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H.X. Phu^a & V.M. Pho^b

^a Institute of Mathematics , 18 Hoang Quoc Viet Road, Hanoi , Vietnam

^b Department of Computer Sciences , Le Qui Don Technical University , 100 Hoang Quoc Viet Road, Hanoi , Vietnam

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Some properties of boundedly perturbed strictly convex quadratic functions

H.X. Phu^{a*} and V.M. Pho^b

^aInstitute of Mathematics, 18 Hoang Quoc Viet Road, Hanoi, Vietnam; ^bDepartment of Computer Sciences, Le Qui Don Technical University, 100 Hoang Quoc Viet Road, Hanoi, Vietnam

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We investigate the problem (\tilde{P}) of minimizing $\tilde{f}(x) := f(x) + p(x)$ subject to $x \in D$, where $f(x) := x^T Ax + b^T x$, A is a symmetric positive definite n -by- n matrix, $b \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is convex and $p: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $\sup_{x \in D} |p(x)| \leq s$ for some given $s < +\infty$. Function p is called a perturbation, but it may also describe some correcting term, which arises when investigating a real inconvenient objective function \tilde{f} by means of an idealized convex quadratic function f . We prove that \tilde{f} is strictly outer Γ -convex for some specified balanced set $\Gamma \subset \mathbb{R}^n$. As a consequence, a Γ -local optimal solution of (\tilde{P}) is global optimal and the difference of two arbitrary global optimal solutions of (\tilde{P}) is contained in Γ . By the property that $x^* - \tilde{x}^* \in \frac{1}{2} \Gamma$ holds if x^* is the optimal solution of the problem of minimizing f on D and \tilde{x}^* is an arbitrary global optimal solution of (\tilde{P}) , we show that the set S_s of global optimal solutions of (\tilde{P}) is stable with respect to the Hausdorff metric $d_H(\cdot, \cdot)$. Moreover, the roughly generalized subdifferentiability of \tilde{f} and a generalization of Kuhn–Tucker theorem for (\tilde{P}) are presented.

Keywords: quadratic programming; bounded perturbation; global minimizer; generalized convexity; stability; subdifferentiability; Kuhn–Tucker theorem

AMS Subject Classifications: 52A01; 47H14; 49K40; 90C20; 90C26; 90C31; 90C46

1. Introduction

Let A be a symmetric positive definite n -by- n matrix, $b \in \mathbb{R}^n$, and

$$f(x) := x^T Ax + b^T x, \quad x \in \mathbb{R}^n.$$

For a given nonempty convex set $D \subset \mathbb{R}^n$, which is not necessarily closed, there arises the convex quadratic program

$$(P) \quad \text{minimize } f(x) \quad \text{subject to } x \in D.$$

*Corresponding author. Email: hxphu@math.ac.vn

Our concern is not the above classical optimization problem, but the following program:

$$(\tilde{P}) \quad \text{minimize } \tilde{f}(x) := f(x) + p(x) \quad \text{subject to } x \in D,$$

where $p: \mathbb{R}^n \rightarrow \mathbb{R}$ is only assumed to be bounded by some given parameter s :

$$\sup_{x \in D} |p(x)| \leq s < +\infty. \quad (1.1)$$

Such a problem (\tilde{P}) may arise when f is some original objective function and p is some perturbation, which comprises additional (deterministic or random) influences to the objective function and errors caused by modelling, measurement, calculation, etc. The particular point is that we restrict ourselves to consider only bounded perturbation satisfying (1.1). An example for bounded perturbation always appears when solving any optimization problem (P) with a computer. Since most real numbers cannot be exactly represented by computers, for most of $x \in D$ the value $f(x) = x^T A x + b^T x$ cannot be exactly computed but only approximated by some floating-point number $\tilde{f}(x)$, and the corresponding function f is neither convex nor quadratic, even not continuous. Then the discontinuous function $p(x) = \tilde{f}(x) - f(x)$ describes the computing errors, and it is reasonable to assume that these computing errors are bounded by some upper bound s , which can be estimated. By using longer floating-point numbers and better algorithms, one can reduce the upper bound s .

Another scenario is that \tilde{f} is the proper objective function and f is just an idealized objective function. It is often the case that functions expressing some practical goals are assumed to be convex or quadratic, or to have some favourable properties which were already intensively investigated or which are more easy to study, although they do not really have these ideal properties. In such a situation, $p(x) = \tilde{f}(x) - f(x)$ is the correcting term, which may be assumed to be bounded (at least on the feasible set D) by some sufficiently small positive number s . An example for this scenario is given by the problem of economic power dispatch, for which it is often assumed that the total cost function is quadratic and strictly convex (see, for instance, [3,5,26,29]). This assumption is a strong idealization, since the real costs may be neither quadratic nor strictly convex. In particular, if the valve-point effect is considered, then the quadratic cost function must be rectified with a correcting term, which is bounded but neither quadratic nor convex, and even not continuously differentiable (see, for instance, [1,27,28]). In this scenario, we will use the idealized convex quadratic function f for investigating the real inconvenient objective function \tilde{f} .

For the sake of shortness, in this article we simply name p as the perturbation, \tilde{f} as the perturbed function and (\tilde{P}) as the perturbed problem, although \tilde{f} may be the proper objective function, p may be the correcting term and (\tilde{P}) may be the proper problem, as explained above.

Since program (P) is both convex and quadratic and since there are already many papers dealing with different stability aspects of perturbed convex or/and quadratic programs, one might ask what is still to do? The first point is that in former investigations perturbations do not change the form or the main characteristics of the original programs. This means that perturbed convex programs remain convex as in [2,8–10,13,22–25,30,31], and perturbed quadratic programs remain quadratic as

in [4,11,12,20]. In this article the perturbed program (\tilde{P}) is neither convex nor quadratic. Moreover, since the perturbation p is only assumed to be bounded by some given positive parameter s , the perturbed objective function \tilde{f} may be nowhere continuous, i.e. it may be quite wild from the analytical point of view. The second point is, beside stability aspects, we also study some other properties of the perturbed program.

In Section 2, we show that, despite of p satisfying (1.1), the strict convexity of f does not disappear completely, but the perturbed function \tilde{f} is still strictly and roughly convex in the following sense. For some given balanced subset $\Gamma \subset \mathbb{R}^n$, $\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *outer Γ -convex* on $D \subset \mathbb{R}^n$ if for all $x_0, x_1 \in D$ satisfying $x_0 - x_1 \notin \Gamma$ there is a closed subset $\Lambda \subset [0, 1]$ containing $\{0, 1\}$ such that

$$[x_0, x_1] \subset \{(1 - \lambda)x_0 + \lambda x_1 \mid \lambda \in \Lambda\} + \frac{1}{2} \Gamma \tag{1.2}$$

and

$$\forall \lambda \in \Lambda \setminus \{0, 1\} : \tilde{f}((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)\tilde{f}(x_0) + \lambda\tilde{f}(x_1). \tag{1.3}$$

(This definition, which was introduced in [18], is a generalization of some kinds of roughly generalized convexity presented in [6,14,15,19].) If

$$\forall \lambda \in \Lambda \setminus \{0, 1\} : \tilde{f}((1 - \lambda)x_0 + \lambda x_1) < (1 - \lambda)\tilde{f}(x_0) + \lambda\tilde{f}(x_1) \tag{1.4}$$

holds instead of (1.3), then \tilde{f} is called *strictly outer Γ -convex*. Theorem 2.2 claims that \tilde{f} is strictly outer Γ -convex for $\Gamma = M(2s)$, where $M(\cdot)$ is defined in (2.4).

Section 3 deals with the consequence of the remaining strict outer Γ -convexity of \tilde{f} . Theorem 3.1 presents a key property of outer Γ -convex functions, namely each Γ -*minimizer* $x^* \in D$ of \tilde{f} , which satisfies

$$\tilde{f}(x^*) = \inf_{x \in (x^* + \Gamma) \cap D} \tilde{f}(x),$$

is a *global minimizer*, i.e.

$$\tilde{f}(x^*) = \inf_{x \in D} \tilde{f}(x).$$

Because of the unruly perturbation p , the existence of Γ -minimizers is hardly warranted. Therefore, we are still interested in Γ -*infimizers*, i.e. in such $x^* \in D$ satisfying

$$\liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) = \inf_{x \in (x^* + \Gamma) \cap D} \tilde{f}(x).$$

Theorem 3.2 says that each Γ -infimizer of \tilde{f} is a *global infimizer*, i.e.

$$\liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x).$$

(Note that in Theorems 3.1 and 3.2 we have to choose $\Gamma = M(2s)$.) Theorems 3.3 and 3.4 assert that, if \tilde{x}_0^* and \tilde{x}_1^* are two arbitrary global minimizers or global infimizers of problem (\tilde{P}) , then $\tilde{x}_0^* - \tilde{x}_1^* \in M(2s)$. This property corresponds to the uniqueness of the minimizer of a strictly convex function.

