quad (3.11)$$

(2) The completely random operator $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ is said to be probabilistic (f, q)-contractive if Φ is probabilistic (f, q)-Lipschitz with $q < 1$.

Remark. If Φ is probabilistic q-Lipschitz, then it is probabilistic (f, q)-Lipschitz for $f(t) = t$. In particular, if Φ is probabilistic q-contractive, then it is probabilistic (f, q)-contractive for $f(t) = t$.

Proposition 3.8. *If $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is probabilistic (f, q)-Lipschitz, then Φ is continuous in probability.*

Proof. Let $t > 0$, and let $g = f^{-1}$ be the inverse function of f. Put $s = g(t)$. For each u, v in $L_0^X(\Omega)$, we have

$$\begin{aligned}\mathbb{P}(\|\Phi u - \Phi v\| > t) &= \mathbb{P}(\|\Phi u - \Phi v\| > f(s)) \\ &\le \mathbb{P}(\|u - v\| > f(s/q)) = \mathbb{P}(g(\|u - v\|) > s/q). \quad (3.12)\end{aligned}$$

Let $r > 0$ be sufficiently small such that $s/r > q$. Then,

$$\mathbb{P}(g(\|u - v\|) > s/q) \le \mathbb{P}(g(\|u - v\|) > r) = \mathbb{P}(\|u - v\| > f(r)). \quad (3.13)$$

From (3.12) and (3.13), we obtain

$$\mathbb{P}(\|\Phi u - \Phi v\| > t) \le \mathbb{P}(\|u - v\| > f(r)).$$

Suppose that (u_n) is a sequence in $L_0^X(\Omega)$ such that $p\text{-}\lim u_n = u$. Since

$$\mathbb{P}(\|\Phi u_n - \Phi u\| > t) \le \mathbb{P}(\|u_n - u\| > f(r)),$$

we get that

$$\lim_n \mathbb{P}(\|\Phi u_n - \Phi u\| > t) = 0.$$

Hence, Φ is continuous in probability. □

Theorem 3.9. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ denote a probabilistic (f, q)-contractive completely random operator.*

(1) *If Φ has a random fixed point, then it has a unique one. Moreover, there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that*

$$M = \sup_{t>0} t^p \mathbb{P}(\|\Phi u_0 - u_0\| > f(t)) < \infty. \quad (3.14)$$

(2) *Assume that there exists c in $(q, 1)$ such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \quad (3.15)$$

Then, condition (3.14) is sufficient for Φ to have a unique random fixed point.

(3) *Assume that for each $t, s > 0$*

$$f(t + s) \ge f(t) + f(s). \quad (3.16)$$

Then, condition (3.14) is also sufficient for Φ to have a unique random fixed point.

Proof. Let $g = f^{-1}$ be the inverse function of f. Then, $g : [0,\infty) \to [0,\infty)$ is increasing with $g(0) = 0, \lim_{t\to\infty} g(t) = \infty$. Condition (3.11) is equivalent to the following:

$$\mathbb{P}(g(\|\Phi u - \Phi v\|) > t) \leq \mathbb{P}(g(\|u - v\|) > t/q). \tag{3.17}$$

Let u_0 in $L_0^X(\Omega)$ such that (3.14) holds. Define a sequence (u_n) in $L_0^X(\Omega)$ by

$$u_{n+1} = \Phi u_n, \quad n = 0, 1, \ldots. \tag{3.18}$$

From (3.17)

$$\begin{aligned}\mathbb{P}(g(\|u_{n+1} - u_n\|) > t) &= \mathbb{P}(g(\|\Phi u_n - \Phi u_{n-1}\|) > t) \\ &\leq \mathbb{P}(g(\|u_n - u_{n-1}\|) > t/q).\end{aligned}$$

By induction, we obtain for each n

$$\mathbb{P}(g(\|u_{n+1} - u_n\|) > t) \leq \mathbb{P}(g(\|u_1 - u_0\|) > t/q^n). \tag{3.19}$$

(1) Let ξ, η be two random fixed points of Φ. Then, for each $t > 0$, we have

$$\mathbb{P}(\|\xi - \eta\| > f(t)) = \mathbb{P}(\|\Phi\xi - \Phi\eta\| > f(t)) \leq \mathbb{P}(\|\xi - \eta\| > f(t/q)).$$

By induction, it follows that

$$\mathbb{P}(\|\xi - \eta\| > f(t)) \leq \mathbb{P}(\|\xi - \eta\| > f(t/q^n)) \quad \text{for all } n.$$

Since $\lim_{n\to\infty} f(t/q^n) = \infty$, we conclude that $\mathbb{P}(\|\xi - \eta\| > f(t)) = 0$ for each $t > 0$. Hence, $g(\|\xi - \eta\|) = 0$ a.s. So, we have $\xi = \eta$ a.s. as claimed.

