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Long-time behavior for 2D non-autonomous g-Navier–Stokes equations

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Abstract. We study the first initial boundary value problem for the 2D non-autonomous g-Navier–Stokes equations in an arbitrary (bounded or unbounded) domain satisfying the Poincaré inequality. The existence of a weak solution to the problem is proved by using the Galerkin method. We then show the existence of a unique minimal finite-dimensional pullback $\mathcal{D}_\sigma$-attractor for the process associated to the problem with respect to a large class of non-autonomous forcing terms. Furthermore, when the force is time-independent and "small", the existence, uniqueness and global stability of a stationary solution are also studied.

1. Introduction. Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. In this paper we study the long-time behavior of solutions to the following 2D non-autonomous g-Navier–Stokes equations in Ω:

$$\begin{cases} \partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla p = f, & x\in\Omega,\ t>\tau, \\ (1/g)\nabla\cdot(gu) = 0, & x\in\Omega,\ t>\tau, \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(\tau,+\infty), \\ u(x,\tau) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t) = (u_1,u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, and u_0 is the initial velocity.

The g-Navier–Stokes equations are a variation of the standard Navier–Stokes equations. More precisely, when $g \equiv 1$ we get the usual Navier–Stokes equations. The 2D g-Navier–Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [11, 12] for a derivation of the 2D g-Navier–Stokes equations from the 3D Navier–Stokes equations and a relationship between them. As mentioned

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in [6], good properties of the 2D g-Navier–Stokes equations can initiate the study of the Navier–Stokes equations on the thin three-dimensional domain $\Omega_g = \Omega \times (0, g)$. In the last few years, the existence and asymptotic behavior of solutions to g-Navier–Stokes equations have been studied extensively (see e.g. [1, 3–6, 11, 12, 16]).

The aim of this paper is to continue the study of the long-time behavior of weak solutions to problem (1.1). When the external force f is time-dependent, we use the theory of pullback attractors. This theory is a natural generalization of the theory of global attractors for autonomous dynamical systems (cf. [15]) and allows considering a number of different problems of non-autonomous dynamical systems and random dynamical systems for a large class of non-autonomous forcing terms. When the force is time-independent and "small", we prove the existence, uniqueness and global stability of a stationary solution. The results obtained, in particular, recover and extend some existing ones for the 2D Navier–Stokes equations in [2, 7, 14, 15] and for 2D g-Navier–Stokes equations in bounded domains in [4].

In order to study problem (1.1), we assume that:

(H1) Ω is an arbitrary (bounded or unbounded) domain in $\mathbb{R}^2$ without any regularity assumption on $\partial\Omega$, provided that the Poincaré inequality holds on Ω: There exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g\,dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 g\,dx, \qquad \forall\phi \in H_0^1(\Omega).$$

(H2) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0\lambda_1^{1/2}.$$

(H3) $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V_g')$ such that

$$\int_{-\infty}^0 e^{\sigma s}\|f(s)\|^2_{V_g'}\,ds < +\infty,$$

where $\sigma < 2\nu\lambda_1\gamma_0$ is a fixed positive number with $\gamma_0 = 1 - |\nabla g|_\infty/(m_0\lambda_1^{1/2}) > 0$.

The paper is organized as follows. In the next section, we recall some auxilliary results on function spaces and inequalities for the nonlinear terms related to the g-Navier–Stokes equations, and abstract results on the existence and the fractal dimension of pullback attractors. In Sections 3 and 4, following the general lines of the proof in [2, 7], we prove the existence and fractal dimension estimates of a unique minimal pullback $\mathcal{D}_\sigma$-attractor for the associated process. The existence, uniqueness and global stability of a stationary solution are studied in the last section under some additional conditions.

2. Preliminary results

2.1. Function spaces and inequalities for the nonlinear terms. Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot vg\, dx, \qquad u, v \in L^2(\Omega, g),$$

and

$$((u, v))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g\, dx, \qquad u = (u_1, u_2),\, v = (v_1, v_2) \in H_0^1(\Omega, g),$$

and norms $|u|^2 = (u, u)_g$, $\|u\|^2 = ((u, u))_g$. Thanks to assumption (H2), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega, g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle \cdot, \cdot \rangle$ for the duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g\, dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u, v, w) = -b(u, w, v). \tag{2.1}$$

Hence

$$b(u, v, v) = 0, \qquad \forall u, v \in V_g. \tag{2.2}$$

Define $A : V_g \to V_g'$ by $\langle Au, v \rangle = ((u, v))_g$, $B : V_g \times V_g \to V_g'$ by $\langle B(u, v), w \rangle = b(u, v, w)$, $Bu = B(u, u)$. Then $D(A) = H^2(\Omega, g) \cap V$ and $Au = -P_g \Delta u$ for all $u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g.

Using Hölder's inequality, the Ladyzhenskaya inequality (when $n = 2$):

$$|u|_{L^4} \le c|u|^{1/2} |\nabla u|^{1/2}, \qquad \forall u \in H_0^1(\Omega),$$

and the interpolation inequalities, as in [13, 14], one can prove the following

LEMMA 2.1. *If $n = 2$, then*

$$
(2.3)\qquad |b(u,v,w)| \le \begin{cases} c_1|u|^{1/2}\|u\|^{1/2}\|v\|\,|w|^{1/2}\|w\|^{1/2}, & \forall u,v,w \in V_g,\\ c_2|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|, & \forall u \in V_g,\ v \in D(A),\ w \in H_g,\\ c_3|u|^{1/2}|Au|^{1/2}\|v\|\,|w|, & \forall u \in D(A),\ v \in V_g,\ w \in H_g,\\ c_4|u|\,\|v\|\,|w|^{1/2}|Aw|^{1/2}, & \forall u \in H_g,\ v \in V_g,\ w \in D(A), \end{cases}
$$

where c_i, $i = 1,\dots,4$, are appropriate constants.

LEMMA 2.2 ([1]). *Let $u \in L^2(\tau,T;V_g)$. Then the function Bu defined by*

$$
(Bu(t),v)_g = b(u(t),u(t),v), \qquad \forall u \in V_g,\ a.e.\ t \in [\tau,T],
$$

belongs to $L^2(\tau,T;V_g')$.

LEMMA 2.3 ([1]). *Let $u \in L^2(\tau,T;V_g)$. Then the function Cu defined by*

$$
(Cu(t),v)_g = \left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u,v\right)_g = b\left(\frac{\nabla g}{g},u,v\right), \qquad \forall v \in V_g,
$$

belongs to $L^2(\tau,T;H_g)$, and hence to $L^2(\tau,T;V_g')$. Moreover,

$$
\|Cu(t)\|_* \le \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u(t)\| \qquad \textit{for a.e. } t \in (\tau,T).
$$

Since

$$
-\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - \left(\frac{\nabla g}{g}\cdot\nabla\right)u,
$$

we have

$$
\begin{aligned}
(-\Delta u,v)_g &= ((u,v))_g + \left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u,v\right)_g\\
&= (Au,v)_g + \left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u,v\right)_g, \qquad \forall u,v \in V_g.
\end{aligned}
$$

2.2. Pullback attractors. Let (X,d) be a metric space. For $A,B \subset X$, we define the Hausdorff semi-distance between A and B by

$$
\operatorname{dist}(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y).
$$

A *process* on X is a two-parameter family of mappings $\{U(t,\tau)\}$ in X having the following properties:

$$
\begin{gathered}
U(t,r)U(r,\tau) = U(t,\tau) \qquad \text{for all } t \ge r \ge \tau,\\
U(\tau,\tau) = \mathrm{Id} \qquad \text{for all } \tau \in \mathbb{R}.
\end{gathered}
$$

The process $\{U(t,\tau)\}$ is said to be *norm-to-weak continuous* if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ as $x_n \to x$ in X, for all $t \ge \tau$, $\tau \in \mathbb{R}$.

Suppose that $\mathcal{B}(X)$ is the family of all non-empty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 2.4. The process $\{U(t,\tau)\}$ is said to be *pullback $\mathcal{D}$-asymptotically compact* if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 2.5. The family of bounded sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is called *pullback $\mathcal{D}$-absorbing* for the process $U(t,\tau)$ if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \le t$ such that

$$\bigcup_{\tau \le \tau_0} U(t,\tau)D(\tau) \subset B(t).$$

Definition 2.6. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a *pullback $\mathcal{D}$-attractor* for $\{U(t,\tau)\}$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;
(2) $\hat{\mathcal{A}}$ is invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t), \text{ for all } t \ge \tau;$$

(3) $\hat{\mathcal{A}}$ is *pullback $\mathcal{D}$-attracting*, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D} \text{ and all } t \in \mathbb{R};$$

(4) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.7 ([8]). *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau)B(\tau)}.$$

We now recall some results on the estimates of the fractal dimension of pullback attractors in [7].

