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On random equations and applications to random fixed point theorems

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Abstract. In this paper, some theorems on the equivalence between the solvability of a random operator equation and the solvability of a deterministic operator equation are presented. As applications and illustrations, some results on random fixed points and random coincidence points in the literature are obtained or extended.

Keywords. Random operator, continuous random operator, multivalued random operator, continuous multivalued random operator, measurable random operator, measurable multivalued random operator, random equation, random fixed point, random coincidence point, common random fixed point.

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1 Introduction and preliminaries

Random fixed point theory for singlevalued and multivalued random operators are stochastic generalizations of classical fixed point theory for singlevalued and multivalued deterministic mappings. It has received much attention in recent years; see, for example, [3], [4], [6], [20], [25], [32], [35], etc. and references therein. Some authors (see, e.g. [4], [29], [30], [35]) have shown that under some assumptions the existence of a deterministic fixed point is equivalent to the existence of a random fixed point. In this case every deterministic fixed point theorem produces a random fixed point theorem.

In this paper we shall deal with random equations for singlevalued and multivalued random operators. The main results of this paper are sufficient conditions ensuring that the solvability of a deterministic equation is equivalent to the solvability of a corresponding random equation. As applications and illustrations, some results on random fixed points and random coincidence points in the literature (e.g. [2], [3], [4], [13], [20], [22], [29], [31] and [32]) are obtained or extended.

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y be Polish spaces (i.e. completely separable metric spaces). We denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, by 2^X the family of all nonempty subsets of X, by $C(X)$ the family of all nonempty closed subsets of X and by $CB(X)$ the family of all nonempty closed and bounded subsets of X. The σ-algebra on $\Omega \times X$ is denoted by $\mathcal{F} \times \mathcal{B}(X)$. Noting that in general $\mathcal{B}(X \times Y)$ contains $\mathcal{B}(X) \times \mathcal{B}(Y)$ and $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if X and Y are Suslin spaces (i.e. X, Y are Hausdorff and the continuous images of Polish spaces). The Hausdorff metric induced by d on $C(X)$ is given by

$$H(A, B) = \max\Big\{ \sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}$$

for $A, B \in C(X)$, where $d(a, B) = \inf_{b\in B} d(a, b)$ is the distance from a point $a \in X$ to a subset $B \subset X$.

Let $(E, \mathcal{A})$ be a measurable space. A mapping $u\colon E \to X$ is said to be $\mathcal{A}$-measurable if $u^{-1}(B) = \{\omega \in E \,|\, u(\omega) \in B\} \in \mathcal{A}$ for any $B \in \mathcal{B}(X)$. If $\xi\colon \Omega \to X$ is $\mathcal{F}$-measurable, then ξ is called an X-valued random variable. A set-valued mapping $F\colon E \to 2^X$ is called a multivalued mapping and it is said to be $\mathcal{A}$-measurable if $F^{-1}(B) = \{\omega \in E \,|\, F(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$ for each open subset B of X (note that in Himmelberg [16] this is called weakly measurable). The graph of F is defined by

$$Gr(F) = \{(\omega, x) \,|\, \omega \in E, x \in F(\omega)\}.$$

An $\mathcal{F}$-measurable multivalued mapping $\Phi\colon \Omega \to 2^X$ is called an X-multivalued random variable.

We recall the concept of random operators and multivalued random operators.

Definition 1.1. (i) A mapping $f\colon \Omega \times X \to Y$ is said to be a random operator if for each $x \in X$, the mapping $f(\cdot, x)$ is a Y-valued random variable, where $f(\cdot, x)$ denotes the mapping $\omega \mapsto f(\omega, x)$.

(ii) A mapping $T\colon \Omega \times X \to 2^Y$ is said to be a multivalued random operator if for each $x \in X$, the mapping $T(\cdot, x)$ is a Y-multivalued random variable, where $T(\cdot, x)$ denotes the mapping $\omega \mapsto T(\omega, x)$.

(iii) The random operator $f\colon \Omega \times X \to Y$ is said to be measurable if the mapping $f\colon \Omega \times X \to Y$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable.

(iv) The multivalued random operator $T\colon \Omega \times X \to 2^Y$ is said to be measurable if the mapping $T\colon \Omega \times X \to 2^Y$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable.

(v) The random operator $f\colon \Omega \times X \to Y$ is said to be continuous if for each ω the mapping $f(\omega, \cdot)$ is continuous, where $f(\omega, \cdot)$ denotes the mapping $x \mapsto f(\omega, x)$.

(vi) The multivalued random operator $T: \Omega \times X \to C(Y)$ is said to be continuous if for each ω the mapping $T(\omega, \cdot)$ is continuous, where $T(\omega, \cdot)$ denotes the mapping $x \mapsto T(\omega, x)$.

