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Refinement CTIN for General Type-2 Fuzzy Logic Systems

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***Abstract*—Triangulated irregular network (TIN) has been used for representing general type-2 fuzzy sets and gained some results of reducing computational complexity. In general, TIN-based algorithms are still complex and difficulty to deploy in applications. So, an approach based on refinement constraint TIN (CTIN) for representing general type-2 fuzzy set is proposed. The paper deals with the use CTIN in general type-2 fuzzy logic systems (GT2FLS). Operations are developed for general type-2 fuzzy sets. A T2FLS is designed and implemented in comparing previous approaches under an application of robot navigation.**

***Index Terms*—Type-2 Fuzzy Sets, Type-2 Fuzzy Logic Systems, Type-2 Fuzzy Inference, Geometric Representation**

I. INTRODUCTION

Type-2 fuzzy logic systems has widely been used in many applications. Almost applications are limited by computational complexity of general T2FLS. Recently, these are many researches arising from reduction the complexity of these systems that depend on representation of type-2 fuzzy sets. There are many approaches to representation of type-2 fuzzy sets. The point-based representation [5], [6], [7], [8] has proposed with operations and computation of type-2 fuzzy sets and systems. Mendel et al [9], [10] proposed the representation based on embedded fuzzy sets, vertical slices. Starczewski [17] introduced a method for complexity reduction of operations on triangular type-2 fuzzy sets. For the this purpose, Coupland et al [2] proposed geometric method for representation type-1 and interval type-2 fuzzy sets, new algorithms for various operations on type-1 and type-2 fuzzy sets and for defuzzification. Coupland et al [3], [4] presented new techniques using upper and lower surfaces for performing logical operations on type-2 fuzzy sets with considering computational speed and accuracy. In [14], [15], authors proposed an approach to TIN-based geometric representation of type-2 fuzzy sets with or without using upper and lower surfaces. Hagras et al [20], [21], [22] proposed an approach to representation of type-2 fuzzy sets using zSlices that be able to gap interval type-2 fuzzy sets and general type-2 fuzzy sets. A similar approach to representation of type-2 fuzzy sets was proposed by Mendel et al [11], [12] based on α - plane representation.

The paper introduces an approach to representation of general type-2 fuzzy sets, called ϵ_u-approximation. A constrained TIN representing T2FS is triangulated from set of points set that satisfy ϵ_u-approximate criterion. This representation is the basis for designing algorithms of general type-2 fuzzy logic systems. This approach is considered under two respects: computational speed and accuracy. An computational process of general type-2 FLSs (GT2FLS) involving fuzzification, inference process, aggregation and defuzzification are proposed in comparing with previous methods. Computation for GT2FLS is able to deploy in real time applications. Because geometric algorithms are only computed at vertices of TIN based on operations of interval type-2 fuzzy sets. Experimental results are shown in comparisons on speed and accuracy with other approaches. The paper also discusses the feasibility of proposed algorithms by implementing RCTIN-based type-2 fuzzy logic systems under avoidance behavior of robot navigation with reported runtime tables.

The paper is organized as follows: II presents an overview on type-2 fuzzy sets and inference, ϵ-approximation representation of type-2 fuzzy sets; III introduces ϵ_u-approximate representation and applications to operations involving meet and join under minimum, negation; IV presents a general T2FLS involving sub-processes such as fuzzification, inference process, aggregation and defuzzification. V introduces implementation and discussion of the feasibility under the experimentation of robot navigation; V is conclusion and future works.

II. GEOMETRIC REPRESENTATION FOR GENERAL TYPE-2 FUZZY SETS

A. Type-2 Fuzzy Sets

A type-2 fuzzy set in X is denoted $\tilde{A}$, and its membership grade of $x \in X$ is $\mu_{\tilde{A}}(x,u), u \in J_x \subseteq [0,1]$, which is a type-1 fuzzy set in [0, 1]. The elements of domain of $\mu_{\tilde{A}}(x,u)$ are called primary memberships of x in $\tilde{A}$ and memberships of primary memberships in $\mu_{\tilde{A}}(x,u)$ are called secondary memberships of x in $\tilde{A}$.

Definition 2.1: A $type-2\ fuzzy\ set$, denoted $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$ where $x \in X$ and $u \in J_x \subseteq [0,1]$, i.e.,

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \qquad (1)$$

or

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u))/(x,u), J_x \subseteq [0,1] \qquad (2)$$

in which $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$.

At each value of x, say $x = x'$, the 2-D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a $vertical\ slice$ of $\mu_{\tilde{A}}(x, u)$. A *secondary membership function* is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x \in X$ and $\forall u \in J_{x'} \subseteq [0, 1]$, i.e.

$$\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x') = \int_{u \in J_{x'}} f_{x'}(u)/u, J_{x'} \subseteq [0, 1] \quad (3)$$

in which $0 \leq f_{x'}(u) \leq 1$.

In manner of embedded fuzzy sets, a type-2 fuzzy sets [9] is union of its type-2 embedded sets, i.e

$$\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j \quad (4)$$

where $n \equiv \prod_{i=1}^{N} M_i$ and $\tilde{A}_e^j$ denoted the j^{th} type-2 embedded set of $\tilde{A}$, i.e.,

$$\tilde{A}_e^j \equiv \{\left(u_i^j, f_{x_i}(u_i^j)\right), i = 1, 2, ..., N\} \quad (5)$$

where $u_i^j \in \{u_{ik}, k = 1, ..., M_i\}$.