Let H be a separable real Hilbert space H. Given a compact set $K \subset H$, and $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimum number of open balls in H with radii ε that are necessary to cover K.

Definition 2.8. For any non-empty compact set $K \subset H$, the *fractal dimension* of K is the number

$$d_{\mathrm{F}}(K) = \limsup_{\varepsilon \downarrow 0} \frac{\log(N_\varepsilon(K))}{\log(1/\varepsilon)}.$$

Consider a separable real Hilbert space $V \subset H$ such that the injection of V in H is continuous, and V is dense in H. We identify H with its topological dual H', and we consider V as a subspace of H', identifying $v \in V$ with the element $f_v \in H'$, defined by

$$f_v(h) = (v, h), \qquad h \in H.$$

Let $F : V \times \mathbb{R} \to V'$ be a given family of non-linear operators such that, for all $\tau \in \mathbb{R}$ and any $u_0 \in H$, there exists a unique function $u(t) = u(t; \tau, u_0)$ satisfying

(2.4)
$$\begin{cases} u \in L^2(\tau, T; V) \cap C([\tau, T]; H), \quad F(u(t), t) \in L^1(\tau, T; V') \quad \text{for all } T > \tau, \\ du/dt = F(u(t), t), \quad t > \tau, \\ u(\tau) = u_0. \end{cases}$$

Let us define

$$U(t, \tau)u_0 = u(t; \tau, u_0), \qquad \tau \leq t,\ u_0 \in H.$$

Fix $T^* \in \mathbb{R}$. We assume that there exists a family $\{A(t) : t \leq T^*\}$ of non-empty compact subsets of H with the invariance property

$$U(t, \tau)A(\tau) = A(t) \qquad \text{for all } \tau \leq t \leq T^*$$

and such that, for all $\tau \leq t \leq T^*$ and any $u_0 \in A(\tau)$, there exists a continuous linear operator $L(t; \tau, u_0) \in \mathcal{L}(H)$ such that

$$(2.5)\quad |U(t,\tau)\overline{u}_0 - U(t,\tau)u_0 - L(t;\tau,u_0)(\overline{u}_0 - u_0)| \leq \gamma(t-\tau, |\overline{u}_0 - u_0|)|\overline{u}_0 - u_0|$$

for all $\overline{u}_0 \in A(\tau)$, where $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\gamma(s, \cdot)$ is non-decreasing for all $s \geq 0$, and

$$(2.6)\qquad \lim_{r \to 0} \gamma(s, r) = 0 \qquad \text{for any } s \geq 0.$$

We assume that, for all $t \leq T^*$, the mapping $F(\cdot, t)$ is Gateaux differentiable in V, i.e., for any $u \in V$ there exists a continuous linear operator $F'(u, t) \in \mathcal{L}(V; V')$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(u + \epsilon v, t) - F(u, t) - \epsilon F'(u, t)v] = 0 \in V'.$$

Moreover, we suppose that the mapping

$$F' : (u, t) \in V \times (-\infty, T^*] \mapsto F'(u, t) \in \mathcal{L}(V; V')$$

is continuous (thus, in particular, for each $t \leq T^*$, the mapping $F(\cdot, t)$ is continuously Fréchet differentiable in V).

Then, for all $\tau \leq T^*$ and $u_0, v_0 \in H$, there exists a unique $v(t) = v(t; \tau, u_0, v_0)$ which is a solution of

$$\begin{cases} v \in L^2(\tau, T; V) \cap C([\tau, T]; H) \quad \text{for all } \tau < T \le T^*, \\ dv/dt = F'(U(t,\tau)u_0, t)v, \quad \tau < t < T^*, \\ v(\tau) = v_0. \end{cases}$$

We make the assumption that

$$v(t; \tau, u_0, v_0) = L(t; \tau, u_0)v_0 \quad \text{for all } \tau \le t \le T^*,\ u_0, v_0 \in A(\tau). \tag{2.7}$$

Let us write, for $j = 1, 2, \ldots,$

$$\widetilde{q}_j = \lim_{T\to+\infty} \sup_{\tau \le T^*} \sup_{u_0 \in A(\tau-T)} \left(\frac{1}{T} \int_{\tau-T}^{\tau} \mathrm{Tr}_j(F'(U(s, \tau - T)u_0, s))\, ds \right),$$

where

$$\mathrm{Tr}_j(F'(U(s,\tau)u_0, s)) = \sup_{v_0^i \in H,\, |v_0^i| \le 1,\, i \le j} \left(\sum_{i=1}^{j} \langle F'(U(s,\tau)u_0, s)e_i, e_i \rangle \right),$$

$e_1, \ldots, e_j$ being an orthonormal basis of the subspace in H spanned by

$$v(s; \tau, u_0, v_0^1), \ldots, v(s; \tau, u_0, v_0^j).$$

THEOREM 2.9 ([7, Theorem 2.2]). *Under the assumptions above, suppose that*

$$\bigcup_{\tau \le T^*} A(\tau) \ \textit{is relatively compact in } H,$$

and there exist q_j, $j = 1, 2, \ldots,$ such that

$$\begin{aligned} &\widetilde{q}_j \le q_j \quad \textit{for any } j \ge 1, \\ &q_{n_0} \ge 0, \quad q_{n_0+1} < 0 \quad \textit{for some } n_0 \ge 1, \\ &q_j \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j) \quad \textit{for all } j = 1, 2, \ldots. \end{aligned}$$

Then

$$d_{\mathrm{F}}(A(\tau)) \le d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}} \quad \textit{for all } \tau \le T^*.$$

3. Existence of pullback attractors. We first prove a result on the existence and uniqueness of a weak solution to problem (1.1).

DEFINITION 3.1. A function u is called a *weak solution* to problem (1.1) on the interval (τ, T) if

$$\begin{cases} u \in L^\infty(\tau, T; H_g) \cap L^2(\tau, T; V_g), \\ \dfrac{d}{dt}u(t) + \nu Au(t) + B(u(t), u(t)) + \nu Cu(t) = f(t) \text{ in } V_g' \quad \text{for a.e. } t \in (\tau, T), \\ u(\tau) = u_0. \end{cases}$$

THEOREM 3.2. *Suppose $u_0 \in H_g$ is given and assumptions* (H1)–(H3) *hold. Then, for any $\tau \in \mathbb{R}$, $T > \tau$ given, problem* (1.1) *has a unique weak* *solution* u *on* (τ, T). *Moreover,*

$$(3.1)\qquad |u(t)|^2 \le e^{-\sigma(t-\tau)}|u_0|^2 + \frac{e^{-\sigma t}}{2\epsilon\nu}\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|_*^2\,ds,$$

where ϵ *is the positive number such that* $\sigma = 2\nu\lambda_1(\gamma_0 - \epsilon)$.

Proof. (i) *Existence.* The existence part is based on Galerkin appoximations, *a priori* estimates, and the compactness method [9]. As it is standard and similar to the case of the Navier–Stokes equations [13], we provide only some basic *a priori* estimates used frequently later. From (1.1), we have

$$(3.2)\qquad \frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 2\langle f(t), u(t)\rangle - 2\nu b\left(\frac{\nabla g}{g}, u(t), u(t)\right).$$

Using Lemma 2.3 and the Cauchy inequality, we get

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \le 2\epsilon\nu\|u(t)\|^2 + \frac{1}{2\epsilon\nu}\|f(t)\|_*^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u(t)\|^2,$$

and hence

$$\frac{d}{dt}|u(t)|^2 + 2\nu(\gamma_0 - \epsilon)\|u(t)\|^2 \le \frac{1}{2\epsilon\nu}\|f(t)\|_*^2,$$

where $\gamma_0 = 1 - |\nabla g|_\infty/(m_0\lambda_1^{1/2})$, and $\epsilon > 0$ is chosen such that $\gamma_0 - \epsilon > 0$. Integrating the last inequality on $[\tau, t]$, $\tau \le t \le T$, we get

$$(3.3)\qquad \begin{aligned} |u(t)|^2 + 2\nu(\gamma_0 - \epsilon)\int_\tau^t \|u(s)\|^2\,ds &\le |u(\tau)|^2 + \frac{1}{2\epsilon\nu}\int_\tau^t \|f(s)\|_*^2\,ds \\ &\le |u_0|^2 + \frac{1}{2\epsilon\nu}\|f\|^2_{L^2(\tau,T;V_g')}. \end{aligned}$$

This inequality implies the estimates of u in the function space $L^2(\tau, T; V_g) \cap L^\infty(\tau, T; H_g)$. By rewriting the equation as

$$(3.4)\qquad \frac{du(t)}{dt} = -\nu Au(t) - B(u(t)) - \nu Cu(t) + f(t),$$

we get the estimate of du/dt in $L^2(\tau, T; V_g')$.