The stability of the set of optimal solutions of the perturbed program is investigated in Section 4. If x^* is the minimizer of problem (P) and \tilde{x}^* is a global infimizer of problem (\tilde{P}), then Theorem 4.1 claims that $x^* - \tilde{x}^* \in \frac{1}{2}M(2s)$. Consequently, $\|x^* - \tilde{x}^*\| \leq \sqrt{2s/\lambda_{\min}}$, where λ_{\min} is the smallest eigenvalue of matrix A (Corollary 4.2). This property is used to deduce in Theorem 4.4 that $d_H(\{x^*\}, S_s) \leq \sqrt{2s/\lambda_{\min}}$, where $d_H(\cdot, \cdot)$ is the Hausdorff distance and S_s is the set of global infimizers of (\tilde{P}), and to infer the stability of the set of optimal solutions of (\tilde{P}) in Corollary 4.5.

In Section 5, Theorem 5.1 describes the roughly generalized subdifferentiability of the perturbed function \tilde{f} and Theorem 5.2 states a generalization of the Kuhn–Tucker theorem for the problem of minimizing $\tilde{f}(x)$ subject to $x \in D$, where $D = \{x \in C \mid g_1(x) \leq 0, \dots, g_m(x) \leq 0\}$.

Throughout this article, $\|\cdot\|$ denotes the n -dimensional Euclidean norm and the following notions are used:

$$\begin{aligned}x_\lambda &:= (1 - \lambda)x_0 + \lambda x_1, \\ [x_0, x_1] &:= \{x_\lambda \mid 0 \leq \lambda \leq 1\}, \\ \bar{B}(x, r) &:= \{x' \in X \mid \|x' - x\| \leq r\}.\end{aligned}$$

2. Outer Γ -convexity

The aim of this section is to show that an arbitrary perturbation p satisfying (1.1) does not completely destroy the strict convexity of the quadratic function $f(x) = x^T A x + b^T x$, but the perturbed function $\tilde{f} = f + p$ is still strictly outer Γ -convex for some suitable set $\Gamma \subset \mathbb{R}^n$.

In [17] we used the convexity modulus $h_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ of f defined by

$$h_1(\gamma) = \inf_{x_0, x_1 \in \mathbb{R}^n, \|x_0 - x_1\| = \gamma} \left(\frac{1}{2}(f(x_0) + f(x_1)) - f\left(\frac{1}{2}(x_0 + x_1)\right) \right)$$

for characterizing the outer γ -convexity of the perturbed function $\tilde{f} = f + p$. In order to improve our results, in this article we do not define the convexity modulus by taking infimum over the entire space, but dependently on individual directions as follows:

$$h_1(\mu, z) := \inf_{x \in \mathbb{R}^n} \left(\frac{1}{2}(f(x) + f(x + \mu z)) - f\left(x + \frac{\mu}{2}z\right) \right), \quad (2.1)$$

where $\mu \in \mathbb{R}$ and $z \in \mathbb{R}^n$, and use it to define

$$m(\delta, z) := \inf \{ \mu > 0 \mid h_1(\mu, z) > \delta \}. \quad (2.2)$$

Since $h_1(\cdot, z)$ is nondecreasing, there holds

$$h_1(\mu, z) > \delta \quad \text{if } \mu > m(\delta, z). \quad (2.3)$$

Consider the balanced set

$$M(\delta) := \{ \mu z \mid z \in \mathbb{R}^n, |\mu| \leq m(\delta, z) \}, \quad (2.4)$$

which plays a central role in this article. Next, we state some basic properties of $m(\cdot, \cdot)$ and $M(\cdot)$.

PROPOSITION 2.1

(a) For any $\delta > 0$ and $z \in \mathbb{R}^n$, there hold

$$m(\delta, z) = 2\sqrt{\delta(z^T Az)^{-1}} \tag{2.5}$$

and

$$\frac{1}{2}(f(x) + f(x + m(\delta, z)z)) - f\left(x + \frac{m(\delta, z)}{2}z\right) = \delta \quad \text{for all } x \in \mathbb{R}^n. \tag{2.6}$$

(b) For any $\delta > 0$, there holds

$$M(\delta) = \{x \in \mathbb{R}^n \mid x^T Ax \leq 4\delta\}. \tag{2.7}$$

(c) Let λ_{\min} and λ_{\max} be the smallest eigenvalue and the greatest eigenvalue of matrix A . Then

$$\bar{B}(0, 2\sqrt{\delta/\lambda_{\max}}) \subset M(\delta) \subset \bar{B}(0, 2\sqrt{\delta/\lambda_{\min}}), \tag{2.8}$$

and $M(\delta) = \bar{B}(0, 2\sqrt{\delta/\lambda_{\min}})$ if and only if $\lambda_{\max} = \lambda_{\min}$.

Proof (a) For any $x, z \in \mathbb{R}^n$ and $\mu > 0$, there holds

$$\begin{aligned} & \frac{1}{2}(f(x) + f(x + \mu z)) - f\left(x + \frac{\mu}{2}z\right) \\ &= \frac{1}{2}(x^T Ax + b^T x + (x + \mu z)^T A(x + \mu z) + b^T(x + \mu z)) \\ & \quad - \left(x + \frac{\mu}{2}z\right)^T A\left(x + \frac{\mu}{2}z\right) - b^T\left(x + \frac{\mu}{2}z\right) \\ &= \frac{\mu^2}{4} z^T Az. \end{aligned}$$

Therefore, due to (2.1) and (2.2),

$$h_1(\mu, z) = \frac{\mu^2}{4} z^T Az$$

and

$$m(\delta, z) = \inf \left\{ \mu > 0 \mid \frac{\mu^2}{4} z^T Az > \delta \right\} = 2\sqrt{\delta(z^T Az)^{-1}}.$$

Moreover, for any $x \in \mathbb{R}^n$, we have

$$\frac{1}{2}(f(x) + f(x + m(\delta, z)z)) - f\left(x + \frac{m(\delta, z)}{2}z\right) = \frac{1}{4}(2\sqrt{\delta(z^T Az)^{-1}})^2 z^T Az = \delta.$$

(b) For $x = \mu z$, (2.5) yields

$$m(\delta, z) = 2\sqrt{\delta(z^T Az)^{-1}} = 2|\mu| \sqrt{\delta(x^T Ax)^{-1}}.$$

Hence, by definition,

$$(x \in M(\delta)) \Leftrightarrow (x = \mu z, |\mu| \leq m(\delta, z) = 2|\mu| \sqrt{\delta(x^T A x)^{-1}}) \Leftrightarrow (x^T A x \leq 4\delta).$$

(c) By the spectral theorem, there is an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of \mathbb{R}^n consisting of unit eigenvectors of A , i.e.

$$\begin{aligned} A e_i &= \lambda_i e_i, \quad \|e_i\| = 1 \quad \text{for } i = 1, 2, \dots, n, \\ e_i^T e_j &= 0 \quad \text{for } i \neq j, \end{aligned}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding real eigenvalues. Then any $x \in \mathbb{R}^n$ can be represented by $x = \sum_{i=1}^n \mu_i e_i$ and there hold

$$\|x\|^2 = x^T x = \sum_{i,j=1}^n \mu_i \mu_j e_i^T e_j = \sum_{i=1}^n \mu_i^2$$

and

$$x^T A x = \sum_{i,j=1}^n \mu_i \mu_j e_i^T A e_j = \sum_{i,j=1}^n \lambda_j \mu_i \mu_j e_i^T e_j = \sum_{i=1}^n (\lambda_i \mu_i^2).$$

Thus we have the following:

$$\lambda_{\min} \|x\|^2 = \lambda_{\min} \sum_{i=1}^n \mu_i^2 \leq x^T A x = \sum_{i=1}^n (\lambda_i \mu_i^2) \leq \lambda_{\max} \sum_{i=1}^n \mu_i^2 = \lambda_{\max} \|x\|^2.$$