Suppose that Φ has a random fixed point ξ. Then take $u_0 = \xi$ and we obtain $M = 0$.

(2) From (3.14), we have

$$\mathbb{P}(g(\|u_1 - u_0\|) > s) \leq \frac{M}{s^p}. \tag{3.20}$$

From (3.19) and (3.20), we get

$$\mathbb{P}(g(\|u_{n+1} - u_n\|) > t) \leq \frac{Mq^{np}}{t^p}. \tag{3.21}$$

Taking $t = c^n$, from (3.21) we get

$$\mathbb{P}(g(\|u_{n+1} - u_n\|) > c^n) \leq M\frac{q^{np}}{c^{np}} \tag{3.22}$$

i.e.

$$\mathbb{P}(\|u_{n+1} - u_n\| > f(c^n)) \leq M\frac{q^{np}}{c^{np}}. \tag{3.23}$$

Since

$$\sum_{n=1}^{\infty} \mathbb{P}(\|u_{n+1} - u_n\| > f(c^n)) \leq M \sum_{n=1}^{\infty} \frac{q^{np}}{c^{np}} < \infty,$$

by the Borel–Cantelli Lemma, there is a set D with probability one such that for each ω in D there is $N(\omega)$

$$\|u_{n+1}(\omega) - u_n(\omega)\| \leq f(c^n) \quad \text{for all } n > N(\omega).$$

By (3.15), we conclude that

$$\sum_{n=1}^{\infty} \|u_{n+1}(\omega) - u_n(\omega)\| < \infty$$

for all ω in D which implies that there exists $\lim u_n(\omega)$ for all ω in D. Consequently, the sequence (u_n) converges a.s. to ξ in $L_0^X(\Omega)$. Since Φ is continuous in probability, from (3.18), letting $n \to \infty$, we get $\xi = \Phi\xi$ a.s.

(3) It is easy to see that for each $t, s > 0$

$$g(s + t) \leq g(t) + g(s).$$

Hence, for $a = \sum_{i=1}^{m} s_i$, we have

$$\begin{aligned}
\mathbb{P}(g(\|u_{n+m} - u_n\|) > a) &\leq \mathbb{P}\left(g\left(\sum_{i=1}^{m} \|u_{n+i} - u_{n+i-1}\| \right) > a \right) \\
&\leq \mathbb{P}\left(\sum_{i=1}^{m} g(\|u_{n+i} - u_{n+i-1}\|) > a \right) \\
&\leq \sum_{i=1}^{m} \mathbb{P}(g(\|u_{n+i} - u_{n+i-1}\|) > s_i).
\end{aligned}$$

From (3.21), we have

$$\mathbb{P}(g(\|u_{n+i} - u_{n+i-1}\|) > s_i) \leq \frac{M q^{(n+i-1)p}}{s_i^p}. \tag{3.24}$$

Put $r = \frac{x}{q}$ where $q < x < 1$ and $s_i = s(r-1)/r^i$. An argument similar to that in the proof of Theorem 3.5 yields

$$\lim_{n\to\infty} \mathbb{P}(g(\|u_{n+m} - u_n\|) > s) = 0 \quad \text{for all } s > 0,$$

so

$$\lim_{n\to\infty} \mathbb{P}(\|u_{n+m} - u_n\| > f(s)) = 0 \quad \text{for all } s > 0.$$

Thus, we obtain

$$\lim_{n\to\infty} \mathbb{P}(\|u_{n+m} - u_n\| > t) = 0 \quad \text{for all } t > 0.$$

Consequently, the sequence (u_n) converges in probability to ξ in $L_0^X(\Omega)$. Since Φ is continuous in probability, letting $n \to \infty$ in (3.18), we get $\xi = \Phi\xi$ a.s. □

4 Applications to random equations

In the last section, we will give some applications of above results to random equations.