(ii) *Uniqueness and continuous dependence.* Assume that $u = u(t; \tau, u_0)$ and $v = v(t; \tau, v_0)$ are two weak solutions of (1.1) with initial data u_0, v_0. Set $w = u - v$. Then

$$w \in L^2(\tau, T; V_g) \cap L^\infty(\tau, T; H_g),$$

and w satisfies

$$\frac{d}{dt}w + \nu Aw + \nu Cw = Bv - Bu,$$
$$w(\tau) = u_0 - v_0.$$

Hence we have

$$\frac{d}{dt}|w(t)|^2 + 2\nu\|w(t)\|^2 + 2\nu b\bigg(\frac{\nabla g}{g}, w(t), w(t)\bigg)$$
$$= 2b(v(t), v(t), w(t)) - 2b(u(t), u(t), w(t)) = -2b(w(t), v(t), w(t)).$$

By Lemma 2.1, we have

$$|2b(w(t), v(t), w(t))| \le 2c|w(t)|\,\|w(t)\|\,\|v(t)\| \le \nu\|w(t)\|^2 + \frac{c^2}{\nu}|w(t)|^2\|v(t)\|^2,$$

and

$$\bigg|2\nu b\bigg(\frac{\nabla g}{g}, w(t), w(t)\bigg)\bigg| \le 2\frac{|\nabla g|_\infty}{m_0}\|w(t)\|\,|w(t)| \le \nu\|w(t)\|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}|w(t)|^2.$$

Therefore,

$$\frac{d}{dt}|w(t)|^2 \le \bigg(\frac{c^2}{\nu}\|v(t)\|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}\bigg)|w(t)|^2.$$

Thus,

$$|w(t)|^2 \le |w(\tau)|^2 \exp\bigg(\int_\tau^t \bigg(\frac{c^2}{\nu}\|v(s)\|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}\bigg)\,ds\bigg).$$

The last estimate implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of solutions on the initial data.

(iii) *The a priori estimate* (3.1). Choose $\epsilon > 0$ in inequality (3.3) such that $\sigma = 2\nu\lambda_1(\gamma_0 - \epsilon)$, where $\gamma_0 = 1 - |\nabla g|_\infty/(m_0\lambda_1^{1/2}) > 0$. Applying the Gronwall lemma in (3.3), we get (3.1). Hence it follows that the solution u can be extended to $[\tau, +\infty)$. ∎

Thanks to Theorem 3.2, we can define a process $U(t,\tau)$ in H_g by

$$U(t,\tau)u_0 = u(t;\tau,u_0), \qquad \tau \le t,\ u_0 \in H_g,$$

where $u(t) = u(t;\tau,u_0)$ is the unique weak solution of problem (1.1) with the initial datum $u(\tau) = u_0$.

We first prove the weak continuity of the process.

LEMMA 3.3. *Let $\{u_{0n}\}$ be a sequence in H_g converging weakly in H_g to an element $u_0 \in H_g$. Then*

(3.5) $\quad U(t,\tau)u_{0n} \rightharpoonup U(t,\tau)u_0$ *weakly in H_g, for all $\tau \le t$,*

(3.6) $\quad U(t,\tau)u_{0n} \rightharpoonup U(t,\tau)u_0$ *weakly in $L^2(\tau,T;V_g)$, for all $\tau < T$.*

Proof. Let $u_n(t) = U(t,\tau)u_{0n}$ and $u(t) = U(t,\tau)u_0$. As in the proof of Theorem 3.1, for all $T > \tau$,

(3.7) $\quad \{u_n\}$ is bounded in $L^\infty(\tau,T;H_g) \cap L^2(\tau,T;V_g)$,

and

$$\{u_n'\} \text{ is bounded in } L^2(\tau,T;V_g').$$

Then, for all $v \in V_g$, and $\tau \le t \le t+a \le T$ with $T > \tau$,

$$(3.8)\qquad (u_n(t+a)-u_n(t),v)_g = \int_t^{t+a} \langle u_n'(s),v\rangle\,ds \le \|v\|a^{1/2}\|u_n'\|_{L^2(\tau,T;V_g')} \le C_T\|v\|a^{1/2},$$

where C_T is positive and independent of n. Then, for $v = u_n(t+a)-u_n(t)$, which belongs to V_g for almost every t, from (3.8) we have

$$|u_n(t+a)-u_n(t)|^2 \le C_T a^{1/2}\|u_n(t+a)-u_n(t)\|.$$

Hence

$$(3.9)\qquad \int_\tau^{T-a} |u_n(t+a)-u_n(t)|^2\,ds \le C_T a^{1/2}\int_\tau^{T-a} \|u_n(t+a)-u_n(t)\|\,dt.$$

Using the Cauchy inequality and (3.7), we deduce from (3.9) that

$$\int_\tau^{T-a} |u_n(t+a)-u_n(t)|^2\,dt \le \widetilde{C}_T a^{1/2}$$

for another positive constant $\widetilde{C}_T$ independent of n. Therefore

$$(3.10)\qquad \lim_{a\to 0}\sup_n \int_\tau^{T-a} \|u_n(t+a)-u_n(t)\|^2_{L^2(\Omega_r,g)}\,dt = 0$$

for all $r > 0$, where $\Omega_r = \{x \in \Omega : |x| < r\}$. Moreover, from (3.7),

$$\{u_n|_{\Omega_r}\} \text{ is bounded in } L^\infty(\tau,T;L^2(\Omega_r,g)) \cap L^2(\tau,T;H^1(\Omega_r,g))$$

for all $r > 0$. Consider now a truncation function $\rho \in C^1(\mathbb{R}^+)$ with $\rho(s) = 1$ in $[0,1]$, and $\rho(s) = 0$ in $[2,+\infty)$. For each $r > 0$, define $v_{n,r}(x) = \rho(|x|^2/r^2)u_n(x)$ for $x \in \Omega_{2r}$. Then, from (3.10), we have

$$\lim_{a\to 0}\sup_n \int_\tau^{T-a} \|u_n(t+a)-u_n(t)\|^2_{L^2(\Omega_{2r},g)}\,dt = 0, \qquad \forall T > \tau,\ \forall r > 0,$$

and $\{v_{n,r}\}$ is bounded in $L^\infty(\tau,T;L^2(\Omega_{2r},g)) \cap L^2(\tau,T;H^1_0(\Omega_{2r},g))$ for all $T > \tau$, $r > 0$. Thus, by the Aubin–Lions lemma [9],

$$\{v_{n,r}\} \text{ is relatively compact in } L^2(\tau,T;L^2(\Omega_{2r},g)), \qquad \forall T > \tau,\ r > 0.$$

It follows that

$$\{u_n|_{\Omega_r}\} \text{ is relatively compact in } L^2(\tau,T;L^2(\Omega_{2r},g)), \qquad \forall T > \tau,\ r > 0.$$

Then, by a diagonal process, we can extract a subsequence $\{u_{n'}\}$ such that

$$
(3.11)\qquad \begin{aligned}
u_{n'} &\to \widetilde{u} \quad \text{weakly}^* \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}; H_g),\\
u_{n'} &\to \widetilde{u} \quad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}; V_g),\\
u_{n'} &\to \widetilde{u} \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega_r, g)),\ r > 0,
\end{aligned}
$$

for some $\widetilde{u} \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H_g) \cap L^2_{\mathrm{loc}}(\mathbb{R}; V_g)$.

The convergences (3.11) allows us to pass to the limit in the equation for $u_{n'}$ to find that $\widetilde{u}$ is a weak solution of (1.1) with $\widetilde{u}(\tau) = u_0$. By the uniqueness of the solutions we must have $\widetilde{u} = u$. Then by a contradiction argument we deduce that the whole sequence $\{u_n\}$ converges to u in the sense of (3.11). This proves (3.6).