Therefore, (2.7) yields

$$\begin{aligned} M(\delta) &= \{x \in \mathbb{R}^n \mid x^T A x \leq 4\delta\} \\ &\subset \{x \in \mathbb{R}^n \mid \lambda_{\min} \|x\|^2 \leq 4\delta\} \\ &= \{x \in \mathbb{R}^n \mid \|x\| \leq 2\sqrt{\delta/\lambda_{\min}}\} \\ &= \bar{B}(0, 2\sqrt{\delta/\lambda_{\min}}) \end{aligned}$$

and

$$\begin{aligned} M(\delta) &\supset \{x \in \mathbb{R}^n \mid \lambda_{\max} \|x\|^2 \leq 4\delta\} \\ &= \{x \in \mathbb{R}^n \mid \|x\| \leq 2\sqrt{\delta/\lambda_{\max}}\} \\ &= \bar{B}(0, 2\sqrt{\delta/\lambda_{\max}}). \end{aligned}$$

In particular, $M(\delta) = \bar{B}(0, 2\sqrt{\delta/\lambda_{\min}})$ if and only if

$$\lambda_{\min} \sum_{i=1}^n \mu_i^2 = \sum_{i=1}^n (\lambda_i \mu_i^2) \quad \text{when} \quad \sum_{i=1}^n \mu_i^2 \leq 4\delta/\lambda_{\min},$$

which holds if and only if $\lambda_{\max} = \lambda_{\min}$. ■

Since $M(\delta) = \{x \in \mathbb{R}^n \mid x^T A x \leq 4\delta\}$ and A is positive definite, $M(\delta)$ is convex, closed and balanced. Moreover, if $\delta > 0$ then $0 \in \mathbb{R}^n$ is an interior point of $M(\delta)$.

We now use $M(\cdot)$ for characterizing the strict outer Γ -convexity of the perturbed function \tilde{f} .

THEOREM 2.2 *Suppose $0 < \sup_{x \in D} |p(x)| \leq s < +\infty$. Then $\tilde{f} = f + p$ is strictly outer Γ -convex on D for $\Gamma = M(2s)$.*

Proof Consider arbitrary $x_0, x_1 \in D$ with $x_0 - x_1 \notin \Gamma = M(2s)$. Let $\bar{\mu} > 0$ be defined by

$$\frac{1}{\bar{\mu}}(x_0 - x_1) \in \partial M(2s), \tag{2.9}$$

where $\partial M(2s)$ denotes the boundary of the set $M(2s)$, and

$$\Lambda = \{0, 1\} \cup \left[\frac{1}{2\bar{\mu}}, 1 - \frac{1}{2\bar{\mu}} \right]. \tag{2.10}$$

Obviously, $x_0 - x_1 \notin \Gamma = M(2s)$ yields $\bar{\mu} > 1$. For $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$, $\lambda = \frac{1}{2\bar{\mu}}$ and $\lambda = 1 - \frac{1}{2\bar{\mu}}$, we have

$$\begin{aligned} x_{\frac{1}{2\bar{\mu}}} - x_0 &= \left(1 - \frac{1}{2\bar{\mu}}\right)x_0 + \frac{1}{2\bar{\mu}}x_1 - x_0 = \frac{1}{2\bar{\mu}}(x_1 - x_0), \\ x_{1-\frac{1}{2\bar{\mu}}} - x_1 &= \frac{1}{2\bar{\mu}}x_0 + \left(1 - \frac{1}{2\bar{\mu}}\right)x_1 - x_1 = \frac{1}{2\bar{\mu}}(x_0 - x_1), \end{aligned} \tag{2.11}$$

i.e. $x_{\frac{1}{2\bar{\mu}}} - x_0 \in \frac{1}{2}M(2s)$ and $x_{1-\frac{1}{2\bar{\mu}}} - x_1 \in \frac{1}{2}M(2s)$, which imply

$$\begin{aligned} [x_0, x_{\frac{1}{2\bar{\mu}}}] &\subset \{x_0, x_{\frac{1}{2\bar{\mu}}}\} + \frac{1}{2}M(2s) \subset \{x_0, x_{\frac{1}{2\bar{\mu}}}\} + \frac{1}{2}\Gamma, \\ [x_{1-\frac{1}{2\bar{\mu}}}, x_1] &\subset \{x_{1-\frac{1}{2\bar{\mu}}}, x_1\} + \frac{1}{2}M(2s) \subset \{x_{1-\frac{1}{2\bar{\mu}}}, x_1\} + \frac{1}{2}\Gamma. \end{aligned}$$

Therefore, by (2.10),

$$[x_0, x_1] = [x_0, x_{\frac{1}{2\bar{\mu}}}] \cup [x_{\frac{1}{2\bar{\mu}}}, x_{1-\frac{1}{2\bar{\mu}}}] \cup [x_{1-\frac{1}{2\bar{\mu}}}, x_1] \subset \{x_\lambda | \lambda \in \Lambda\} + \frac{1}{2}\Gamma,$$

i.e. (1.2) is fulfilled.

Now consider an arbitrary fixed $\lambda \in [\frac{1}{2\bar{\mu}}, 1 - \frac{1}{2\bar{\mu}}]$ and denote

$$\lambda' = \lambda - \frac{1}{2\bar{\mu}}, \quad \lambda'' = \lambda + \frac{1}{2\bar{\mu}}.$$

Since $\frac{1}{2\bar{\mu}} < \frac{1}{2} < 1 - \frac{1}{2\bar{\mu}}$, there holds $\lambda' > 0$ or $\lambda'' < 1$. Therefore, it follows from the strict convexity of f that

$$\begin{aligned} f(x_{\lambda'}) &\leq (1 - \lambda')f(x_0) + \lambda'f(x_1), \\ f(x_{\lambda''}) &\leq (1 - \lambda'')f(x_0) + \lambda''f(x_1), \end{aligned}$$

where ' $<$ ' holds at least in one inequality. Hence,

$$\begin{aligned} f(x_{\lambda'}) + f(x_{\lambda''}) &< (2 - \lambda' - \lambda'')f(x_0) + (\lambda' + \lambda'')f(x_1) \\ &= 2(1 - \lambda)f(x_0) + 2\lambda f(x_1). \end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned} x_{\lambda''} - x_{\lambda'} &= (1 - \lambda'')x_0 + \lambda''x_1 - (1 - \lambda')x_0 - \lambda'x_1 \\ &= -\frac{1}{\bar{\mu}}x_0 + \frac{1}{\bar{\mu}}x_1 \\ &= \frac{1}{\bar{\mu}}(x_1 - x_0), \end{aligned}$$

i.e. by (2.9), $x_{\lambda''} - x_{\lambda'} \in \partial M(2s)$. Thus, (2.4) and (2.6) imply $m(2s, x_1 - x_0) = 1/\bar{\mu}$, $x_{\lambda'} + m(2s, x_1 - x_0)(x_1 - x_0) = x_{\lambda''}$ and

$$\frac{1}{2}(f(x_{\lambda'}) + f(x_{\lambda''})) - f\left(\frac{1}{2}(x_{\lambda'} + x_{\lambda''})\right) = 2s.$$

Since

$$\begin{aligned} \frac{1}{2}(x_{\lambda''} + x_{\lambda'}) &= \frac{1}{2}((1 - \lambda'')x_0 + \lambda''x_1 + (1 - \lambda')x_0 + \lambda'x_1) \\ &= \frac{1}{2}((2 - \lambda' - \lambda'')x_0 + (\lambda' + \lambda'')x_1) \\ &= (1 - \lambda)x_0 + \lambda x_1 \\ &= x_{\lambda}, \end{aligned}$$

we have

$$\frac{1}{2}(f(x_{\lambda'}) + f(x_{\lambda''})) - f(x_{\lambda}) = 2s.$$

Combining this with (2.12) yields

$$(1 - \lambda)f(x_0) + \lambda f(x_1) - f(x_{\lambda}) > 2s,$$

and following, by (1.1),

$$\begin{aligned} (1 - \lambda)\tilde{f}(x_0) + \lambda\tilde{f}(x_1) - \tilde{f}(x_{\lambda}) &\geq (1 - \lambda)(f(x_0) - s) + \lambda(f(x_1) - s) - (f(x_{\lambda}) + s) \\ &= (1 - \lambda)f(x_0) + \lambda f(x_1) - f(x_{\lambda}) - 2s \\ &> 0, \end{aligned}$$

i.e. (1.4) holds true. Hence, \tilde{f} is strictly outer Γ -convex on D for $\Gamma = M(2s)$. \blacksquare

Due to (2.8), we have

$$M(2s) \subset \bar{B}(0, 2\sqrt{2s/\lambda_{\min}}). \quad (2.13)$$

Therefore, it follows from Theorem 2.2 that \tilde{f} is strictly outer Γ -convex on D for $\Gamma = \bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$ (see Proposition 2.1 in [18]). That means, for $\gamma = 2\sqrt{2s/\lambda_{\min}}$, \tilde{f} is outer γ -convex in the sense of [19] and strictly and roughly γ -convex in the sense of [16].