Let $\tilde{A}, \tilde{B}$ be type-2 fuzzy sets whose secondary membership grades are $f_x(u), g_x(w)$, respectively. Theoretic operations of type-2 fuzzy sets such as union, intersection and complement are described [6] as follows:

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) = \int_u \int_v (f_x(u) \star g_x(w))/(u \vee w) \quad (6)$$

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) = \int_u \int_v (f_x(u) \star g_x(w))/(u \star w) \quad (7)$$

$$\mu_{\tilde{A}}(x) = \mu_{\neg \tilde{A}}(x) = \int_u (f_x(u))/(1 - u) \quad (8)$$

where $\vee$, $\star$ are t-cornorm, t-norm, respectively. Type-2 fuzzy sets are called an interval type-2 fuzzy sets if the secondary membership function $f_{x'}(u) = 1 \ \forall u \in J_x$.

B. Delaunay Triangulation

A *topographic surface* σ is the image of a real bivariate function f defined over a domain D in the Euclidean plane, as

$$\sigma = \{(x, u, f(x, u)) \big| (x, u) \in D\} \quad (9)$$

A polyhedral model is the image of a piecewise-line function f described on a partition of D into polygonal regions $\{T_1, ..., T_k\}$ and the image of f over each region $T_i (i = 1, ..., k)$ is a planar patch. If all of T_is $(i = 1, .., k)$ are triangles then the polyhedral model is called a $Triangulated\ Irregular\ Network$ (TIN). Hence, σ may be represented approximately by a TIN, as

$$\sigma \tilde{=} \sum_{i=1}^{k} \{(x, u, f_i(x, u)) \big| (x, u) \in T_i\}, \bigcup_{i=1}^{k} T_i \equiv D \quad (10)$$

where f_is $(i = 1, ..., k)$ are planar equations.

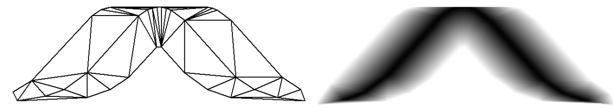

Fig. 1. Example of representation of a type-2 Gaussian fuzzy sets

The $Delaunay\ triangulation$ of a set V of points in IR^2 is a subdivision of the convex hull of V into triangles having their vertices at points of V, and such that triangles are as much aquiangular as possible. More formally, a triangulation τ of V is a Delaunay triangulation if and only if, for any triangle t of τ, the circumcircle of t does not contain any point of V in its interior. This property is called the $empty\ circle\ property$ of the Delaunay triangulation.

C. Geometric Representation of Type-2 Fuzzy Sets

In [15], geometric representation of type-2 fuzzy sets is presented as an approach based on TIN. Concept of ε-approximation set is used to approximately discretize the membership grades of type-2 fuzzy sets into triangular patches. ϵ-approximation is defined as follows:

Definition 2.2: A type-2 fuzzy set is called ϵ-approximation set, denoted $\tilde{A}^*$, of $\tilde{A}$ in continuous domain D if

$$\|\mu_{\tilde{A}}(x, u) - \mu_{\tilde{A}^*}(x, u)\| \leq \varepsilon, (x, u) \in D \quad (11)$$

The theorem on representation of a type-2 fuzzy set referenced as *Approximation Theorem* is presented as follows:

Theorem 2.1 (Approximation Theorem): Let $\tilde{A}$ be type-2 fuzzy set with membership grade $\mu_{\tilde{A}}(x, u)$ in continuous domain D. There exists a type-2 fuzzy set with membership grade is a TIN $T_{\tilde{A}}$, denoted $\tilde{A_T}$, so that $\tilde{A_T}$ is ϵ-approximation set of $\tilde{A}$, i.e,

$$\|\mu_{\tilde{A}}(x, u) - \mu_{\tilde{A_T}}(x, u)\| < \epsilon, \ \ \forall (x, u) \in D. \quad (12)$$

Fig. 1 is the TIN consisting of 36 vertices and 48 faces that represents approximately of Gaussian type-2 fuzzy sets with $\epsilon = 0.1$. The primary membership function is a Gaussian with fixed deviation and mean $m_k \in [m_1, m_2]$ and the secondary membership function is a triangular membership function that its apex is at the Gaussian function with mean $(m_1 + m_2)/2$. At the point (x, u) that its color is darker, the value of $\mu_{\tilde{A}}(x, u)$ is closer 1.0, otherwise $\mu_{\tilde{A}}(x, u)$ is closer 0.0.

III. Refinement CTIN Based Type-2 Fuzzy Sets

TIN-based representation of type-2 FS gives nice results, but computational complexity is still large. To gain more effective computational method, refinement constrain triangulated irregular network (CTIN) is proposed to represent a general T2FS. This section introduces the approach to representation using refinement CTIN and applications to operations of T2FS.