Now, from the strong convergence in (3.11) we also infer that $u_n(t)$ converges strongly in $L^2(\Omega_r, g)$ to $u(t)$ for a.e. $t \ge \tau$ and all $r > 0$. Hence for all $v \in \mathcal{V}$,

$$(u_n(t), v)_g \to (u(t), v)_g \quad \text{for a.e. } t \in \mathbb{R}.$$

Moveover, from (3.9) and (3.10), we see that $\{(u_n(t), v)\}$ is equibounded and equicontinuous on $[\tau, T]$ for all $T > 0$. Therefore

$$(u_n(t), v)_g \to (u(t), v)_g, \quad \forall t \in \mathbb{R},\ \forall v \in \mathcal{V}.$$

Finally, (3.5) follows from the fact that $\mathcal{V}$ is dense in H_g. ■

Let $\mathcal{R}_\sigma$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$(3.12)\qquad \lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0,$$

and denote by $\mathcal{D}_\sigma$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(H_g)$ such that $D(t) \subset B(0, \hat{r}(t))$ for some $\hat{r}(t) \in \mathcal{R}_\sigma$, where $B(0, r)$ denotes the close ball in H_g, centered at zero with radius r.

Now, we can prove one of the main results of the paper.

THEOREM 3.4. *Suppose that conditions* (H1)–(H3) *hold. Then there exists a unique pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ for the process $\{U(t, \tau)\}$ associated to problem* (1.1).

Proof. Let $\tau \in \mathbb{R}$ and $u_0 \in H_g$ be fixed, and denote

$$u(t) = u(t; \tau, u_0) = U(t, \tau)u_0 \quad \text{for all } t \ge \tau.$$

In order to apply Theorem 2.7, we will check the conditions in the abstract theorem.

(i) *The process $U(t, \tau)$ has a family $\hat{B}$ of pullback $\mathcal{D}_\sigma$-absorbing sets.* Let $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$ be given. From (3.1), we get

$$(3.13)\qquad |U(t, \tau)u_0|^2 \le e^{-\sigma(t-\tau)}\hat{r}(\tau) + \frac{e^{-\sigma t}}{2\epsilon\nu} \int_{-\infty}^{t} e^{\sigma s} \|f(s)\|_*^2\, ds$$

for all $u_0 \in D(\tau)$ and all $t \ge \tau$.

Define $R_\sigma(t) \in \mathcal{R}_\sigma$ by

$$R_\sigma^2(t) = \frac{e^{-\sigma t}}{\epsilon\nu} \int_{-\infty}^{t} e^{\sigma s} \|f(s)\|_*^2 \, ds, \tag{3.14}$$

and consider the family $\hat{\mathcal{B}}_\sigma$ of closed balls in H_g defined by $B_\sigma(t) = B(0, R_\sigma(t))$. It is straightforward to check that $\hat{\mathcal{B}}_\sigma \in \mathcal{D}_\sigma$, and moreover, by (3.12) and (3.13), the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing for the process $U(t,\tau)$.

(ii) $U(t,\tau)$ *is pullback* $\mathcal{D}_\sigma$*-asymptotically compact.* Fix $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$, a sequence $\tau_n \to -\infty$, a sequence $u_{0n} \in D(\tau_n)$ and $t \in \mathbb{R}$. We must prove that from the sequence $\{U(t,\tau_n)u_{0n}\}$ we can extract a subsequence that converges in H_g.

As the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing, for each integer $k \geq 0$, there exists a $\tau_{\hat{D}}(k) \leq t - k$ such that

$$U(t-k,\tau)D(\tau) \subset B_\sigma(t-k) \quad \text{for all } \tau \leq \tau_{\hat{D}}(k), \tag{3.15}$$

so that for $\tau_n \leq \tau_{\hat{D}}(k)$,

$$U(t-k,\tau_n)u_{0n} \subset B_\sigma(t-k).$$

Thus, $\{U(t-k,\tau_n)u_{0n}\}$ is weakly precompact in H_g and since $B_\sigma(t-k)$ is closed and convex, there exist a subsequence $\{(\tau_{n'}, u_{0n'})\} \subset \{(\tau_n, u_{0n})\}$ and a sequence $\{w_k : k \geq 0\} \subset H_g$ such that for all $k \geq 0$, $w_k \in B_\sigma(t-k)$, and

$$U(t-k,\tau_{n'})u_{0n'} \rightharpoonup w_k \quad \text{weakly in } H_g. \tag{3.16}$$

Note that from the weak continuity of $U(t,\tau)$ established in Lemma 3.3, we have

$$\begin{aligned} w_0 &= \underset{n'\to\infty}{w\text{-lim}}\, U(t,\tau_{n'})u_{0n'} = \underset{n'\to\infty}{w\text{-lim}}\, U(t,t-k)U(t-k,\tau_{n'})u_{0n'} \\ &= U(t,t-k)\, \underset{n'\to\infty}{w\text{-lim}}\, U(t-k,\tau_{n'})u_{0n'} = U(t,t-k)w_k, \end{aligned}$$

where w-lim denotes the limit taken in the weak topology of H_g. Thus

$$U(t,t-k)w_k = w_0 \quad \text{for all } k \geq 0. \tag{3.17}$$

Now, from (3.16), by the lower semicontinuity of the norm, we have

$$|w_0| \leq \liminf_{n'\to\infty} |U(t,\tau_{n'})u_{0n'}|.$$

If we now prove that also

$$\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{0n'}| \leq |w_0|, \tag{3.18}$$

then we will have

$$\lim_{n'\to\infty} |U(t,\tau_{n'})u_{0n'}| = |w_0|,$$

and this, together with the weak convergence, will imply the strong convergence in H_g of $U(t,\tau_{n'})u_{0n'}$ to w_0.

In order to prove (3.18), define $[\cdot\,,\cdot]_g : V_g \times V_g \to \mathbb{R}$ by

(3.19)
$$[u,v]_g = \nu((u,v))_g + \frac{\nu}{2}\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u,v\Big) + \frac{\nu}{2}\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)v,u\Big) - \frac{\sigma}{2}(u,v)_g,$$

for all $u,v \in V_g$. Clearly, $[\cdot\,,\cdot]_g$ is bilinear and symmetric. Moreover, from the fact that $\|u\|^2 \geq \lambda_1|u|^2$, we have

$$\begin{aligned}[u]^2 \equiv [u,u]_g &= \nu\|u\|^2 + \nu\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u,u\Big) - \frac{\sigma}{2}|u|^2 \\ &\geq \nu\|u\|^2 - \nu\Big(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{\gamma_0}{2}\Big)\|u\|^2 = \epsilon\nu\|u\|^2,\end{aligned}$$

where we have used Lemma 2.3 and the facts that $\sigma = 2\nu\lambda_1(\gamma_0-\epsilon)$ and $\gamma_0 = 1 - |\nabla g|_\infty/(m_0\lambda_1^{1/2})$. Hence

$$\frac{\epsilon}{\nu}\|u\|^2 \leq [u]^2 \leq \nu\|u\|^2, \qquad \forall u \in V_g. \tag{3.20}$$

Thus, $[\cdot\,,\cdot]_g$ defines an inner product in V_g with the norm $[\cdot] = [\cdot\,,\cdot]_g^{1/2}$, which is equivalent to the norm $\|\cdot\|$ in V_g.

Now, from (3.2), we get

$$\frac{d}{dt}|u(t)|^2 + \sigma|u(t)|^2 + 2[u(t)]^2 \leq 2\langle f(t),u(t)\rangle.$$

Hence

$$|u(t)|^2 \leq |u_0|^2 e^{-\sigma(t-\tau)} + 2\int_\tau^t (\langle f(s),u(s)\rangle - [u(s)]^2)\,ds,$$

which can be rewritten as

$$\begin{aligned}|U(t,\tau)u_0|^2 \leq{}& |u_0|^2 e^{\sigma(\tau-t)} \\ &+ 2\int_\tau^t e^{\sigma(s-t)}(\langle f(s),U(s,\tau)u_0\rangle - [U(s,\tau)u_0]^2)\,ds\end{aligned} \tag{3.21}$$

for all $\tau \leq t$, and all $u_0 \in H_g$. Thus, for all $k \geq 0$ and all $\tau_{n'} \leq t-k$,

$$\begin{aligned}|U(t,\tau_{n'})u_{0n'}|^2 &= |U(t,t-k)U(t-k,\tau_{n'})u_{0n'}|^2 \\ &\leq e^{-\sigma k}|U(t-k,\tau_{n'})u_{0n'}|^2 \\ &\quad + 2\int_{t-k}^t e^{\sigma(s-t)}\langle f(s),U(s,t-k)U(t-k,\tau_{n'})u_{0n'}\rangle\,ds \\ &\quad - 2\int_{t-k}^t e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0n'}]^2\,ds.\end{aligned} \tag{3.22}$$