Theorem 4.1. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a probabilistic (f, q)-Lipschitz completely random operator where f is a function satisfying either* (3.15) *or* (3.16)*. Consider a random equation of the form*

$$\Phi u - \lambda u = \eta, \tag{4.1}$$

where λ is a real number and η is a random variable in $L_0^X(\Omega)$.

Assume that

$$|\lambda| \geq \sup_{t>0} \frac{f(\frac{q}{q'}t)}{f(t)}, \tag{4.2}$$

where $q' \in (0, 1)$. Then equation (4.1) *has a unique random solution if and only if there exist a random variable u_0 in $L_0^X(\Omega)$ and a number $p > 0$ such that*

$$M = \sup_{t>0} t^p \mathbb{P}(\|\Phi u_0 - \lambda u_0 - \eta\| > |\lambda| f(t)) < \infty. \tag{4.3}$$

Proof. Define a completely random operator Ψ by

$$\Psi u = \frac{\Phi u - \eta}{\lambda}.$$

Let $g = f^{-1}$ be the inverse function of f. Then, $g : [0, \infty) \to [0, \infty)$ is continuous, increasing with $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$. For each $t > 0$, there exists t' so that $f(t') = |\lambda| f(t)$, i.e. $t' = g(|\lambda| f(t))$. So, we have

$$\begin{aligned}
\mathbb{P}(\|\Psi u - \Psi v\| > f(t)) &= \mathbb{P}(\|\Phi u - \Phi v\| > |\lambda| f(t)) \\
&= \mathbb{P}(\|\Phi u - \Phi v\| > f(t')) \\
&\leq \mathbb{P}(\|u - v\| > f(t'/q)) \\
&= \mathbb{P}(\|u - v\| > f((t/q')(q't'/qt))).
\end{aligned}$$

From (4.2), we receive $|\lambda| f(t) \geq f(\frac{q}{q'}t)$. Then, we deduce $g(|\lambda| f(t)) \geq \frac{q}{q'}t$. So, $t' \geq \frac{q}{q'}t$ and $\frac{q't'}{qt} \geq 1$. Direct computation shows that

$$\mathbb{P}(\|\Psi u - \Psi v\| > f(t)) \leq \mathbb{P}(\|u - v\| > f(t/q'))$$

which implies that Ψ is (f, q')-contractive.

By Theorem 3.9, Ψ has a unique random fixed point ξ which implies that equation (4.1) has a unique solution ξ. □

Corollary 4.2. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ denote a probabilistic q-Lipschitz completely random operator. Consider the random equation*

$$\Phi u - \lambda u = \eta, \tag{4.4}$$

where λ is a real number satisfying $|\lambda| > q$ and η is a random variable in $L_p^X(\Omega)$, $p > 0$. Then, the random equation (4.4) has a unique solution if and only if there exists a random variable u_0 in $L_0^X(\Omega)$ such that

$$E\,\|\Phi u_0 - \lambda u_0\|^p < \infty. \tag{4.5}$$

Proof. Suppose that equation (4.4) has a solution ξ. Then, condition (4.5) holds for $u_0 = \xi$.

Conversely, suppose that there exists a random variable u_0 in $L_0^X(\Omega)$ such that (4.5) holds. So, Φ is (f, q)-Lipschitz where $f(t) = t$ is the function satisfying (3.15). Take $q < s < |\lambda|$. Then $q' = q/s < 1$ and

$$|\lambda| > s = \frac{q}{q'} = \frac{f(\frac{q}{q'}t)}{f(t)}.$$

Moreover, for each $t > 0$

$$t^p \mathbb{P}(\|\Phi u_0 - \lambda u_0 - \eta\| > |\lambda| t) \le \frac{E\|\Phi u_0 - \lambda u_0 - \eta\|^p}{|\lambda|^p} < \infty$$

since

$$E(\|\Phi u_0 - \lambda u_0 - \eta\|^p) \le C_p E(\|\Phi u_0 - \lambda u_0\|^p) + C_p\, E\,\|\eta\|^p < \infty,$$

where C_p is a constant. Hence, condition (4.3) is satisfied. By Theorem 4.1, we conclude that equation (4.4) has a unique random solution. □