A. Representation of T2FSs Using Refinement CTIN

TIN is formed from set of points in the 3-dimensional space with satisfying Delaunay criterion, called Delaunay triangulation. CTIN is TIN satisfying Delaunay criterion with each input segment appears as an edge of triangulation. Input of CTIN is a planar straight line graph (PSLG). A PSLG is a set of vertices and segments that satisfies two constraints. First, for each segment contained in a PSLG, the PSLG must also contain the two vertices that serve as endpoints for that segment. Second, segments are permitted to intersect only at their endpoints. Fig. (2) depicts an example of PSLG with 13 vertices and 9 segments.

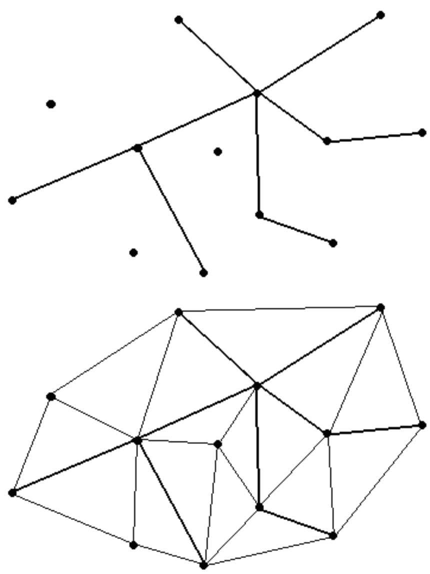

Fig. 2. An example of PSLG and its CTIN

To represent a type-2 fuzzy set using CTIN, a PSLG is approximately retrieved from type-2 fuzzy set $\tilde{A}$. The following is some definitions of ε_u criterion that points out how to retrieve a PSLG.

Definition 3.1 (ε_u approximation): On α-cut plane at z level of the type-2 fuzzy set $\tilde{A}$, the point $P'_{(x,u')} \in \alpha$ is called a ε_u-approximate point of $P_{(x,u)} \in \alpha$ if $\|u - u'\| < \varepsilon$.

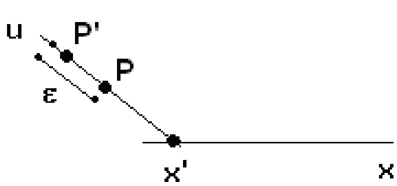

Fig. 3. The P' point is ε_u-approximate point of P.

Let type-2 fuzzy set $\tilde{A}$ and $n+1$ α-planes at levels $\{z_i = \frac{i}{n}, i = 0, 1, ..., n\}$. A contour at the level z_i of $\tilde{A}$, denoted $\tilde{Z}_i$, is a poly-line that involves its vertices on α-plane z_i. A contour $\tilde{Z}_i, (i = 0, 1, ..., n)$ is called ε_u-approximate contour of type-2 fuzzy set $\tilde{A}$ if each point $P'_{(x,u')} \in \tilde{Z}_i$ is also ε_u-approximate point of $P_{(x,u)} \in \tilde{A}$ at the level z_i. Thence, an ε_u-approximate type-2 fuzzy set is described as follows:

Definition 3.2 (ε_u-approximate T2FS): A type-2 fuzzy set $\tilde{A}'$ is called ε_u-approximate type-2 fuzzy set at n levels of $\tilde{A}$ if $n+1$ its contours $(Z_0, Z_1, ..., Z_n)$ are ε_u-approximate contour of $\tilde{A}$.

At each level k of $\tilde{A}$, contour Z_k is an interval type-2 fuzzy set, thus upper (lower) contour, denoted UC_k (LC_k), is upper (lower) bound segments of the contour. In case of $k = 0$, UC_0 (LC_0) is upper (lower) membership function (UMF/LMF) of interval type-2 fuzzy set that is an overlap of the FOU of $\tilde{A}$.

Definition 3.3: (Refinement CTIN) A TIN representing $\tilde{A}$ is called *refinement CTIN* of $\tilde{A}$ at K levels, denoted $T_{\tilde{A}}$, if $T_{\tilde{A}}$ is generated from PSLG involving set of vertices V and set of segments L, in which $\forall v \in V$ be ε_u-approximate points and L is set of segments extracted from K ε_u-approximate contours of $\tilde{A}$.

Hence, a general type-2 fuzzy set is able to be represented by a refinement CTIN as the following theorem.

Theorem 3.1: Let $\tilde{A}$ be a type-2 fuzzy set with continuous membership grade $\mu_{\tilde{A}}(x)$ in continuous domain D and enough small ε_u. These exists *refinement* CTIN that represents $\tilde{A}$ with ε_u-approximation at K levels.