Since $\Gamma = \bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$ is a ball in the Euclidean space \mathbb{R}^n , it is, in general, simpler to determine and to describe than $\Gamma = M(2s)$. Let e_{\min} be a unit eigenvector of A corresponding to the minimal eigenvalue λ_{\min} . If D is large enough to contain a pair of points x_0 and x_1 satisfying $x_0 - x_1 = 2\sqrt{2s/\lambda_{\min}} e_{\min}$, then $\Gamma = \bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$

is the smallest ball for which $\tilde{f} = f + p$ is strictly outer Γ -convex for any perturbation p satisfying $0 < \sup_{x \in D} |p(x)| \leq s < +\infty$. Indeed, if $\gamma < 2\sqrt{2s/\lambda_{\min}}$ and $\Gamma = \bar{B}(0, \gamma)$, then $x_0 - x_1 \notin \Gamma$ and, by choosing a perturbation p satisfying

$$p((1 - \lambda)x_0 + \lambda x_1) = \begin{cases} -s & \text{for } \lambda \in \{0, 1\}, \\ s & \text{for } \lambda \in]0, 1[, \end{cases}$$

it is easy to verify for all $\lambda \in]0, 1[$ that

$$\begin{aligned} \tilde{f}((1 - \lambda)x_0 + \lambda x_1) &= f((1 - \lambda)x_0 + \lambda x_1) + s \\ &> (1 - \lambda)f(x_0) + \lambda f(x_1) - 2s + s \\ &= (1 - \lambda)\tilde{f}(x_0) + \lambda\tilde{f}(x_1), \end{aligned}$$

i.e. \tilde{f} cannot be outer Γ -convex for $\Gamma = \bar{B}(0, \gamma)$.

By Proposition 2.1, if $\lambda_{\min} < \lambda_{\max}$ then $M(2s)$ is properly contained in the ball $\bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$. Hence, in general, the strict outer Γ -convexity with respect to $\Gamma = M(2s)$ is stronger than the one with respect to $\Gamma = \bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$.

A basic property of convex functions is that all lower level sets are convex. Outer Γ -convex functions also possess a similar property, namely all lower level sets are outer Γ -convex. As in [18], a set S is said to be *outer Γ -convex* if for all $x_0, x_1 \in S$ there exists a closed subset $\Lambda \subset [0, 1]$ containing $\{0, 1\}$ such that $\{x_\lambda \mid \lambda \in \Lambda\} \subset S$ and $[x_0, x_1] \subset \{x_\lambda \mid \lambda \in \Lambda\} + \frac{1}{2}\Gamma$. For the perturbed function $\tilde{f} = f + p$, we have the following property.

PROPOSITION 2.3 *Suppose $0 < \sup_{x \in D} |p(x)| \leq s < +\infty$. Then each lower level set $\{x \in D \mid \tilde{f}(x) \leq \alpha\}$ of \tilde{f} is outer Γ -convex for $\Gamma = M(2s)$.*

Proof Due to Proposition 3.3 in [18], if \tilde{f} is outer Γ -convex then each lower level set of \tilde{f} is outer Γ -convex. Combining this fact with Theorem 2.2, we get the desired conclusion. ■

3. Global minimal solutions

In this section, we investigate some typical properties of global optimal solutions of the perturbed function $\tilde{f} = f + p$, which is related to the strict outer Γ -convexity of \tilde{f} .

Recall that A is a symmetric positive definite n -by- n matrix, $b \in \mathbb{R}^n$, $f(x) = x^T A x + b^T x$, and $D \subset \mathbb{R}^n$ is convex. Instead of the optimization problem

$$(P) \quad \text{minimize } f(x) \quad \text{subject to } x \in D,$$

we deal with the perturbed problem

$$(\tilde{P}) \quad \text{minimize } \tilde{f}(x) = f(x) + p(x) \quad \text{subject to } x \in D,$$

where $p: \mathbb{R}^n \rightarrow \mathbb{R}$ is only assumed to satisfy

$$0 < \sup_{x \in D} |p(x)| \leq s < +\infty. \tag{3.1}$$

A typical property of the convex program (P) is that a local minimum is global minimum. Because of the unruly perturbation p , problem (\tilde{P}) cannot have the

above-mentioned property. However, since \tilde{f} is outer Γ -convex as proved in the preceding section, (\tilde{P}) still possesses the following similar property.

THEOREM 3.1 *Suppose $\Gamma = M(2s)$ and $x^* \in D$ is a Γ -minimizer of \tilde{f} , i.e.*

$$\tilde{f}(x^*) = \inf_{x \in (x^* + \Gamma) \cap D} \tilde{f}(x). \quad (3.2)$$

Then x^ is a global minimizer of \tilde{f} , i.e.*

$$\tilde{f}(x^*) = \inf_{x \in D} \tilde{f}(x). \quad (3.3)$$

Proof For $\Gamma = M(2s)$ and $s > 0$, Proposition 2.1 yields that the origin $0 \in \mathbb{R}^n$ is an interior point of Γ . By Theorem 2.2, \tilde{f} is outer Γ -convex. Therefore, Theorem 3.6 in [18] implies that (3.3) follows from (3.2). \blacksquare

Since the considered perturbation p is not assumed to be lower semicontinuous, the perturbed problem (\tilde{P}) may not have minimizers. Therefore, it is more realistic to consider infimizers instead of minimizers, as done in the following theorem.

THEOREM 3.2 *Suppose $\Gamma = M(2s)$ and $x^* \in D$ is a Γ -infimizer of \tilde{f} , i.e.*

$$\liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) = \inf_{x \in (x^* + \Gamma) \cap D} \tilde{f}(x). \quad (3.4)$$

Then x^ is a global infimizer of \tilde{f} , i.e.*

$$\liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x). \quad (3.5)$$

Proof Assume the contrary that (3.5) is not true, i.e. there exists an $x_0 \in D \setminus (x^* + \Gamma)$ with

$$\sigma := \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) - \tilde{f}(x_0) > 0. \quad (3.6)$$

Denote

$$\begin{aligned} \beta &:= \|x_0 - x^*\| - \sqrt{2s/\lambda_{\max}}, \\ \zeta &:= -\sigma\sqrt{2s/\lambda_{\max}}, \\ \varepsilon &:= \frac{1}{4}(-\beta + \sqrt{\beta^2 - 4\zeta}), \end{aligned} \quad (3.7)$$

where λ_{\max} is the greatest eigenvalue of matrix A . Due to Proposition 2.1, $\tilde{B}(0, 2\sqrt{2s/\lambda_{\max}})$ is contained in $M(2s)$. Therefore, $x_0 - x^* \notin \Gamma = M(2s)$ implies that $\|x_0 - x^*\| > 2\sqrt{2s/\lambda_{\max}}$ and thus $\beta > 0$. Since $\varepsilon > 0$ and $x_0 - x^* \notin M(2s)$, we can choose $x_1 \in D \cap (x^* + \frac{1}{2}M(2s))$ such that

$$\|x_1 - x^*\| < \varepsilon, \quad \tilde{f}(x_1) < \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) + \varepsilon \quad (3.8)$$

and

$$x_0 - x_1 \notin \Gamma, \quad x_1 + \frac{1}{2}M(2s) \subset x^* + M(2s) = x^* + \Gamma. \quad (3.9)$$

Our aim is to show that there exists a $\lambda \in [0, 1]$ satisfying

$$x_\lambda = (1 - \lambda)x_0 + \lambda x_1 \in (x^* + \Gamma) \cap D, \quad \tilde{f}(x_\lambda) < \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x), \quad (3.10)$$

in contradiction to (3.4). To this goal, first observe that

$$\frac{1}{2}(-\beta - \sqrt{\beta^2 - 4\zeta}) < 0 < \varepsilon < \frac{1}{2}(-\beta + \sqrt{\beta^2 - 4\zeta}),$$

i.e. the ε defined in (3.7) lies between the two real roots of the quadratic equation $\varepsilon^2 + \beta\varepsilon + \zeta = 0$. Therefore,

$$\varepsilon^2 + \beta\varepsilon + \zeta = \varepsilon^2 + (\|x_0 - x^*\| - \sqrt{2s/\lambda_{\max}})\varepsilon - \sigma\sqrt{2s/\lambda_{\max}} < 0,$$

which implies

$$\sigma > \left(\frac{\|x_0 - x^*\| + \varepsilon}{\sqrt{2s/\lambda_{\max}}} - 1 \right) \varepsilon. \quad (3.11)$$