By (3.15), $U(t-k,\tau_{n'})u_{0n'} \in B_\sigma(t-k)$ for all $\tau_{n'} \leq \tau_{\hat{D}}(k)$, $k \geq 0$, we have

$$\limsup_{n'\to\infty} e^{-\sigma k}|U(t-k,\tau_{n'})u_{0n'}|^2 \le e^{-\sigma k}R_\sigma^2(t-k), \qquad k \ge 0. \tag{3.23}$$

As $U(t-k,\tau_{n'})u_{0n'} \rightharpoonup w_k$ weakly in H_g, from Lemma 3.3 we have

$$U(\cdot,t-k)U(t-k,\tau_{n'})u_{0n'} \rightharpoonup U(\cdot,t-k)w_k \text{ weakly in } L^2(t-k,t;V_g). \tag{3.24}$$

Taking into account that, in particular, $e^{\sigma(s-t)}f(s) \in L^2(t-k,t;V_g')$, from (3.24) we obtain

$$\begin{aligned}\lim_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)U(t-k,\tau_{n'})u_{0n'}\rangle\, ds \\ = \int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)w_k\rangle\, ds.\end{aligned} \tag{3.25}$$

Moreover, since $[\cdot]$ is a norm in V_g equivalent to $\|\cdot\|$ and

$$0 < e^{-\sigma k} \le e^{\sigma(s-t)} \le 1 \quad \text{ for all } s \in [t-k,t],$$

we see that

$$\Big(\int_{t-k}^{t} e^{-\sigma(t-s)}[\cdot]^2\, ds\Big)^{1/2}$$

is a norm in $L^2(t-k,t;V_g)$ equivalent to the usual norm. Hence from (3.24) we deduce that

$$\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2\, ds \le \liminf_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0n'}]^2\, ds.$$

Hence

$$\begin{aligned}\limsup_{n'\to\infty} -2\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0n'}]^2\, ds \\ = -\liminf_{n'\to\infty} 2\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0n'}]^2\, ds \\ \le -2\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2\, ds.\end{aligned} \tag{3.26}$$

We can now pass to the lim sup as n' goes to ∞ in (3.22), and take (3.23), (3.25) and (3.26) into account to obtain

$$\begin{aligned}\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{0n'}|^2 \le e^{-\sigma k}R_\sigma^2(t-k) \\ + 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle f(s), U(s,t-k)w_k\rangle - [U(s,t-k)w_k]^2\big)\, ds.\end{aligned} \tag{3.27}$$

On the other hand, from (3.21) applied to (3.17) we find that

$$|w_0| = |U(t,t-k)w_k|^2 = |w_k|^2 e^{-\sigma k} + 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle f(s), U(s,t-k)w_k\rangle - [U(s,t-k)w_k]^2\big)\,ds. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\begin{aligned}\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{0n'}|^2 &\le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2 - |w_k|^2 e^{-\sigma k}\\ &\le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2,\end{aligned}$$

and thus, taking into account that

$$e^{-\sigma k}R_\sigma^2(t-k) = \frac{e^{-\sigma t}}{\epsilon\nu}\int_{-\infty}^{t-k} e^{\sigma s}\|f(s)\|_*^2\,ds \to 0 \quad \text{as } k\to+\infty,$$

we easily obtain (3.18) from the last inequality. ■

REMARK 3.5. When $g \equiv \text{const} > 0$, we formally get the results for the 2D non-autonomous Navier–Stokes equations. Notice that the result of Theorem 3.4 improves the existing one for the Navier–Stokes equations in the sense that the external force $f \in L^2_{\text{loc}}(\mathbb{R}; V_g')$ only need satisfy

$$\int_{-\infty}^{0} e^{\sigma s}\|f(s)\|_*^2\,ds < +\infty, \qquad \text{where } \sigma < 2\nu\lambda_1,$$

compared with the condition $\sigma = \nu\lambda_1$ for the Navier–Stokes equations [2] (this condition is recovered if we take $g \equiv 1$ and $\epsilon = 1/2$).

4. Fractal dimension estimates of the pullback attractor. Observe that problem (1.1) can be written in the form (2.4) by taking

$$F(u,t) = -\nu Au(t) - Bu(t) - \nu Cu(t) + f(t).$$

Then it follows immediately that for all $t \in \mathbb{R}$, the mapping $F(\cdot,t)$ is Gateaux differentiable in V_g with

$$F'(u,t)v = -\nu Av - B(u,v) - B(v,u) - \nu Cv, \qquad u,v \in V_g,$$

and the mapping $F' : (u,t) \in V_g \times \mathbb{R} \mapsto F'(u,t) \in \mathcal{L}(V_g; V_g')$ is continuous.

Evidently, for any $\tau \in \mathbb{R}$, $u_0, v_0 \in H_g$, there exists a unique solution $v(t) = v(t;\tau,u_0,v_0)$ of the problem

$$\begin{cases} v \in L^2(\tau,T;V_g) \cap C([\tau,T];H_g),\\ \dfrac{dv}{dt} = -\nu Av(t) - B(U(t,\tau)u_0, v(t)) - B(v(t), U(t,\tau)u_0) - \nu Cv(t), \ \tau < t,\\ v(\tau) = v_0. \end{cases} \tag{4.1}$$

From now on we suppose that

$$f \in L^{\infty}(-\infty, T^*; V_g') \quad \text{for some } T^* \in \mathbb{R}. \tag{4.2}$$

LEMMA 4.1. *Suppose that conditions* (H1)–(H3) *and* (4.2) *hold. Then the pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}$ obtained in Theorem* 3.4 *satisfies*

$$\bigcup_{\tau \le T*} A(\tau) \textit{ is relatively compact in } H_g. \tag{4.3}$$

Proof. Denoting $M = \|f\|_{L^{\infty}(-\infty,T^*;V_g')}$, from (3.14) we have

$$R_\sigma^2(t) \le \frac{Me^{-\sigma t}}{\epsilon\nu} \int_{-\infty}^{t} e^{\sigma\xi}\, d\xi = \frac{M}{\epsilon\nu\sigma},$$

and consequently

$$B^* := \bigcup_{\tau \le T^*} B_\sigma(\tau) \text{ is bounded in } H_g,$$

where $B_\sigma(\tau) = B(0, R_\sigma(\tau))$.

Denote by $\mathcal{M}$ the set of all $y \in H_g$ such that there exist a sequence $\{(t_n, \tau_n)\} \subset \mathbb{R}^2$ satisfying

$$\tau_n \le t_n \le T^*, \quad n \ge 1, \quad \lim_{n\to\infty} (t_n - \tau_n) = +\infty,$$

and a sequence $\{u_{0n}\} \subset B^*$ such that $\lim_{n\to\infty} |U(t, \tau_n)u_{0n} - y| = 0$.

It is easy to see that $A(t) \subset \mathcal{M}$ for all $t \le T^*$. If we prove that $\mathcal{M}$ is relatively compact in H_g, then (4.3) follows immediately.

Let $\{y_k\} \subset \mathcal{M}$. For each $k \ge 1$, we can take $(t_k, \tau_k) \in \mathbb{R}^2$ and an element $u_{0k} \in B^*$ such that $t_k \le T^*, t_k - \tau_k \ge k$ and $|U(t_k, \tau_k)u_{0k} - y_k| \le 1/k$. Using (4.2), by arguments as in Proposition 3.4 in [7], we can extract from $\{y_k\}$ a subsequence that converges in H_g. ∎

LEMMA 4.2. *Suppose that conditions* (H1)–(H3) *and* (4.2) *hold. Then the process $U(t, \tau)$ associated to problem* (1.1) *has the quasidifferentiability properties* (2.5)–(2.7), *with $v(t) = v(t; \tau, u_0, v_0)$ defined by* (4.1).

Proof. By (4.2) and Lemma 4.1, there exists a constant $C > 1$ such that

$$\|f\|^2_{L^{\infty}(-\infty,T^*;V_g')} \le C\nu^3, \quad |u_0|^2 \le C\nu^2 \quad \text{for all } u_0 \in \bigcup_{\tau \le T^*} A(\tau). \tag{4.4}$$

Fix $\tau \le T^*$, $u_0, \overline{u}_0 \in A(\tau)$, denote $u(t) = U(t, \tau)u_0$, $\overline{u}(t) = U(t, \tau)\overline{u}_0$ and let $v(t)$ be the solution of (4.1) with $v_0 = \overline{u}_0 - u_0$.