Proof:

Call $T_{\tilde{A}}$ is refinement CTIN that is generated from a PSLG with set of vertices V and set of segments L extracted from $\tilde{A}$ at K levels. Suppose that exists a point $P_{(x_p,u_p,z_p)}$ on $T_{\tilde{A}}$ not to satisfy ε_u-approximate criteria. Thus P is belong to one of α-planes $Z_0, Z_1, ..., Z_{K-1}$. $T_{\tilde{A}}$ is refinement CTIN then P is inside the edge (for example v_iv_j) of $T_{\tilde{A}}$. Denote $P_0(x_{P_0}, u_{P_0}, z_{P_0})$ be the point on the edge v_iv_j that distance to respective point of $\tilde{A}$ in u-axis is maximum, i.e.

$$\{\| u_{P_0} - u \|\} \rightarrow max, z_{P_0} = \mu_{\tilde{A}}(x_{P_0}, u) \qquad (13)$$

Note that P_0 is able to coincide with P. Call t_k, t_l are two adjacent triangles of v_iv_j. The edge v_iv_j is split into two sub-edge v_iP_0 and P_0v_j, so t_k, t_l are divided into four sub-triangle. According to supposition, $\mu_{\tilde{A}}(x)$ is continuous in domain D then distance from the respective point P' of P in new edge v_iP_0 or P_0v_j of $T_{\tilde{A}}$ to respective point on $\tilde{A}$ is closer than P, i.e.

$$\| u_{P'} - u \| < \| u_P - u \|, z_P = z_P = \mu_{\tilde{A}}(x_P, u) \qquad (14)$$

If these still exists a point P inside edges v_iP_0 or P_0v_j that not to satisfy ε_u-approximation then the edge is split into sub-edges as above way. The end of each splitting step, the P is closely moved to respective point of $\tilde{A}$. Because of the continuousness of $\mu_{\tilde{A}}(x)$, these no exists any point inside sub-edge that not to satisfy ε_u-approximation after the limited incremental splitting steps.

■

Example: Let $\tilde{A}$ be Gaussian type-2 fuzzy set with $I = 5$ levels, i.e. $z_i = i/I, i = 0, 1, .., I$. Upper (lower) membership functions at each level are described as follows:

$$f_U^i(x) = (0.9 + \frac{0.1(I-i)}{I})e^{(-\frac{x^2}{2(\sigma_1 - i*(\sigma_1-\sigma_2)/(2I))^2})}$$
$$f_L^i(x) = (0.9 - \frac{0.1(I-i)}{I})e^{(-\frac{x^2}{2(\sigma_2 + i*(\sigma_1-\sigma_2)/(2I))^2})} \quad (15)$$

Fig. 4 is to describe CTIN based type-2 fuzzy set that represent $\tilde{A}$ as the formula (15).

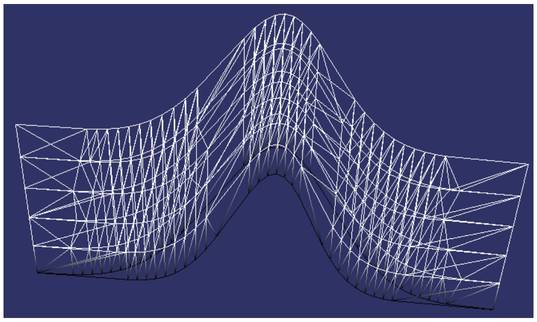

Fig. 4. Example of ε_u approximate representation of $\tilde{A}$

B. Operations

Theoretic operations of type-2 fuzzy sets are described as formulations (6)-(8). Arcording to [6], operations of n type-2 fuzzy sets are computed using max t-conorm and min t-norm as follows:

Join operation:

$$\mu_{\sqcup_{i=1}^n F_i}(\theta) = \begin{cases} \wedge_{i=1}^n \mu_{F_i}(\theta) & \theta < \upsilon_1 \\ \wedge_{i=k+1}^n \mu_{F_i}(\theta) & \upsilon_k \le \theta < \upsilon_{k+1}, 1 \le k < n \\ \vee_{i=1}^n \mu_{F_i}(\theta) & \theta > \upsilon_n \end{cases} \quad (16)$$

Meet operation:

$$\mu_{\sqcap_{i=1}^n F_i}(\theta) = \begin{cases} \vee_{i=1}^n \mu_{F_i}(\theta) & \theta < \upsilon_1 \\ \wedge_{i=1}^k \mu_{F_i}(\theta) & \upsilon_k \le \theta < \upsilon_{k+1}, 1 \le k < n \\ \wedge_{i=1}^n \mu_{F_i}(\theta) & \theta \ge \upsilon_n \end{cases} \quad (17)$$

in which $\wedge$ is a t-norm and $\vee$ is a t-conorm, υ_i is a value such that $\mu_{F_i}(\upsilon_i)$ is at the apex of $\mu_{F_i}(x)$, $i = 1, ..., n$.

Suppose that $\tilde{A}_i (i = 1, ..., n)$ are ε_u type-2 fuzzy sets based on refinement CTIN with L levels. Call V_i is vertices of $\tilde{A}_i$ and $V = \cup_{i=1}^n V_i$. Join (meet) operations of n type-2 fuzzy sets $\tilde{A}_i$ using max t-conorm and min t-norm are described as the following theorem.

Theorem 3.2: Let $\tilde{A}_i (i = 1, .., n)$ be ε_u type-2 fuzzy sets based on refinement CTIN with L levels. Call I_k is set of intersection points between k^{th}-level contours of $\tilde{A}_i$. Join (meet) operation of $\tilde{A}_i$, called $\tilde{B}$, is computed according to the formula (16) ((17) for meet operation) at $v \in V \cup (\cup_{k=0}^K I_k)$. B is ε_u type-2 fuzzy sets based on refinement CTIN with L levels.