Take again $\bar{\mu} > 1$ defined by (2.9), i.e. $\frac{1}{\bar{\mu}}(x_0 - x_1) \in \partial M(2s)$. The inclusion

$$\bar{B}(0, 2\sqrt{2s/\lambda_{\max}}) \subset M(2s)$$

yields

$$\frac{1}{\bar{\mu}}\|x_0 - x_1\| \geq 2\sqrt{2s/\lambda_{\max}}.$$

Combining with

$$\|x_0 - x_1\| \leq \|x_0 - x^*\| + \|x^* - x_1\| < \|x_0 - x^*\| + \varepsilon,$$

we get

$$2\bar{\mu} \leq \frac{\|x_0 - x_1\|}{\sqrt{2s/\lambda_{\max}}} < \frac{\|x_0 - x^*\| + \varepsilon}{\sqrt{2s/\lambda_{\max}}}.$$

This along with (3.11) yield $\sigma > (2\bar{\mu} - 1)\varepsilon$, and the following:

$$\frac{1}{2\bar{\mu}}(-\sigma) + \left(1 - \frac{1}{2\bar{\mu}}\right)\varepsilon < 0. \quad (3.12)$$

In the proof of Theorem 2.2, we have already proved for $x_0 - x_1 \notin \Gamma$, which is warranted by (3.9), and $\lambda = 1 - \frac{1}{2\bar{\mu}}$ that

$$\tilde{f}(x_{1-\frac{1}{2\bar{\mu}}}) < \frac{1}{2\bar{\mu}}\tilde{f}(x_0) + \left(1 - \frac{1}{2\bar{\mu}}\right)\tilde{f}(x_1).$$

Hence, by (3.6), (3.8) and (3.12), there holds

$$\begin{aligned} \tilde{f}(x_{1-\frac{1}{2\bar{\mu}}}) &< \frac{1}{2\bar{\mu}}\tilde{f}(x_0) + \left(1 - \frac{1}{2\bar{\mu}}\right)\left(\liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) + \varepsilon\right) \\ &= \frac{1}{2\bar{\mu}}\left(\tilde{f}(x_0) - \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x)\right) + \left(1 - \frac{1}{2\bar{\mu}}\right)\varepsilon + \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) \\ &= \frac{1}{2\bar{\mu}}(-\sigma) + \left(1 - \frac{1}{2\bar{\mu}}\right)\varepsilon + \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x) \\ &< \liminf_{x \in D, x \rightarrow x^*} \tilde{f}(x). \end{aligned} \quad (3.13)$$

On the other hand, (2.9) and (2.11) imply that

$$x_{1-\frac{1}{2\mu}} - x_1 = \frac{1}{2\mu} (x_0 - x_1) \in \frac{1}{2} M(2s),$$

i.e. by (3.9),

$$x_{1-\frac{1}{2\mu}} \in x_1 + \frac{1}{2} M(2s) \subset x^* + \Gamma. \quad (3.14)$$

Properties (3.13) and (3.14) mean that we get the desired contradiction stated in (3.10) for $\lambda = 1 - \frac{1}{2\mu}$. Thus, (3.5) must be true. ■

Note that the proof of Theorem 3.2 may be shorter by applying Theorem 3.1, whose proof in turn uses Theorem 2.2. It is our intention to present a longer proof but without using Theorem 2.2.

It follows from Theorems 3.1 and 3.2 and (2.13) that if $x^* \in D$ is a Γ -minimizer (or Γ -infimizer) of \tilde{f} for $\Gamma = \bar{B}(0, 2\sqrt{2s/\lambda_{\min}})$, then x^* is a global minimizer (or global infimizer, respectively).

A typical property of strictly convex functions is that their minimizer is unique. In [16] we have already dealt with a similar property of strictly and roughly convex-like functions. Next, we present a similar typical property of strictly Γ -convex functions.

THEOREM 3.3 *Let \tilde{x}_0^* and \tilde{x}_1^* be two arbitrary global minimizers of problem (\tilde{P}) . Then $\tilde{x}_0^* - \tilde{x}_1^* \in M(2s)$.*

Proof Assume the contrary that $\tilde{x}_0^* - \tilde{x}_1^* \notin M(2s)$. By Theorem 2.2, \tilde{f} is strictly outer Γ -convex for $\Gamma = M(2s)$. By definition, there exists a closed subset $\Lambda \subset]0, 1[$ containing $\{0, 1\}$ and satisfying (1.2) and (1.4), i.e.

$$[\tilde{x}_0^*, \tilde{x}_1^*] \subset \{(1-\lambda)\tilde{x}_0^* + \lambda\tilde{x}_1^* \mid \lambda \in \Lambda\} + \frac{1}{2} \Gamma \quad (3.15)$$

and

$$\forall \lambda \in \Lambda \setminus \{0, 1\} : \tilde{f}((1-\lambda)\tilde{x}_0^* + \lambda\tilde{x}_1^*) < (1-\lambda)\tilde{f}(\tilde{x}_0^*) + \lambda\tilde{f}(\tilde{x}_1^*) = \inf_{x \in D} \tilde{f}(x). \quad (3.16)$$

The property $\tilde{x}_0^* - \tilde{x}_1^* \notin M(2s) = \Gamma$ and (3.15) imply that $\Lambda \setminus \{0, 1\} \neq \emptyset$. Therefore, by (3.16), there exists a $\lambda \in]0, 1[$ such that $\tilde{f}((1-\lambda)\tilde{x}_0^* + \lambda\tilde{x}_1^*) < \inf_{x \in D} \tilde{f}(x)$, a contradiction. Hence, $\tilde{x}_0^* - \tilde{x}_1^* \in M(2s)$. ■

Since A is positive definite, (2.7) yields that while s tends to $0 \in \mathbb{R}$ the set $M(2s)$ shrinks to $\{0\} \subset \mathbb{R}^n$, therefore Theorem 3.3 implies that the diameter of the set of global minimizers of $\tilde{f} = f + p$ converges to zero, too.

What happens if global minimizers are replaced by global infimizers to be more realistic, as explained before Theorem 3.2? In general, the distance between the global infimizers of a strictly outer Γ -convex function may be unbounded. For instance, the function $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) = [x] - x, \quad x \in \mathbb{R},$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$, is strictly outer Γ -convex on $D = \mathbb{R}$ for $\Gamma = [-1, 1]$, and each integer number is a global infimizer of g . Hence, $\tilde{x}_0^* - \tilde{x}_1^* \in \Gamma$ does not hold true for any pair of global infimizers \tilde{x}_0^* and \tilde{x}_1^* of g .

But for the particular function $\tilde{f}(x) = x^T Ax + b^T x + p(x)$, where A is positive definite and p satisfies (3.1), we get the following result for global infimizers, which is similar to Theorem 3.3.