Theorem 4.3. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a function such that $f(0) = 0$, $f(t) < t$ and*

$$h(t) = \inf_{s \ge t} \frac{f(s)}{s} > 0 \quad \text{for all } t > 0.$$

Let $\Phi, \Psi : L_0^X(\Omega) \to L_0^X(\Omega)$ be two completely random operators satisfying the following conditions:

(a) *$\Psi(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$,*

(b) *$\Phi(L_0^X(\Omega)) \subset \Psi(L_0^X(\Omega))$,*

(c) *Φ, Ψ are continuous in probability,*

(d) *for any u, v in $L_0^X(\Omega)$ and $t > 0$, we have*

$$\mathbb{P}(\|\Phi u - \Phi v\| > t) \le \mathbb{P}(\|\Psi u - \Psi v\| - f(\|\Psi u - \Psi v\|) > t).$$

Then, the random equation

$$\Phi u = \Psi u \tag{4.6}$$

has a solution if and only if there exist a random variable u_0 *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that*

$$E\,\|\Phi u_0 - \Psi u_0\|^p < \infty. \tag{4.7}$$

Proof. If (4.6) has a solution ξ, then (4.7) holds with $u_0 = \xi$ and any $p > 0$.

Conversely, suppose that (4.7) holds. By condition (a), there exists an X-valued random variable u_1 such that $\Psi u_1 = \Phi u_0$. By induction, there is a sequence of X-valued random variables (u_n) such that $\Psi u_n = \Phi u_{n-1}, n \geq 1$. Let $\xi_n = \Psi u_n$ $(n = 1, 2, \dots)$. We will show that (ξ_n) is a Cauchy sequence in probability.

Set

$$g(t) = 1 - \frac{f(t)}{t}, \quad t > 0.$$

We have $f(t) = (1 - g(t))t$ and $g(t) \in (0, 1)$ for all $t > 0$.

For any $u, v \in L_0^X(\Omega)$, we have

$$\mathbb{P}(\|\Phi u - \Phi v\| > t) \leqslant \mathbb{P}(\|\Psi u - \Psi v\| - f(\|\Psi u - \Psi v\|) > t).$$

Equivalently,

$$\mathbb{P}(\|\Phi u - \Phi v\| > t) \leqslant \mathbb{P}(g(\|\Psi u - \Psi v\|)\|\Psi u - \Psi v\| > t). \tag{4.8}$$

Fix $t > 0$. For each $s \geq t$, we have

$$g(s) = 1 - \frac{f(s)}{s} \leq 1 - h(t) = q(t).$$

Since $g(t) < 1$, we get $\{g(\|\Psi u - \Psi v\|)\|\Psi u - \Psi v\| > t\} \subset \{\|\Psi u - \Psi v\| > t\}$. Hence,

$$\begin{aligned}
\mathbb{P}(\|\Phi u - \Phi v\| > t) &\leq \mathbb{P}(g(\|\Psi u - \Psi v\|)\|\Psi u - \Psi v\| > t) \\
&= \mathbb{P}(g(\|\Psi u - \Psi v\|)\|\Psi u - \Psi v\| > t, \|\Psi u - \Psi v\| > t) \\
&\leq \mathbb{P}(q(t)\|\Psi u - \Psi v\| > t, \|\Psi u - \Psi v\| > t) \\
&\leq \mathbb{P}(q(t)\|\Psi u - \Psi v\| > t) \\
&= \mathbb{P}(\|\Psi u - \Psi v\| > t/q(t)) = \mathbb{P}(\|\Psi u - \Psi v\| > t/q),
\end{aligned}$$

where $q = q(t)$. Note that $q < 1$ since $h(t) > 0$. We obtain

$$\begin{aligned}
\mathbb{P}(\|\xi_{n+1} - \xi_n\| > t) &= \mathbb{P}(\|\Phi u_n - \Phi u_{n-1}\| > t) \\
&\leq \mathbb{P}(\|\Psi u_n - \Psi v_{n-1}\| > t/q) \\
&= \mathbb{P}(\|\xi_n - \xi_{n-1}\|) > t/q).
\end{aligned}$$