The following is the algorithm for computing join, meet operations using max t-conorm, min t-norm of n refinement CTIN-based type-2 fuzzy sets.

Algorithm 3.1 (Join/Meet Operations): .

Input: n RCTIN-based type-2 fuzzy sets $\tilde{A}_i$, $(i = 1, ..., n)$.
Output: $\tilde{B} = \sqcup_{i=1}^n \tilde{A}_i$ ($\tilde{B} = \sqcap_{i=1}^n \tilde{A}_i$ for meet operation).

1) For each level $k = \overline{0, L}$.
   a) Compute I_k set of intersection points of k^{th} contours of $\tilde{A}_i$.
   b) For each $v \in I_k$
      If $v \notin \tilde{A}_i$ then insert v into the $\tilde{A}_i$'s TIN.
2) For each level $k = \overline{0, L}$.
   For each vertex $v \in V$ belong to k^{th} contour
   a) Compute vertex $v' \in \tilde{B}$ of join (meet) operation according to (16) or (17).
   b) If v' is not a vertex of $\tilde{A}_i$s then reject v'.
   c) Otherwise add v' into $V_{\tilde{B}}^k$, i.e. set of vertices at k^{th} level of $\tilde{B}$.
3) Triangulate set of vertices $V_{\tilde{B}} = \cup_{k=0}^L V_{\tilde{B}}^k$ with PSLG (segments in $V_{\tilde{B}}^k$).

Example: Let $\tilde{A}_1, \tilde{A}_2$ be two RCTIN-based fuzzy sets described as the formula (15) with $m_1 = 1.0, m_2 = 1.5, \sigma_{11} = 0.4, \sigma_{12} = 0.3, \sigma_{21} = 0.5, \sigma_{22} = 0.3$. Fig. (5) show results of join/meet operations.

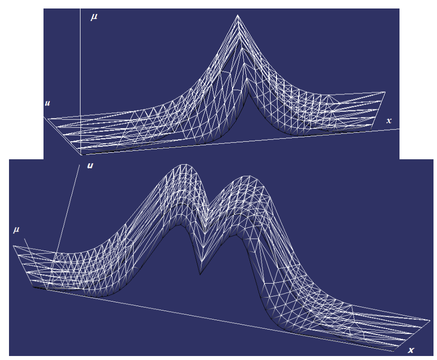

Fig. 5. Example of join/meet operations.

IV. Refinement CTIN Based Type-2 Fuzzy Logic Systems

A. Rule base and fuzzifier

Fig. 6 shows the structure of a RCTIN-based type-2 FLS. The difference from general type-2 FLS contains only geometric defuzzification in the output processing block.

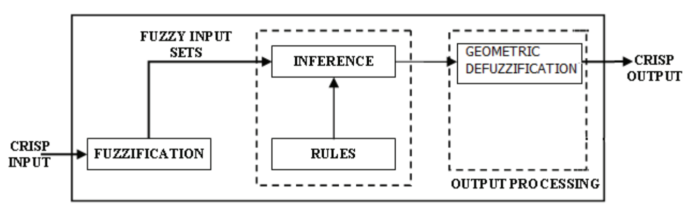

Fig. 6. The structure of RCTIN based type-2 fuzzy logic systems

The singleton fuzzifier maps the crisp input into a RCTIN-based type-2 fuzzy set. We consider only singleton fuzzification, for which the input fuzzy set has only a single point of nonzero membership. The "IF-THEN" rules has the form, for l^{th} rule, as follows:

R^l : IF x_1 is $\tilde{F}_1^l$ and x_2 is $\tilde{F}_2^l$ and ... and x_p is $\tilde{F}_p^l$ THEN y is $\tilde{G}^l$

where: x_is are inputs; $\tilde{F}_i^l$s are antecedent sets (i=1, 2, ..., p); $\tilde{G}^l$ is the output.

Example: Consider the RCTIN-based type-2 fuzzy set that described as the formula (15). Using singleton fuzzifier at $x = 0.8$, output of the fuzzifier is a RCTIN-based type-2 fuzzy set as the Fig. 7.

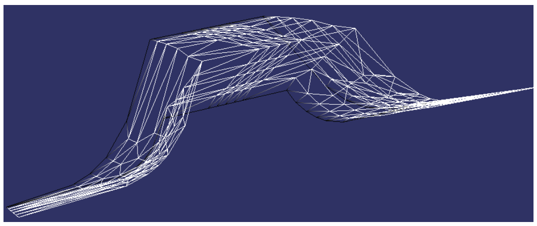

Fig. 7. Example on output of fuzzifier

B. Inference Engine

Inference engine combines rules and gives a mapping from input type-2 fuzzy sets to output type-2 fuzzy sets. Multiple antecedents in rules are connected by the meet operation. The membership grades in the input sets are combined with those in the output sets using the $sup-star$ composition. To do this one needs to find meets and joins of type-2 sets, as well as compositions of type-2 relations. Note that mentioned type-2 fuzzy sets are RCTIN-based type-2 fuzzy sets.

The output of the inference engine is a RCTIN-based type-2 fuzzy set. We do defuzzification by finding its geometric centroid, that is described more detail in the next section.