THEOREM 3.4 *Let \tilde{x}_0^* and \tilde{x}_1^* be two arbitrary global infimizers of problem (\tilde{P}) . Then $\tilde{x}_0^* - \tilde{x}_1^* \in M(2s)$.*

Proof (a) Assume that x^* is the minimizer of problem (P) . By Theorem 4.1 (which will be proved in the next section), $\tilde{x}_0^* - x^* \in \frac{1}{2} M(2s)$ and $x^* - \tilde{x}_1^* \in \frac{1}{2} M(2s)$. Since $M(2s)$ is convex, it follows that

$$\tilde{x}_0^* - \tilde{x}_1^* = (\tilde{x}_0^* - x^*) + (x^* - \tilde{x}_1^*) \in M(2s).$$

(b) Assume that problem (P) has no minimizer. Since $f(x) = x^T Ax + b^T x$ and A is positive definite, it is only possible if D is not closed. In this case, we replace the set D by its closure $\text{cl } D$ and the substitute problem of minimizing f on $\text{cl } D$ has now exactly one optimal solution, say x^* . In addition, we change the function p outside of D by

$$p(x) = s \quad \text{for } x \notin D,$$

which does not violate the only characteristic property of p , namely

$$0 < \sup_{x \in D} |p(x)| \leq \sup_{x \in \text{cl } D} |p(x)| = \sup_{x \in \mathbb{R}^n} |p(x)| \leq s < +\infty.$$

Obviously,

$$\begin{aligned} \inf_{x \in \text{cl } D \setminus D} \tilde{f}(x) &= \inf_{x \in \text{cl } D \setminus D} (f(x) + p(x)) \\ &= s + \inf_{x \in \text{cl } D \setminus D} f(x) \\ &\geq s + \inf_{x \in \text{cl } D} f(x) \end{aligned}$$

and

$$\begin{aligned} s + \inf_{x \in D} f(x) &= \inf_{x \in D} (f(x) + s) \\ &\geq \inf_{x \in D} (f(x) + p(x)) \\ &= \inf_{x \in D} \tilde{f}(x). \end{aligned}$$

Since $\inf_{x \in \text{cl } D} f(x) = \inf_{x \in D} f(x)$ holds because of the continuity of f , it follows that

$$\inf_{x \in \text{cl } D \setminus D} \tilde{f}(x) \geq \inf_{x \in D} \tilde{f}(x). \tag{3.17}$$

This along with

$$\liminf_{x \in D, x \rightarrow \tilde{x}_0^*} \tilde{f}(x) = \liminf_{x \in D, x \rightarrow \tilde{x}_1^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x),$$

yield

$$\liminf_{x \in \text{cl } D, x \rightarrow \tilde{x}_0^*} \tilde{f}(x) = \liminf_{x \in \text{cl } D, x \rightarrow \tilde{x}_1^*} \tilde{f}(x) = \inf_{x \in \text{cl } D} \tilde{f}(x),$$

i.e. \tilde{x}_0^* and \tilde{x}_1^* are also global infimizers of the problem of minimizing \tilde{f} on $\text{cl } D$. Hence, due to part (a), $\tilde{x}_0^* - \tilde{x}_1^* \in M(2s)$. ■

Note that the assertion of Theorems 3.3 and 3.4 cannot be further improved. To illustrate this fact, just consider the example with

$$f(x) = x^T A x \quad \text{for } x \in D = \mathbb{R}^n, \quad (3.18)$$

i.e. $b = 0 \in \mathbb{R}^n$, and

$$p(x) = \begin{cases} s - f(x) & \text{for } x \in \frac{1}{2}M(2s), \\ 0 & \text{otherwise.} \end{cases} \quad (3.19)$$

Then (2.7) yields

$$f(x) \begin{cases} \in [0, 2s] & \text{for } x \in \frac{1}{2}M(2s) \\ > 2s & \text{otherwise.} \end{cases}$$

Therefore, $|p(x)| \leq s$ for all $x \in \mathbb{R}^n$ and

$$\tilde{f}(x) = \begin{cases} s & \text{for } x \in \frac{1}{2}M(2s) \\ f(x) > 2s & \text{otherwise.} \end{cases} \quad (3.20)$$

Hence, each global infimizer of (\tilde{P}) is a global minimizer and

$$\frac{1}{2}M(2s) = \{\tilde{x}^* \mid \tilde{x}^* \text{ is a global infimizer of } (\tilde{P})\} \quad (3.21)$$

and

$$M(2s) = \{\tilde{x}_0^* - \tilde{x}_1^* \mid \tilde{x}_0^*, \tilde{x}_1^* \text{ are global infimizers of } (\tilde{P})\},$$

i.e. there exists no other set which is smaller than $M(2s)$ and contains the difference $\tilde{x}_0^* - \tilde{x}_1^*$ of any pair of global infimizers of (\tilde{P}) .

Inclusion (2.13) and Theorem 3.4 immediately imply the following corollary.

COROLLARY 3.5 *Let \tilde{x}_0^* and \tilde{x}_1^* be two arbitrary global infimizers of problem (\tilde{P}) . Then $\|\tilde{x}_0^* - \tilde{x}_1^*\| \leq 2\sqrt{2s/\lambda_{\min}}$.*

4. Stability

In this section we investigate the relation between the global optimal solutions of the original problem (P) and of the perturbed problem (\tilde{P}) ; in particular, the stability of the set of global optimal solutions of (\tilde{P}) with respect to the Hausdorff metric. Although the following results are only formulated for global infimizers of (\tilde{P}) , they are also valid for its global minimizers because each global minimizer is a global infimizer.

Next, we estimate the distance between the minimizer of problem (P) and any global infimizer of problem (\tilde{P}) . Since a local minimizer of the strictly convex function f on the convex set D is global and it is unique, we simply call the minimizer of (P) if it exists.

THEOREM 4.1 *Let x^* be the minimizer of problem (P) and \tilde{x}^* be any global infimizer of problem (\tilde{P}) . Then $\tilde{x}^* - x^* \in \frac{1}{2}M(2s)$.*

Proof Assume the contrary that $\tilde{x}^* - x^* \notin \frac{1}{2}M(2s)$. Then there exist an $s' > s$ and a neighbourhood $U(\tilde{x}^*)$ of \tilde{x}^* such that

$$\left(x^* + \frac{1}{2}M(2s')\right) \cap U(\tilde{x}^*) = \emptyset. \quad (4.1)$$

Consider an arbitrary $\tilde{x} \in U(\tilde{x}^*) \cap D$ and denote $\tilde{z} = \tilde{x} - x^*$. Since $f(x^*) \leq f(x)$ for all $x \in D$ while both x^* and \tilde{x} are contained in the convex set D , there holds

$$\begin{aligned} 0 &\leq \left. \frac{d}{dt} f(x^* + t\tilde{z}) \right|_{t=0} \\ &= \left. \frac{d}{dt} (\tilde{z}^T A \tilde{z} t^2 + (2Ax^* + b)^T \tilde{z} t + x^{*T} Ax^* + b^T x^*) \right|_{t=0} \\ &= (2Ax^* + b)^T \tilde{z}. \end{aligned}$$

In consequence, we have

$$\begin{aligned} f(x^* + \tilde{z}) + f(x^* - \tilde{z}) &= (\tilde{z}^T A \tilde{z} + (2Ax^* + b)^T \tilde{z} + x^{*T} Ax^* + b^T x^*) \\ &\quad + (\tilde{z}^T A \tilde{z} - (2Ax^* + b)^T \tilde{z} + x^{*T} Ax^* + b^T x^*) \\ &= 2(\tilde{z}^T A \tilde{z} + x^{*T} Ax^* + b^T x^*) \\ &= 2(f(x^* + \tilde{z}) - (2Ax^* + b)^T \tilde{z}) \\ &\leq 2f(\tilde{x}), \end{aligned}$$

which implies that

$$f(\tilde{x}) - f(x^*) \geq \frac{1}{2} (f(x^* + \tilde{z}) + f(x^* - \tilde{z})) - f(x^*).$$

On the other hand, $\tilde{z} = \tilde{x} - x^* \notin \frac{1}{2}M(2s')$ because $\tilde{x} \notin (x^* + \frac{1}{2}M(2s'))$ follows from (4.1) and $\tilde{x} \in U(\tilde{x}^*) \cap D$. Therefore, it follows from (2.1) and (2.3) and (2.4) that

$$\frac{1}{2} (f(x^* + \tilde{z}) + f(x^* - \tilde{z})) - f(x^*) > 2s'.$$

As a result, we obtain $f(\tilde{x}) - f(x^*) > 2s'$, which yields immediately

$$\tilde{f}(\tilde{x}) - \tilde{f}(x^*) = f(\tilde{x}) - f(x^*) + p(\tilde{x}) - p(x^*) > 2s' - 2s = 2(s' - s) > 0. \quad (4.2)$$

This relation is valid, as chosen above, for any $\tilde{x} \in U(\tilde{x}^*) \cap D$. Thus,

$$\liminf_{x \in D, x \rightarrow \tilde{x}^*} \tilde{f}(x) - \tilde{f}(x^*) \geq 2(s' - s) > 0,$$

i.e. \tilde{x}^* cannot be a global infimizer of problem (\tilde{P}) , a contradiction of the assumption. Hence, $\tilde{x}^* - x^* \in \frac{1}{2}M(2s)$ must be true. ■

The assertion of Theorem 4.1 cannot be further improved. This fact can be demonstrated by example (3.18)–(3.20) again, for which $x^* = 0 \in \mathbb{R}^n$ is the unique minimizer of (P) and (3.21) yields

$$\begin{aligned} \frac{1}{2} M(2s) &= \{\tilde{x}^* \mid \tilde{x}^* \text{ is a global infimizer of } (\tilde{P})\} \\ &= \{\tilde{x}^* - x^* \mid \tilde{x}^* \text{ is a global infimizer of } (\tilde{P})\}, \end{aligned}$$

i.e. there exists no other set which is smaller than $\frac{1}{2}M(2s)$ and contains all such differences $\tilde{x}^* - x^*$.