By similar arguments as in the proof of Theorem 3.5, it implies that (ξ_n) is a Cauchy sequence in probability. Hence, there exists ξ in $L_0^X(\Omega)$ such that

$$p\text{-}\lim \xi_n = \xi.$$

From the assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $\Psi u^* = \xi$. So, we have

$$\begin{aligned}\mathbb{P}(\|\Psi u_{n+1} - \Phi u^*\| > t) &= \mathbb{P}(\|\Phi u_n - \Phi u^*\| > t)\\ &\le \mathbb{P}(\|\Psi u_n - \Psi u^*\| - f(\|\Psi u_n - \Psi u^*\|) > t)\\ &\le \mathbb{P}(\|\xi_n - \xi\| > t).\end{aligned}$$

Let $n \to \infty$. We receive $\mathbb{P}(\|\xi - \Phi u^*\| > t) = 0$ implying $\Phi u^* = \xi$ a.s. Hence, u^* is a solution of the random equation (4.6). □

Remark. The following simple example shows that the random equation (4.6) needs not to have a unique solution.

Example 4.4. Define two completely random operators $\Phi, \Psi : L_0^{\mathbb{R}}(\Omega) \to L_0^{\mathbb{R}}(\Omega)$ by $\Phi u = k|u| + \eta$, $\Psi u = |u|$, where η is a positive random variable, $k \in (0,1)$.

It is easy to check that Φ, Ψ satisfy all assumptions of Theorem 4.3 with

$$f(t) = k't, \quad k' \in (0, 1-k),$$

and the random equation (4.6) has two solutions

$$\xi_1 = \frac{1}{1-k}\eta, \quad \xi_2 = -\frac{1}{1-k}\eta.$$

Corollary 4.5. *Let Φ, Ψ be two completely random operators satisfying the conditions stated in Theorem* 4.3 *and Φ, Ψ commute, i.e. $\Phi\Psi u = \Psi\Phi u$ for any random variable u in $L_0^X(\Omega)$. Then, Φ and Ψ have a unique common random fixed point if and only if there exist u_0 in $L_0^X(\Omega)$ and $p > 0$ such that* (4.7) *holds.*

Proof. If Φ and Ψ have a common random fixed point ξ, then (4.7) holds with $u_0 = \xi$ and any $p > 0$.

Conversely, suppose that (4.7) holds. By Theorem 4.3, there exists ξ such that $\Phi\xi = \Psi\xi$. Put $\theta = \Phi\xi = \Psi\xi$. For $t > 0$, we have

$$\begin{aligned}\mathbb{P}(\|\Phi\theta - \theta\| > t) &= \mathbb{P}(\|\Phi\theta - \Phi\xi\| > t)\\ &\le \mathbb{P}(\|\Psi\theta - \Psi\xi\| > t/q)\\ &= \mathbb{P}(\|\Psi\Phi\xi - \theta\| > t/q)\\ &= \mathbb{P}(\|\Phi\Psi\xi - \theta\| > t/q) = \mathbb{P}(\|\Phi\theta - \theta\| > t/q).\end{aligned}$$

By induction, it follows that $\mathbb{P}(\|\Phi\theta - \theta\| > t) \le \mathbb{P}(\|\Phi\theta - \theta\| > t/q^n)$ for any n in $\mathbb{N}$. Letting $n \to \infty$, we have $\mathbb{P}(\|\Phi\theta - \theta\| > t) = 0$ for any $t > 0$. Thus, $\Phi\theta = \theta$, i.e. θ is a random fixed point of Φ. We have

$$\Psi\theta = \Psi\Phi\xi = \Phi\Psi\xi = \Phi\theta = \theta.$$

So θ is also a random fixed point of Ψ.

Let θ_1 and θ_2 be two common random fixed points of Φ and Ψ. For each $t > 0$, we have

$$\begin{aligned} \mathbb{P}(\|\theta_1 - \theta_2\| > t) &= \mathbb{P}(\|\Phi\theta_1 - \Phi\theta_2\| > t) \\ &\le \mathbb{P}(\|\Psi\theta_1 - \Psi\theta_2\| > t/q) \\ &= \mathbb{P}(\|\theta_1 - \theta_2\| > t/q) \le \cdots \le \mathbb{P}(\|\theta_1 - \theta_2\| > t/q^n). \end{aligned}$$

Letting $n \to \infty$, we have

$$\mathbb{P}(\|\theta_1 - \theta_2\| > t) = 0 \quad \text{for all } t > 0.$$

Hence $\theta_1 = \theta_2$ a.s. □

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