The following is the inference processing of RCTIN-based type-2 FLS:

Consider a RCTIN-based type-2 FLS having p inputs, $x_1 \in X_1, x_2 \in X_2, ..., x_p \in X_p$, and one output $y \in Y$. Suppose that it has M rules where the l^{th} rule has the form

$$R^l : \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ AND ... AND } x_p \text{ is } \tilde{F}_p^l \text{ THEN } y \text{ is } \tilde{G}^l. \quad (18)$$

This rule represents a RCTIN-based type-2 fuzzy relation between the input space $X_1 \times X_2 \times ... \times X_p$ and the output space Y of the FLS. The membership function of this type-2 relation is denoted as $\mu_{\tilde{F}_1^l \times ... \times \tilde{F}_p^l \rightarrow \tilde{G}^l}(x, y)$, where $\tilde{F}_1^l \times ... \times \tilde{F}_p^l$ denotes the Cartesian product of $\tilde{F}_1^l$, ..., $\tilde{F}_p^l$, and $x = \{x_1, x_2, ..., x_p\}$.

When an input $\mathbf{x}'$ is applied, the composition of the fuzzy set $\tilde{X}'$ to which $\mathbf{x}''$ belongs and the rule R^l is found by using the extended sup-star composition

$$\mu_{\tilde{X}' \circ \tilde{F}_1^l \times ... \times \tilde{F}_p^l \rightarrow \tilde{G}^l}(y) = \sqcap_{x \in \tilde{X}'}[\mu_{\tilde{X}'}(x) \sqcap \mu_{\tilde{F}_1^l \times ... \times \tilde{F}_p^l \rightarrow \tilde{G}^l}(x, y)] \quad (19)$$

We denote $\tilde{X}' \circ \tilde{F}_1^l \times ... \times \tilde{F}_p^l \rightarrow \tilde{G}^l$ as $\tilde{B}^l$, the output set corresponding to the l^{th} rule. We use singleton fuzzification and the product or minimum implication, inference process of

Fig. 8. Centroid computations of $\tilde{A}$ by partitioning the space of uncertainty

rule l^{th} is described as follows:

$$\begin{aligned} \mu_{\tilde{B}^l}(y) &= \mu_{\tilde{F}_1^l}(x_1) \sqcap \mu_{\tilde{F}_2^l}(x_2) \sqcap ... \sqcap \mu_{\tilde{F}_p^l}(x_p) \sqcap \mu_{\tilde{G}^l}(y) \\ &= \mu_{\tilde{G}^l}(y) \sqcap [\sqcap_{i=1}^p \mu_{\tilde{F}_i^l}(x_i)] \end{aligned} \quad (20)$$

The output sets of rules are combined in a type-2 fuzzy set by using $join$ operation, i.e.

$$\mu_{B'}(y) = \sqcup_{i=1}^K \mu_{\tilde{B}^i}(y) \quad (21)$$

in which K is number of fired rules and $\mu_{\tilde{B}^i}(y)$ is output of i^{th} fired rule.

Consider a vertex v of the TIN that represents output RCTIN-based type-2 fuzzy set of l^{th} rule. Suppose that v belongs to Z_k contour. After inference process, abscissa x, z of v are no change, abscissa u of v is computed as follows:

If $u_v = max_u\{\mu_{\tilde{G}}(x_v, u)\}$ then

$$u'_v = u_v \star [max_y\{Z^k_{\tilde{F}_1}(x_1)\} \star ... \star max_y\{Z^k_{\tilde{F}_1}(x_1)\}] \quad (22)$$

If $u_v = min_u\{\mu_{\tilde{G}}(x_v, u)\}$ then

$$u'_v = u_v \star [min_y\{Z^k_{\tilde{F}_1}(x_1)\} \star ... \star min_y\{Z^k_{\tilde{F}_1}(x_1)\}] \quad (23)$$

The algorithm for inference engine is described as follows:

Algorithm 4.1 (Inference engine): .

Input: $n+1$ RCTIN-based type-2 fuzzy sets $\tilde{A}_i, \tilde{G}$, ($i = 1, ..., n$).

Output: RCTIN-based type-2 fuzzy set $\tilde{G}$ is output of inference process.

1) For each contour $Z_k, k = \overline{0, L}$.
   a) For each vertex v is on the Z_k
   b) if v belongs to upper surface, i.e $u_v = max_u\{\mu_{\tilde{G}}(x_v, u)\}$
      computing u'_v as the formula (22).
   c) otherwise
      computing u'_v as the formula (23).

C. Defuzzification

Defuzzification is the most complex process of theoretic type-2 fuzzy logic systems. This section is described a defuzzification based on finding center of gravity of the space of uncertainty.

Let V_i with center of gravity $cog_i = (x_{ci}, y_{ci}, z_{ci})$ and mass m_i ($i = 1, \ldots, P$) are objects in three-dimensions space. The gravity center of the system of massive objects is computed as follows:

$$\mathbf{c} = \frac{\sum_{i=1}^P \mathbf{cog}_i * m_i}{\sum_{i=1}^P m_i} \quad (24)$$

If mass density of objects is equilateral then equation (24) is rewritten as follows:

$$\mathbf{c} = \frac{\sum_{i=1}^P \mathbf{cog}_i * V_i * md}{\sum_{i=1}^P V_i * md} = \frac{\sum_{i=1}^P \mathbf{cog}_i * V_i}{\sum_{i=1}^P V_i} \quad (25)$$

where V_is are volumes of massive objects and dm is their mass density.