It is easy to see that

$$|u(t)|^2 + 2\nu(\gamma_0 - \epsilon) \int_{\tau}^{t} \|u(s)\|^2\, ds \le |u_0|^2 + \frac{1}{2\epsilon\nu} \int_{\tau}^{t} \|f(s)\|_*^2\, ds, \tag{4.5}$$

where $\gamma_0 = 1 - |\nabla g|_\infty/(m_0\lambda_1^{1/2})$, and $\epsilon > 0$ is chosen such that $\gamma_0 - \epsilon > 0$.

Taking into account (4.4), we easily deduce from (4.5) that

$$\int_\tau^t \|u(s)\|^2\, ds \le \frac{C\nu}{2(\gamma_0-\epsilon)}(1+t-\tau) \quad \text{for all } \tau \le t \le T^*. \tag{4.6}$$

Denoting

$$w(t) = \overline{u}(t) - u(t), \quad \tau \le t,$$

we have

$$\begin{aligned}\frac{d}{dt}|w(t)|^2 + 2\nu\|w(t)\|^2 + 2\bigg(\frac{\nu}{g}(\nabla g\cdot\nabla)w(t), w(t)\bigg)&\\ = -2b(\overline{u}(t),\overline{u}(t),w(t)) &+ 2b(u(t),u(t),w(t))\\ = 2b(w(t),u(t),w(t)).&\end{aligned}$$

Since

$$\begin{aligned}|2b(w(t),u(t),w(t))| &\le 2c|w(t)|\,\|w(t)\|\,\|u(t)\|\\ &\le \frac{\nu}{2}\|w(t)\|^2 + \frac{2c^2}{\nu}|w(t)|^2\|u(t)\|^2,\end{aligned}$$

and

$$\begin{aligned}\bigg|2\bigg(\frac{\nu}{g}(\nabla g\cdot\nabla)w(t), w(t)\bigg)\bigg| &\le 2\nu\frac{|\nabla g|_\infty}{m_0}\|w(t)\||w(t)|\\ &\le \frac{\nu}{2}\|w(t)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}|w(t)|^2,\end{aligned}$$

we have

$$\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 \le \bigg(\frac{2c^2}{\nu}\|u(t)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\bigg)|w(t)|^2. \tag{4.7}$$

In particular,

$$|w(t)|^2 \le |w(\tau)|^2 \exp\bigg(\int_\tau^t\bigg(\frac{2c^2}{\nu}\|u(s)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\bigg)\,ds\bigg).$$

Thus, by using (4.6),

$$|w(t)|^2 \le |w(\tau)|^2\exp(C_1(1+t-\tau)) \quad \text{for all } \tau\le t\le T^*, \tag{4.8}$$

where $C_1 = \max\{Cc^2/(\gamma_0-\epsilon) + 2\nu|\nabla g|_\infty^2/m_0^2, 1\}$.

Now, from (4.7) and (4.8), we have

$$\begin{aligned}\nu\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2 + \int_\tau^t \Big(\frac{2c^2}{\nu}\|u(s)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\Big)|w(s)|^2\,ds \\ &\le |w(\tau)|^2 + \int_\tau^t \Big(\frac{2c^2}{\nu}\|u(s)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\Big)|w(\tau)|^2\exp(C_1(1+s-\tau))\,ds \\ &\le |w(\tau)|^2\Big[1+\exp(C_1(1+t-\tau))\int_\tau^t\Big(\frac{2c^2}{\nu}\|u(s)\|^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\Big)\,ds\Big],\end{aligned}$$

and thus, by (4.4), we have

$$\begin{aligned}\nu\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2[1+\exp(C_1(1+t-\tau))C_1(1+t-\tau)] \\ &\le |w(\tau)|^2[1+C_1(1+t-\tau)]\exp(C_1(1+t-\tau)) \\ &\le |w(\tau)|^2\exp(2C_1(1+t-\tau)).\end{aligned}\tag{4.9}$$

Let $z(t)$ be defined by

$$z(t) = \overline{u}(t) - u(t) - v(t) = w(t) - v(t), \quad t \ge \tau.$$

Evidently, $z(t)$ satisfies

$$\begin{cases} z \in L^2(\tau,T;V_g)\cap L^\infty(\tau,T;H_g)\cap C([\tau,T];H_g) \quad \text{for all } t>\tau, \\ \dfrac{dz}{dt} = -\nu Az(t) - B(\overline{u}(t),\overline{u}(t)) + B(u(t),u(t)) + B(u(t),v(t)) \\ \qquad\quad + B(v(t),u(t)) - \nu Cz(t), \quad t>\tau, \\ z(\tau)=0. \end{cases}$$

It is easy to see that

$$\begin{aligned}-B(\overline{u}(t),\overline{u}(t)) &+ B(u(t),u(t)) + B(u(t),v(t)) + B(v(t),u(t)) \\ &= -B(u(t),z(t)) - B(z(t),u(t)) - B(w(t),w(t)),\end{aligned}$$

and consequently, for all $t>\tau$,

$$\begin{aligned}\frac{d}{dt}|z|^2 + 2\nu\|z\|^2 &= -2b(z,u,z) - 2b(w,w,z) - 2\Big(\frac{\nu}{g}(\nabla g\cdot\nabla)z, z\Big) \\ &\le -2b(z,u,z) - 2b(w,w,z) + 2\nu\frac{|\nabla g|_\infty}{m_0}\|z\|\,|z| \\ &\le \frac{\nu}{2}\|z\|^2 + \frac{2c^2}{\nu}\|u\|^2|z|^2 + \frac{\nu}{2}\|z\|^2 + \frac{2c^2}{\nu}\|w\|^2|w|^2 + \nu\|z\|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}|z|^2 \\ &= 2\nu\|z\|^2 + \frac{2c^2}{\nu}\|u\|^2|z|^2 + \frac{2c^2}{\nu}\|w\|^2|w|^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2}|z|^2.\end{aligned}\tag{4.10}$$

Integrating (4.10) from τ to t, and using the fact that $z(\tau) = 0$, we have

$$|z(t)|^2 \leq \frac{2c^2}{\nu} \int_\tau^t \|w\|^2 |w|^2 \, ds + \int_\tau^t \left(\frac{2c^2}{\nu} \|u\|^2 + \frac{\nu |\nabla g|_\infty^2}{m_0^2} \right) |z|^2 \, ds, \quad \forall t \geq \tau,$$

and consequently, by Gronwall's lemma,

$$|z(t)|^2 \leq \frac{2c^2}{\nu} \int_\tau^t \|w\|^2 |w|^2 \, ds \exp \left[\int_\tau^t \left(\frac{2c^2}{\nu} \|u\|^2 + \frac{\nu |\nabla g|_\infty^2}{m_0^2} \right) ds \right].$$

From (4.8) we obtain

$$|z(t)|^2 \leq \frac{2c^2}{\nu} |w(\tau)|^2 \exp(2C_1(1 + t - \tau)) \int_\tau^t \|w(s)\|^2 \, ds.$$

Plugging (4.9) into the last estimate, we obtain

$$|z(t)|^2 \leq \frac{2c^2}{\nu^2} |w(\tau)|^4 \exp(4C_1(1 + t - \tau)),$$

i.e., (2.5)–(2.7) hold with

$$\gamma(s, r) = \frac{\sqrt{2}\, cr}{\nu} \exp(2C_1(1 + s)),$$

where $C_1 > 1$. ∎

We now prove the main result in this section.

THEOREM 4.3. *Suppose that conditions* (H1)–(H3) *and* (4.2) *hold. Then the pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ satisfies*

$$d_{\mathrm{F}}(A(\tau)) \leq \max \left(1, \frac{\kappa \|f\|_{L^\infty(-\infty, T^*; V_g')}^2}{16 \nu^4 (\gamma_0 - \epsilon)^2 \epsilon^2 \lambda_1} \right) \quad \textit{for all } \tau \in \mathbb{R},$$

where $\gamma_0 = 1 - |\nabla g|_\infty^2 / (m_0 \lambda_1^2) > 0$, and $\epsilon > 0$ is the number such that $\sigma = 2\nu\lambda_1(\gamma_0 - \epsilon)$.