Inclusion (2.13) and Theorem 4.1 immediately imply the following corollary.

COROLLARY 4.2 *Let x^* be the minimizer of problem (P) and \tilde{x}^* be any global infimizer of problem (\tilde{P}) . Then $\|x^* - \tilde{x}^*\| \leq \sqrt{2s/\lambda_{\min}}$.*

Applying the above result, we can estimate the Hausdorff distance

$$d_H(S_0, S_s) = \max\{\sup_{x \in S_0} \inf_{y \in S_s} \|x - y\|, \sup_{y \in S_s} \inf_{x \in S_0} \|x - y\|\} \quad (4.3)$$

between the set S_0 of minimizers of the original problem (P) and the set S_s of global infimizers of the perturbed problem (\tilde{P}) .

Of course, we are only interested in the case where both S_0 and S_s are nonempty. Since D is not assumed to be closed, it is not *a priori* sure whether S_0 and S_s are nonempty. Therefore, the existence of global optimal solutions must be explicitly assumed or ensured, as done in the following.

LEMMA 4.3 *Assume that problem (P) has a minimizer called x^* and*

$$\left(x^* + \frac{1}{2} M(2s)\right) \cap D \text{ is closed.} \quad (4.4)$$

Then there exist global infimizers of problem (\tilde{P}) .

Proof If $\tilde{f}(x^*) = \inf_{x \in D} \tilde{f}(x)$ then x^* is a global infimizer of (\tilde{P}) we look for. Otherwise, we can choose a sequence (x_i) in D such that

$$\tilde{f}(x^*) > \tilde{f}(x_i) \geq \inf_{x \in D} \tilde{f}(x) \quad \text{for all } i \in \mathbb{N} \text{ and } \lim_{i \rightarrow +\infty} \tilde{f}(x_i) = \inf_{x \in D} \tilde{f}(x).$$

Since the relation (4.2) is valid for any $\tilde{x} \in D \setminus (x^* + \frac{1}{2} M(2s))$, i.e.

$$\tilde{f}(\tilde{x}) > \tilde{f}(x^*) \quad \text{for all } \tilde{x} \in D \setminus \left(x^* + \frac{1}{2} M(2s)\right), \quad (4.5)$$

the entire sequence (x_i) must be contained in the set $(x^* + \frac{1}{2} M(2s)) \cap D$, which is compact. Hence, we can assume without loss of generality that (x_i) converges to a point $\tilde{x}^* \in D$, which implies that

$$\liminf_{x \in D, x \rightarrow \tilde{x}^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x),$$

i.e. \tilde{x}^* is a global infimizer of (\tilde{P}) . ■

The preceding lemma is a preparation for the next theorem.

THEOREM 4.4 Assume that problem (P) has a minimizer called x^* and

$$(x^* + \bar{B}(0, r)) \cap D \text{ is closed for some given } r > 0. \tag{4.6}$$

If

$$\sup_{x \in D} |p(x)| \leq s \leq \frac{1}{2} r^2 \lambda_{\min}, \tag{4.7}$$

then the set S_s of global infimizers of (\tilde{P}) is nonempty and

$$d_H(\{x^*\}, S_s) \leq \sqrt{2s/\lambda_{\min}}. \tag{4.8}$$

Proof Inequality (4.7) implies that $\sqrt{2s/\lambda_{\min}} \leq r$. Therefore, it follows from (2.13) that

$$\frac{1}{2} M(2s) \subset \bar{B}(0, \sqrt{2s/\lambda_{\min}}) \subset \bar{B}(0, r).$$

Since

$$\left(x^* + \frac{1}{2} M(2s)\right) \cap D = \left(x^* + \frac{1}{2} M(2s)\right) \cap ((x^* + \bar{B}(0, r)) \cap D),$$

(4.6) yields (4.4). Hence, by Lemma 4.3, S_s is nonempty. By applying Corollary 4.2 for (4.3) and $S_0 = \{x^*\}$, we finally get (4.8). ■

COROLLARY 4.5 Assume that problem (P) has a minimizer called x^* and (4.6) holds true. For all $\varepsilon > 0$, if

$$\sup_{x \in D} |p(x)| \leq s < \delta := \frac{1}{2} (\min\{\varepsilon, r\})^2 \lambda_{\min}, \tag{4.9}$$

then $d_H(\{x^*\}, S_s) < \varepsilon$.

Proof On the one hand, (4.9) implies $s < \frac{1}{2} r^2 \lambda_{\min}$, i.e. (4.7) holds true. Therefore, by Theorem 4.4, we get (4.8). On the other hand, (4.9) yields $s < \frac{1}{2} \varepsilon^2 \lambda_{\min}$. Hence, it follows from (4.8) that $d_H(\{x^*\}, S_s) \leq \sqrt{2s/\lambda_{\min}} < \varepsilon$. ■

The above result describes the stability of the set S_s of global optimal solutions (in a generalized sense as global infimizers) of the perturbed problem (\tilde{P}) . It says that $d_H(\{x^*\}, S_s)$ tends to zero when $s \rightarrow 0$.

5. Generalized subdifferentiability and optimality condition

As usual, a convex function $g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *subdifferentiable* at $x^* \in D$ if there exists a so-called *subgradient* $\eta \in \mathbb{R}^n$ satisfying (see, e.g. [21])

$$g(x) \geq g(x^*) + \eta^T(x - x^*) \quad \text{for all } x \in D. \tag{5.1}$$

For the convex function $g(x) = f(x) = x^T A x + b^T x$, $x \in D$, $\eta = \nabla f(x^*) = 2Ax^* + b$ is a subgradient at x^* and it is the only subgradient if x^* is in the interior of D . Since p is only assumed to be bounded by some given parameter s , the perturbed function $\tilde{f} = f + p$ is no more subdifferentiable in the above classical sense. Our aim is to

show, similarly to [18], that \tilde{f} is subdifferentiable in a generalized sense. In pursuit of this aim, we transform (5.1) equivalently to

$$g(x^*) - \eta^T x^* \leq g(x) - \eta^T x \quad \text{for all } x \in D$$

and replace the term on the left with $\inf_{x' \in (x^* + \Gamma) \cap D} (\tilde{f}(x') - \eta^T x')$ for some suitable balanced set Γ and the term on the right with $\tilde{f}(x) - \eta^T x$ to get

$$\inf_{x' \in (x^* + \Gamma) \cap D} (\tilde{f}(x') - \eta^T x') \leq \tilde{f}(x) - \eta^T x \quad \text{for all } x \in D,$$

which states the definition of the *roughly generalized subgradient* η of \tilde{f} . In such a way, the *roughly generalized subdifferentiability* of the function \tilde{f} may be described as in Theorem 5.1. Note that we only use the information $\sup_{x \in D} |p(x)| \leq s$ of p , i.e. p may be unspecified outside of D . Therefore, \tilde{f} is only given on D although f is well known on the entire space \mathbb{R}^n . That is why we consider the subdifferentiability of \tilde{f} only on D .