Let $\tilde{A}$ be RCTIN-based type-2 fuzzy set that contains N vertices $v_k(k = 1, \ldots, N)$ and M faces $f_k(k = 1, \ldots, M)$. In order to compute the center of gravity of the uncertainty space of $\tilde{A}$, $T_{\tilde{A}}$ is divided into 3-dimensional patches based on triangles of TIN. For each triangle, we make a triangular prism in which its vertices are the three vertices of the triangle and the three points of these vertices are projected on 2-dimensional plane (x, u). Next, triangular prisms are also divided into three tetrahedrons. Hence, the TIN of $\tilde{A}$ is divided into $3M$ tetrahedrons. Let t_k with center of gravity $c_k = (x_k, u_k, y_k)$ and volume V_k is the k^{th} tetrahedrons $(k = 1, \ldots, 3M)$. Because mass density of tetrahedrons is equilateral, the center of gravity of system of tetrahedrons is computed by using 25 as follows:

$$\mathbf{c} = \frac{\sum_{i=1}^{3M} \mathbf{c}_i * V_i}{\sum_{i=1}^{3M} V_i} \tag{26}$$

For a tetrahedron with vertices $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13})$, $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23})$, $\mathbf{v}_3 = (v_{31}, v_{32}, v_{33})$, and $\mathbf{v}_4 = (v_{41}, v_{42}, v_{43})$, the volume is computed using a dot product (.) and a cross product ($\times$), yielding

$$\mathbf{V} = \frac{|(\mathbf{v}_1 - \mathbf{v}_4).((\mathbf{v}_2 - \mathbf{v}_4) \times (\mathbf{v}_3 - \mathbf{v}_4))|}{6} \tag{27}$$

Algorithm 4.2 (Defuzzification): .
Input: $T_{\tilde{A}}$ is a T2FS.
Output: $fDefuzz$ is output value.
For each triangle t_k of $T_{\tilde{A}}$

1) Make triangular prism with with six vertices: $v_1, v_2, v_3, v_{1'}, v_{2'}, v_{3'}$ in which abscissa y of $v_{1'}, v_{2'}, v_{3'}$ is zero.
2) Divide the triangular prism into three tetrahedrons in which their vertices are $(v_{1'}, v_1, v_2, v_3)$, $(v_{1'}, v_2, v_{2'}, v_3)$ and $(v_{1'}, v_3, v_{2'}, v_{3'})$.
3) Compute centroid of each tetrahedron.
4) Compute volume of each tetrahedron as equation (27).
5) Compute centroid of the space of uncertainty of $\tilde{A}$ as equation (26).
6) Abscissa $fDefuzz = x$ of the centroid is the output of type-reduction and defuzzification.

V. Experimental Results

A. Comparisons

Algorithms are experimented to get comparisons in runtime and accuracy between the proposed method and previous methods. On runtime, the proposed representation is compared with TIN-based representation [15]. All of algorithms are implemented on Laptop HP DV9500 Core 2 Duo T7500 RAM 2GB with statistics summarised in the following table.

TABLE I
Sum of cumulative time of 100 replications (in seconds)

	Meet	Join	Negation	Inference
TIN-based representation	2.278	2.293	0.003	1.880
The proposed method	0.0118	0.0205	0.000015	0.0128

On the manner of accuracy, the proposed method (RCTIN) is compared with zSlice-based representation [18], [19]. I is number of slices/contours of a zSlice type-2 fuzzy sets or RCTIN type-2 fuzzy sets. Results on accuracy, summarised in the table II, show that RCTIN type-2 fuzzy sets accurately represent the original type-2 fuzzy sets in case primary and secondary membership grades is piece-wise linear functions.

TABLE II
Comparison on errors in representation

I	3	5	10	50	100
zSlice (triangle)	0.0251	0.0156	0.0075	0.0015	0.00075
RCTIN (triangle)	0.0	0.0	0.0	0.0	0.0
zSlice (Gaussian)	0.0623	0.0374	0.0187	0.00374	0.00187
RCTIN (Gaussian)	0.0039	0.0023	0.0012	0.00023	0.00012

B. Problems

We implement RCTIN-based type-2 fuzzy logic systems with collision avoidance behavior of robot navigation. The fuzzy logic systems have two inputs: the extended fuzzy directional relation [23] and range to obstacle, the output is angle of deviation (AoD). The behavior is experimented in comparisons between type-1 FLS and RCTIN-based general type-2 FLS. The fuzzy rule has the form as following:

IF FDR is $\tilde{A}_i$ AND Range is $\tilde{B}_i$ THEN AoD is $\tilde{C}_i$
where $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$ are type-2 fuzzy sets of antecedent and consequent, respectively.