Proof. For $u_0, \xi_1, \ldots, \xi_m \in H_g$, we suppose $v_j(t) = L(t, \tau; u_0)\xi_j$. Let $\{e_1(t), \ldots, e_m(t)\}$ be an orthonormal basis in H_g of the subspace spanned by $\{v_1(t), \ldots, v_m(t)\}$. Since $v_j(t) \in V_g$ for a.e. $t \geq \tau$, we can assume $e_j(t) \in V_g$ for a.e. $t \geq \tau$. Then it is not difficult to see that

$$\begin{aligned} \text{(4.11)} \qquad \operatorname{Tr}_m(F'(U(s, \tau)u_0, s)) &= \sum_{i=1}^m \langle F'(U(s, \tau)u_0, s) e_i, e_i \rangle \\ &= -\nu \sum_{i=1}^m \|e_i\|^2 - \sum_{i=1}^m b(e_i, U(s, \tau)u_0, e_i) - \sum_{i=1}^m \left(\frac{\nu}{g} (\nabla g \cdot \nabla) e_i, e_i \right) \end{aligned}$$

for a.e. $s \geq \tau$.

Using the explicit expression for b, we have

$$\Big|\sum_{i=1}^{m} b(e_i,u,e_i)\Big| = \Big|\sum_{i=1}^{m}\int_{\Omega}\sum_{k,l=1}^{2} e_{ik}(x)D_l u_k(x)e_{il}(x)g(x)\,dx\Big|$$
$$\le \int_{\Omega} |\operatorname{grad} u(x)|\rho(x)g(x)\,dx,$$

where

$$|\operatorname{grad} u(x)| = \Big\{\sum_{l,k=1}^{2} |D_l u_k(x)|^2\Big\}^{1/2}, \qquad \rho(x) = \sum_{i=1}^{m}\sum_{k=1}^{2}(e_{ik}(x))^2.$$

Therefore,

$$\Big|\sum_{i=1}^{m} b(e_i,u,e_i)\Big| \le \int_{\Omega} |\operatorname{grad} u(x)|\rho(x)g(x)\,dx \le \|u\|\,|\rho| \tag{4.12}$$

by the Schwarz inequality. Also, we obtain

$$\Big|\sum_{i=1}^{m}\Big(\frac{\nu}{g}(\nabla g\cdot\nabla)e_i,e_i\Big)\Big| \le \sum_{i=1}^{m}\frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|e_i\|^2. \tag{4.13}$$

We recall that the dependence on s has been omitted and in fact $u = u(s,x)$, $\rho = \rho(s,x)$, etc. From (4.11)–(4.13), we get

$$\operatorname{Tr}_m(F'(U(s,\tau)u_0,s) \le -\nu\Big(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)\sum_{i=1}^{m}\|e_i\| + \|U(t,\tau)u_0\|\,|\rho|$$
$$= -\nu\gamma_0\sum_{i=1}^{m}\|e_i\| + \|U(t,\tau)u_0\|\,|\rho|.$$

Since $\{e_i\}$ are orthonormal in H_g, hence in $L^2(\Omega,g)$, and belong to $V_g \hookrightarrow H_0^1(\Omega,g)$, by the Lieb–Thirring inequality (see [15, Theorem A.3.1] with $n=2$, $p=2$, $m=1$ and in particular [15, Example A5.1]), there exists a constant κ depending only on the shape of Ω such that

$$|\rho(s)|^2 = \int_{\Omega}\rho(s,x)^2 g(x)\,dx \le \kappa\sum_{i=1}^{m}\|e_i\|^2.$$

Hence,

$$\operatorname{Tr}_m(F'(U(s,\tau)u_0,s) \le -\nu\gamma_0\sum_{i=1}^{m}\|e_i\| + \|U(t,\tau)u_0\||\rho|$$
$$\le -\nu\gamma_0\sum_{i=1}^{m}\|e_i\|^2 + \|U(t,\tau)u_0\|\Big(\kappa\sum_{i=1}^{m}\|e_i\|^2\Big)^{1/2}$$
$$\le -\nu(\gamma_0-\epsilon)\sum_{i=1}^{m}\|e_i\|^2 + \frac{\kappa}{4\nu\epsilon}\|U(t,\tau)u_0\|^2$$

$$\le -\nu(\gamma_0-\epsilon)\lambda_1\sum_{i=1}^{m}|e_i|^2+\frac{\kappa}{4\nu\epsilon}\|U(t,\tau)u_0\|^2$$
$$=-\nu(\gamma_0-\epsilon)\lambda_1 m+\frac{\kappa}{4\nu\epsilon}\|U(t,\tau)u_0\|^2,$$

where we have used the fact that $|e_i|=1$. On the other hand,

$$\frac{d}{dt}|U(s,\tau)u_0|^2+2\nu(\gamma_0-\epsilon)\|U(s,\tau)u_0\|^2\le\frac{\|f(s)\|_*^2}{2\nu\epsilon},$$

and if we denote $M=\|f\|^2_{L^\infty(-\infty,T^*;V_g')}$ we have

$$\int_{\tau-T}^{\tau}\|U(s,\tau)u_0\|^2\,ds\le\left(\frac{MT}{4\nu^2\epsilon}+\frac{|u_0|^2}{2\nu}\right)(\gamma_0-\epsilon)^{-1},\qquad t\ge\tau.$$

Using Theorem 2.9, we obtain

(4.14)

$$\widetilde{q}_m\le\limsup_{T\to+\infty}\sup_{u_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\mathrm{Tr}_m(F'(U(s,\tau-T)u_0,s))\,ds$$
$$\le-\nu(\gamma_0-\epsilon)\lambda_1 m+\frac{\kappa}{4\nu\epsilon}\limsup_{T\to+\infty}\sup_{u_0\in A(\tau-T)}\left(\frac{M}{4\nu^2\epsilon}+\frac{|u_0|^2}{2\nu T}\right)(\gamma_0-\epsilon)^{-1}$$
$$\le-\nu(\gamma_0-\epsilon)\lambda_1 m+\frac{\kappa M}{16\nu^3\epsilon^2(\gamma_0-\epsilon)}.$$

We now consider two cases: if $\kappa M<16\nu^4(\gamma_0-\epsilon)^2\epsilon^2\lambda_1$, then taking

$$q_m=-\nu(\gamma_0-\epsilon)\lambda_1(m-1),\qquad m=1,2,\ldots,$$

and $n_0=1$, we can apply Theorem 2.9 to obtain

$$d_{\mathrm F}(A(\tau))\le 1\quad\text{ for all }\tau\le T^*;$$

if $\kappa M\ge 16\nu^4(\gamma_0-\epsilon)^2\epsilon^2\lambda_1$, then taking

$$q_m=-\nu(\gamma_0-\epsilon)\lambda_1 m+\frac{\kappa M}{16\nu^3\epsilon^2(\gamma_0-\epsilon)},\qquad m=1,2,\ldots,$$

and

$$n_0=1+\left[\frac{\kappa M}{16\nu^4(\gamma_0-\epsilon)^2\epsilon^2\lambda_1}-1\right],$$

where $[r]$ denotes the integer part of a real number r, we obtain

$$d_{\mathrm F}(A(\tau))\le\frac{\kappa\|f\|^2_{L^\infty(-\infty,T^*;V_g')}}{16\nu^4(\gamma_0-\epsilon)^2\epsilon^2\lambda_1}\quad\text{ for all }\tau\le T^*.$$

Finally, since $U(t,\tau)$ is Lipschitz in $A(\tau)$, it follows from [10, Proposition 13.9] that $d_{\mathrm F}(A(t))$ is bounded for every $t\ge\tau$ with the same bound. ∎

REMARK 4.4. When $g \equiv 1$ and $\epsilon = 1/2$, we recover the result in [7] for the usual 2D non-autonomous Navier–Stokes equations:

$$d_{\mathrm{F}}(A(\tau)) \leq \max\left(1, \frac{\kappa\|f\|^2_{L^\infty(-\infty,T^*;V_g')}}{\nu^4\lambda_1}\right) \quad \text{for all } \tau \in \mathbb{R}.$$

5. Global stability of stationary solutions. Assuming now that the external force f is independent of time t, in this section we are looking for solutions of the following problem:

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & x \in \Omega, \\ (1/g)\nabla\cdot(gu) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Given $f \in V_g'$, a weak solution to stationary problem (5.1) is a function $u \in V_g$ satisfying

$$\nu((u,v))_g + b(u,u,v) = \langle f, v\rangle \quad \text{for all test functions } v \in V_g.$$

THEOREM 5.1. *For every $f \in V_g'$, there exists at least one weak solution of problem* (5.1). *Moreover, if f belongs to H_g, all weak solutions belong to $D(A)$. Finally, if*

$$\nu^2\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2 > \frac{c_1}{\lambda_1^{1/2}}\|f\|_*, \tag{5.2}$$

where c_1 is the constant in (2.3), *then the weak solution of problem* (5.1) *is unique.*