THEOREM 5.1 *Suppose $0 < \sup_{x \in D} |p(x)| \leq s < +\infty$ and $\tilde{f}(x) = x^T A x - b^T x + p(x)$. Then, for any $x^* \in D$, there holds*

$$\inf_{x' \in (x^* + \frac{1}{2}M(2s)) \cap D} (\tilde{f}(x') - (2Ax^* + b)^T x') \leq \tilde{f}(x) - (2Ax^* + b)^T x \quad \text{for all } x \in D. \quad (5.2)$$

In particular, if D is closed and p is lower semicontinuous, then for any $x^ \in D$ there exists*

$$\tilde{x}^* \in \left(x^* + \frac{1}{2} M(2s) \right) \cap D \quad (5.3)$$

such that

$$\tilde{f}(\tilde{x}^*) - (2Ax^* + b)^T \tilde{x}^* = \min_{x' \in (x^* + \frac{1}{2}M(2s)) \cap D} (\tilde{f}(x') - (2Ax^* + b)^T x') \quad (5.4)$$

and

$$\tilde{f}(\tilde{x}^*) - (2Ax^* + b)^T \tilde{x}^* \leq \tilde{f}(x) - (2Ax^* + b)^T x \quad \text{for all } x \in D, \quad (5.5)$$

or, equivalently,

$$\tilde{f}(x) \geq \tilde{f}(\tilde{x}^*) + (2Ax^* + b)^T (x - \tilde{x}^*) \quad \text{for all } x \in D.$$

Proof Consider any fixed $x^* \in D$ and the function $\bar{f}: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\bar{f}(x) := f(x) - (2Ax^* + b)^T x = x^T A x - 2x^{*T} A x. \quad (5.6)$$

Since \bar{f} is strictly convex and $\nabla \bar{f}(x^*) = 0 \in \mathbb{R}^n$, x^* is the only minimizer of \bar{f} on \mathbb{R}^n . This fact does not change if x is only restricted to D . Applying (4.5) for \bar{f} instead of f and $\bar{f} + p$ instead of $\tilde{f} = f + p$, we get

$$\bar{f}(\tilde{x}) + p(\tilde{x}) > \bar{f}(x^*) + p(x^*) \quad \text{for all } \tilde{x} \in D \setminus \left(x^* + \frac{1}{2} M(2s) \right).$$

By (5.6), we have

$$\tilde{f}(x^*) - (2Ax^* + b)^T x^* < \tilde{f}(\tilde{x}) - (2Ax^* + b)^T \tilde{x} \quad \text{for all } \tilde{x} \in D \setminus \left(x^* + \frac{1}{2} M(2s)\right),$$

which yields immediately

$$\inf_{x' \in (x^* + \frac{1}{2} M(2s)) \cap D} \left(\tilde{f}(x') - (2Ax^* + b)^T x'\right) < \tilde{f}(\tilde{x}) - (2Ax^* + b)^T \tilde{x}$$

for all $\tilde{x} \in D \setminus \left(x^* + \frac{1}{2} M(2s)\right)$

and, therefore, (5.2).

In particular, if D is closed and p is lower semicontinuous, then the set $(x^* + \frac{1}{2} M(2s)) \cap D$ is compact and the function $x \mapsto \tilde{f}(x) - (2Ax^* + b)^T x$ is lower semicontinuous. Therefore, there exists \tilde{x}^* satisfying (5.3) and (5.4). Then (5.5) follows from (5.2) and (5.4). ■

Finally, we state a generalization of the Kuhn–Tucker theorem for the problem of minimizing $\tilde{f}(x) = f(x) + p(x)$ subject to $x \in D$, where

$$D = \{x \in C \mid g_1(x) \leq 0, \dots, g_m(x) \leq 0\}, \tag{5.7}$$

$C \subset \mathbb{R}^n$ is closed and convex, and all functions g_1, \dots, g_m are convex and continuous on C . As usual, $\partial g_i(x)$ denotes the subdifferential of g_i and $N(x \mid C)$ denotes the normal cone to the set C at the point x .

THEOREM 5.2 *Suppose that D is defined by (5.7).*

- (a) *If $\tilde{x}^* \in D$ is a global infimizer of problem (\tilde{P}) , then there exist an $x^* \in (\tilde{x}^* + \frac{1}{2} M(2s)) \cap D$ and Lagrange multipliers $v_0 \geq 0, v_1 \geq 0, \dots, v_m \geq 0$, not all zero, such that*

$$\begin{aligned} 0 &\in v_0(2Ax^* + b) + v_1 \partial g_1(x^*) + \dots + v_m \partial g_m(x^*) + N(x^* \mid C), \\ v_i g_i(x^*) &= 0, \quad i = 1, \dots, m. \end{aligned} \tag{5.8}$$

If the Slater condition is fulfilled, i.e.

$$\exists x \in C : g_1(x) < 0, \dots, g_m(x) < 0, \tag{5.9}$$

then $v_0 \neq 0$ and it can be assumed that $v_0 = 1$.

- (b) *If $x^* \in D$ satisfies (5.8) for $v_0 = 1$, then there exists an $\tilde{x}^* \in (x^* + \frac{1}{2} M(2s)) \cap D$ which is a global infimizer of problem (\tilde{P}) .*

Proof (a) Assume that $\tilde{x}^* \in D$ is a global infimizer of problem (\tilde{P}) . Since D is closed, $f(x) = x^T Ax + b^T x$ and A is positive definite, problem (P) has exactly one global minimizer $x^* \in D$. Due to Theorem 4.1, $x^* \in (\tilde{x}^* + \frac{1}{2} M(2s)) \cap D$. Moreover, by Theorem 2' in [7, p. 69], there exist Lagrange multipliers $v_0 \geq 0, v_1 \geq 0, \dots, v_m \geq 0$, not all zero, such that

$$\begin{aligned} 0 &\in v_0 \partial f(x^*) + v_1 \partial g_1(x^*) + \dots + v_m \partial g_m(x^*) + N(x^* \mid C), \\ v_i g_i(x^*) &= 0, \quad i = 1, \dots, m, \end{aligned}$$

where $v_0 \neq 0$ if (5.9) is fulfilled. This yields (5.8) since $\partial f(x^*) = \{2Ax^* + b\}$.

(b) Assume that $x^* \in D$ satisfies (5.8) for $v_0 = 1$. By Theorem 2' in [7, p. 69], x^* is optimal to (P). Since both $M(2s)$ and D are closed, (4.4) is satisfied. Hence, by Lemma 4.3, problem (\tilde{P}) has at least a global infimizer $\tilde{x}^* \in D$. By Theorem 4.1, $\tilde{x}^* \in (x^* + \frac{1}{2} M(2s)) \cap D$. ■

Due to Theorem 5.1, the expression $2Ax^* + b$ in (5.8) represents a roughly generalized gradient of \tilde{f} at x^* with respect to $\Gamma = \frac{1}{2} M(2s)$. If we denote the roughly generalized subdifferential of \tilde{f} by $\partial_{\Gamma} \tilde{f}$, then the first row of (5.8) reads as follows:

$$0 \in v_0 \partial_{\Gamma} \tilde{f}(x^*) + v_1 \partial g_1(x^*) + \cdots + v_m \partial g_m(x^*) + N(x^* | C),$$

which is just a form of necessary optimality conditions we used to know.

Now assume that $C = \mathbb{R}^n$ and all g_1, \dots, g_m are continuously differentiable convex functions on \mathbb{R}^n . Then (5.8) becomes

$$\begin{aligned} v_0(2Ax^* + b) + v_1 \nabla g_1(x^*) + \cdots + v_m \nabla g_m(x^*) &= 0, \\ v_i g_i(x^*) &= 0, \quad i = 1, \dots, m. \end{aligned} \quad (5.10)$$

Because of (2.13), $\tilde{x}^* \in (x^* + \frac{1}{2} M(2s)) \cap D$ means

$$\tilde{x}^* - x^* \in \frac{1}{2} M(2s) \subset \bar{B}(0, \sqrt{2s/\lambda_{\min}}),$$

i.e. $\|\tilde{x}^* - x^*\| \leq \sqrt{2s/\lambda_{\min}}$. Therefore, the first part of Theorem 5.2 says that, for sufficiently small $s > 0$, if $\tilde{x}^* \in D$ is a global infimizer of the perturbed problem (\tilde{P}) then it almost satisfies the familiar necessary optimality condition for the original problem (P) as follows:

$$\begin{aligned} v_0(2A\tilde{x}^* + b) + v_1 \nabla g_1(\tilde{x}^*) + \cdots + v_m \nabla g_m(\tilde{x}^*) &\approx 0, \\ v_i g_i(\tilde{x}^*) &\approx 0, \quad i = 1, \dots, m. \end{aligned} \quad (5.11)$$

Replacing '=' in (5.10) with '≈' in (5.11) means no essential weakness for application, since in practical computation with computers the condition '=0' must be replaced with '≈0' anyway.

The second part of Theorem 5.2 says that, for sufficiently small $s > 0$, if $x^* \in D$ satisfies the necessary optimality condition (5.10) for the original problem (P), then it approximates a global infimizer $\tilde{x}^* \in D$ of the perturbed problem (\tilde{P}).

Theorem 5.2 explains why and how the Kuhn–Tucker optimality condition can still work for strictly convex quadratic programs in spite of computing errors occurring in objective functions, which may be somewhat wild but sufficiently small.

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