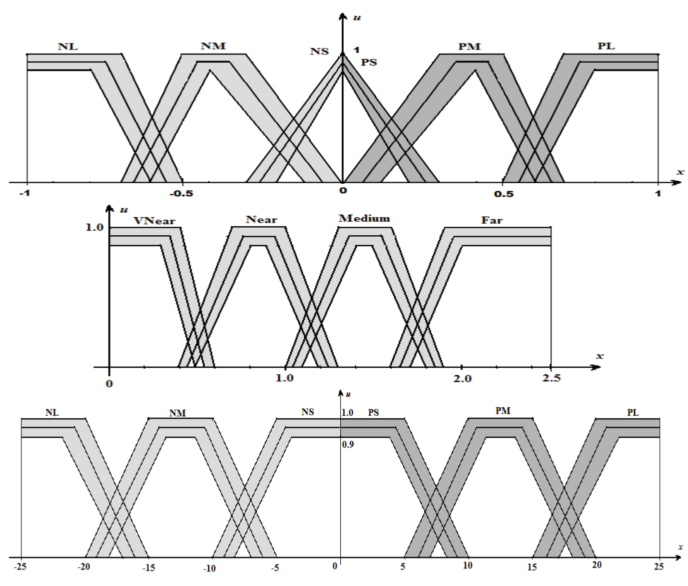

Fig. 9. Membership grades of CA behavior.

The fuzzy directional relation has six linguistic values (NLarge, NMedium, NSmall, PSmall, PMedium and PLarge). The range from robot to obstacle is divided in three subsets: VNear, Near, Medium and Far. The output of fuzzy if-then is a linguistic variable representing for angle of deviation, has six linguistic variables the same the fuzzy directional relation with the different membership functions. Linguistic values are general type-2 fuzzy subsets that membership functions are described in Fig. 9. Membership grades are general type-2 fuzzy sets with vertical slide is triangular in which $z = 0$ for all points on upper/lower MF and $z = 1$ for all points on middle MF. For type-1 behavior, linguistic variables of inputs and output are the same type-2 behavior and their membership functions are upper MF of type-2 membership functions. RCTIN-based type-2 fuzzy subsets are triangulated at 5 levels. The rule-base is described in the table III.

TABLE III
THE RULE BASE OF COLLISION AVOIDANCE BEHAVIOR

FDR	Range	AoD	FDR	Range	AoD
NS	VN	PL	PS	VN	NL
NS	N	PL	PS	N	NL
NS	M	PM	PS	M	NM
NS	F	PS	PS	F	NS
NM	VN	PM	PM	VN	NM
NM	N	PM	PM	N	NM
NM	M	PM	PM	M	NM
NM	F	PS	PM	F	NS
NL	VN	PM	PL	VN	NM
NL	N	PM	PL	N	NM
NL	M	PS	PL	M	NS
NL	F	PS	PL	F	NS

RCTIN-based type-2 FLS is tested on whole of input spaces that is discretized with step of 0.1 in which $FDR \in [-1, 1]$ and $Range \in [0.5, 2.5]$. Fig. 10 shows inference surface of avoidance behavior based on RCTIN Type-2 FLS. Results is summarized as follows:

TABLE IV
EXPERIMENTAL RESULTS OF AVOIDANCE BEHAVIOR

	Type-1 FLS	RCTIN Type-2 FLS
Number of discretized points	441	441
Run-time (in milliseconds) for all of discretized input space	92	524
Minimum AoD	-21.111	-20.54
25%-tile AoD	-4.405	-4.676
Median AoD	-3.889	0
75%-tile AoD	4.222	5.591
Maximum AoD	18.359	18.058
Mean AoD	-0.7178	-0.3172

VI. CONCLUSION

An approach to RCTIN-based representation of generalized type-2 fuzzy sets is presented with implementations of avoidance behavior. The basis for this approach is ε_u representation of type-2 membership grades. Geometric operations involving join under minimum, meet under minimum, negation, inference and defuzzification are developed base on this representation. Geometric computation of RCTIN-based general type-2 fuzzy logic system are implemented under avoidance behavior of robot navigation. Results in comparisons with previous approach show that the proposed approach considerably reduce computational complexity.

The next goal is to implement the general type-2 FLS in applications under other hardware such as FGPA or embedded systems. And other approach is to use RCTIN for intelligent systems such as ANFIS or TSK fuzzy logic systems.

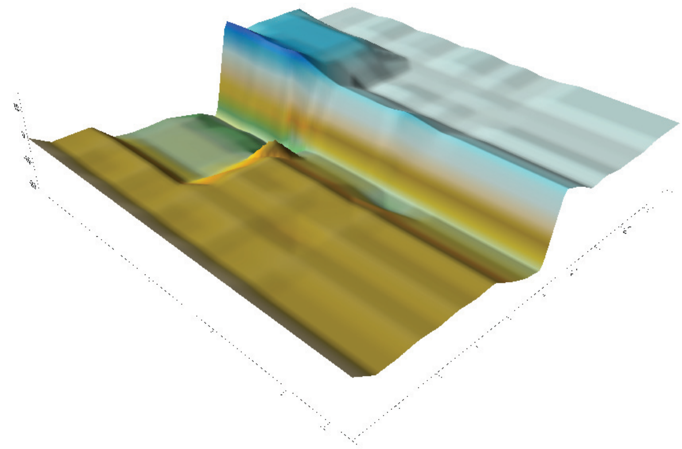

Fig. 10. Inference Surface of Type-2 avoidance behavior

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