Proof. Let us consider an orthonormal basis $\{w_j\} \subset V$ of H_g consisting of eigenfunctions of the Stokes problem in Ω with the homogeneous Dirichlet condition. The subspace of V_g spanned by $w_1,\dots,w_m$ will be denoted V_m. Consider the projector $P_m : H_g \to V_m$ given by

$$P_m u = \sum_{i=1}^m (u,w_i)_g w_i,$$

and define

$$u^m = \sum_{i=1}^m \gamma_{mi} w_i,$$

where

$$\nu((u^m,w_i)) + \nu b\left(\frac{\nabla g}{g}, u^m, w_i\right) + b(u^m,u^m,w_i) = \langle f, w_i\rangle \tag{5.3}$$

for every v in V_m. Equation (5.3) is also equivalent to

$$\nu A u^m + P_m B u^m + \nu P_m C u^m = P_m f. \tag{5.4}$$

The existence of a solution u^m of (5.3) follows from the Brouwer fixed point theorem as in the case of stationary Navier–Stokes equations (for details, see [13, p. 164]). Taking $v = u^m$ in (5.3) and taking into account (2.2), we get

$$\nu\|u^m(t)\|^2 = \langle f, u^m\rangle - \nu b\left(\frac{\nabla g}{g}, u^m, u^m\right) \le \|f\|_*\|u^m\| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u^m\|^2.$$

Hence

$$\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\|u^m\| \le \|f\|_*. \tag{5.5}$$

We extract from $\{u^m\}$ a sequence $\{u^{m'}\}$ which converges weakly in V_g to some limit u. If Ω is bounded, then the injection of V_g into H_g is compact. Thus, this convergence holds also in the norm of H_g:

$$u^{m'} \to u \quad \text{weakly in } V_g, \text{ and strongly in } H_g,$$

up to a subsequence. Passing to the limit in (5.3) with the sequence m', we find that u is a weak solution of (5.1). If Ω is unbounded, the injection of V_g into H_g is no longer compact. However, this difficulty can be overcome by using arguments as in [13, pp. 168–171].

To prove the second statement, we note that if $u \in V_g$, then $Bu \in V_g^{-1/2}$ and $Cu \in V_g^{-1/2}$. Hence $u = (1/\nu)A^{-1}(f - Bu - \nu Cu) \in V_g^{3/2}$ since $f \in H_g$. Therefore, $Bu \in H_g$ and $Cu \in H_g$, and thus u is in $D(A)$.

For the uniqueness of solutions, let us assume that u_1 and u_2 are two solutions of (5.3). Setting $u = u_1 - u_2$, we have

$$\nu\|u\|^2 + \nu b\left(\frac{\nabla g}{g}, u, u\right) = b(u_2, u_2, u) - b(u_1, u_1, u) = -b(u, u_2, u).$$

By Lemma 2.1,

$$\begin{aligned}|b(u, u_2, u)| &\le c_1|u|\,\|u\|\,\|u_2\| \le \frac{c_1}{\lambda_1{}^{1/2}}\|u\|^2\|u_2\| \\ &\le \frac{c_1}{\lambda_1{}^{1/2}}\frac{1}{\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}\|f\|_*\|u\|^2,\end{aligned}$$

where we have used the fact that $\|u\|^2 \ge \lambda_1|u|^2$ and inequality (5.5) for the solution u_2. By Lemma 2.3,

$$\left|b\left(\frac{\nabla g}{g}, u, u\right)\right| \le \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2.$$

Therefore,

$$\left(\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) - \frac{c_1}{\lambda_1^{1/2}}\frac{1}{\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}\|f\|_*\right)\|u\|^2 \le 0,$$

and $u = 0$, i.e. $u_1 = u_2$, if (5.2) holds. ∎

THEOREM 5.2. *Given $f \in H_g$, assume that*

$$\nu\left(\frac{\lambda_1\left(1-\frac{|\nabla g|_\infty^2}{m_0^2\lambda_1}\right)}{c_2'}\right)^{3/4} > \frac{2}{\nu}\left(1+\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}\right)|f| + \frac{c_2^2}{2\nu^5\lambda_1^{3/2}\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^3}|f|^3, \tag{5.6}$$

where c_2 is the constant in Lemma 2.1 and $c_2' = \frac{3}{4}(c_2/\lambda_1^{1/2})^{4/3}$. Then the weak solution of (5.1) (*denoted by u_∞*) *is unique. If u is any weak solution of problem* (1.1) *with $u_0 \in H_g$ arbitrary and $f(t) \equiv f$ for all t, then*

$$u(t) \to u_\infty \quad \text{in } H_g \text{ as } t \to \infty.$$

Proof. Let $w(t) = u(t) - u_\infty$. We have

$$\frac{dw(t)}{dt} + \nu Aw(t) + \nu Cw(t) + Bu(t) - Bu_\infty = 0,$$

and, taking the scalar product with $w(t)$,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \nu b\left(\frac{\nabla g}{g}, w(t), w(t)\right) + b(u(t), u(t), w(t)) - b(u_\infty, u_\infty, w(t)) = 0.$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 \\
&\quad = -b(w(t), u_\infty, w(t)) - \nu b\left(\frac{\nabla g}{g}, w(t), w(t)\right) \\
&\quad \le \frac{c_2}{\lambda_1^{1/2}}|w(t)|^{3/2}\|w(t)\|^{1/2}|Au_\infty| + \nu\frac{|\nabla g|_\infty}{m_0}\|w(t)\|\,|w(t)| \\
&\quad \le \frac{\nu}{4}\|w(t)\|^2 + \frac{c_2'}{\nu^{1/3}}|w(t)|^2|Au_\infty|^{4/3} + \frac{\nu}{4}\|w(t)\|^2 + \nu\frac{|\nabla g|_\infty^2}{m_0^2}|w(t)|^2,
\end{aligned}$$

where we have used (2.3), the inequality $|Au_\infty| \ge \lambda_1^{1/2}\|u_\infty\|$, the Young inequality, and $c_2' = \frac{3}{4}(c_2/\lambda_1^{1/2})^{4/3}$. Therefore,

$$\frac{d}{dt}|w(t)|^2 + \left(\nu\lambda_1 - \frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3} - \nu\frac{|\nabla g|_\infty^2}{m_0^2}\right)|w(t)|^2 \le 0. \tag{5.7}$$

If

$$\bar{\nu} = \nu\lambda_1 - \frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3} - \nu\frac{|\nabla g|_\infty^2}{m_0^2} > 0, \tag{5.8}$$

then (5.7) shows that $|w(t)|$ decays exponentially to 0 as $t \to \infty$:

$$|w(t)|^2 \leq |w(0)|e^{-\bar{\nu}t}, \qquad w(0) = u_0 - u_\infty.$$

Since $u_\infty \in D(A)$, we have

$$\begin{aligned}
\nu|Au_\infty| &\leq |f| + |Bu_\infty| + \nu|Cu_\infty| \\
&\leq |f| + c_2\|u_\infty\|^{3/2}|Au_\infty|^{1/2} + \frac{\nu|\nabla g|_\infty}{m_0}\|u_\infty\| \\
&\leq |f| + \frac{\nu}{2}|Au_\infty| + \frac{c_2^2}{2\nu}\|u_\infty\|^3 + \frac{\nu|\nabla g|_\infty}{m_0}\|u_\infty\| \\
&\leq |f| + \frac{\nu}{2}|Au_\infty| + \frac{c_2^2}{2\nu^4\lambda_1^{3/2}} \frac{1}{\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^3}|f|^3 \\
&\quad + \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} \frac{1}{\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}|f|,
\end{aligned}$$

where we have used (5.5) and the fact that $\|f\|_* \leq \frac{1}{\lambda_1^{1/2}}|f|$. Hence

(5.9)
$$|Au_\infty| \leq \frac{2}{\nu}\left(1 + \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} \frac{1}{\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}\right)|f| + \frac{c_2^2}{2\nu^5\lambda_1^{3/2}} \frac{1}{\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^3}|f|^3.$$

Using (5.9), we obtain a sufficient condition for (5.8), which is exactly (5.6).

If we replace $u(t)$ by another stationary solution u_∞^* of (5.1) in the computations leading to (5.7), we obtain instead of (5.7),

$$\bar{\nu}|u_\infty - u_\infty^*|^2 \leq 0.$$

This implies the uniqueness of the stationary solution if (5.6) holds. ∎

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