For later convenience, we list the following three theorems.

Theorem 1.2 ([16, Theorem 6.1]). *Let X be a separable metric space, Y a metric space and $f: \Omega \times X \to Y$ such that $f(\cdot, x)$ is measurable for each x and $f(\omega, \cdot)$ is continuous for each ω. Then f is measurable.*

Theorem 1.3 ([16, Theorem 3.3]). *Let X be a separable metric space and let $F: \Omega \to C(X)$ be a multivalued mapping. Then the following three statements are equivalent:*

a) *F is $\mathcal{F}$-measurable;*

b) *For each x, the mapping $\omega \mapsto d(x, F(\omega))$ is $\mathcal{F}$-measurable;*

c) *$Gr(F)$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable.*

Theorem 1.4 ([16, Theorem 5.7]). *Suppose that X is a Suslin space and that $F: \Omega \to 2^X$ is a multivalued mapping. If $Gr(F)$ is measurable, then there is an X-valued random variable $\xi: \Omega \to X$ such that $\xi(\omega) \in F(\omega)$ a.s.*

2 Random equations

Definition 2.1. Let $f, g: \Omega \times X \to Y$ be random operators. Consider the random equation of the form

$$f(\omega, x) = g(\omega, x). \tag{2.1}$$

We say that equation (2.1) has a deterministic solution for almost all ω if there is a set D of probability one such that for each $\omega \in D$ there exists $u(\omega) \in X$ such that

$$f(\omega, u(\omega)) = g(\omega, u(\omega)).$$

An X-valued random variable $\xi: \Omega \to X$ is said to be a random solution of equation (2.1) if

$$f(\omega, \xi(\omega)) = g(\omega, \xi(\omega)) \quad \text{a.s.}$$

Clearly, if equation (2.1) has a random solution, then it has a deterministic solution for almost all ω. However, the following simple example shows that the converse is not true.

Example 2.2. Let $\Omega = [0,1]$ and let $\mathcal{F}$ be the family of subsets $A \subset \Omega$ with the property that either A is countable or the complement A^c is countable. Define a probability measure P on $\mathcal{F}$ by

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that $(\Omega, \mathcal{F}, P)$ forms a complete probability space. Let $X = [0,1]$. Define two mappings $f, g \colon \Omega \times X \to X$ by

$$f(\omega, x) = \begin{cases} x & \text{if } \omega = x, \\ 1 & \text{otherwise,} \end{cases}$$

$$g(\omega, x) = \begin{cases} x & \text{if } \omega = x, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that f, g are random operators and for each $\omega \in \Omega$, $u(\omega) = \omega$ is a solution of equation (2.1). Suppose that ξ is a random solution of equation (2.1). Then $\xi(\omega) = \omega$ a.s. Hence, the mapping $u \colon \Omega \to X$ defined by $u(\omega) = \omega$ must be $\mathcal{F}$-measurable. For $B = [0, 1/2) \in \mathcal{B}(X)$ we have $u^{-1}(B) = B = [0, 1/2) \notin \mathcal{F}$ showing that u is not $\mathcal{F}$-measurable and we get a contradiction.

The following theorem gives a sufficient condition on f, g ensuring that the existence of a deterministic solution for almost all ω is equivalent to the existence of a random solution.

Theorem 2.3. *Let X, Y be Polish spaces and $f, g \colon \Omega \times X \to Y$ measurable random operators. Then the random equation $f(\omega, x) = g(\omega, x)$ has a random solution if and only if it has a solution for almost all ω.*

Moreover, if for almost all ω the equation $f(\omega, \cdot) = g(\omega, \cdot)$ has a unique solution, then the random equation $f(\omega, x) = g(\omega, x)$ has a random unique solution.

Proof. It suffices to prove the part "if".

Suppose that the random equation $f(\omega, x) = g(\omega, x)$ has a solution for almost all ω. Without lost of generality, we suppose that it has a solution $u(\omega)$ for all ω. Define a mapping $F \colon \Omega \to 2^{X \times Y}$ by

$$F(\omega) = \{(x, y) \mid x \in X, f(\omega, x) = g(\omega, x) = y\}.$$

Because of $(u(\omega), v(\omega)) \in F(\omega)$, where $v(\omega) = f(\omega, u(\omega))$, F has non-empty values for all ω, so F is a multivalued mapping. We shall show that F has a measurable graph.

By Theorem 1.3, f and g have measurable graphs, i.e. $Gr(f), Gr(g) \in (\mathcal{F} \times \mathcal{B}(X)) \times \mathcal{B}(Y)$. We have

$$\begin{aligned} Gr(f) &= \{(\omega, x, y) \mid \omega \in \Omega, x \in X, f(\omega, x) = y\}, \\ Gr(g) &= \{(\omega, x, y) \mid \omega \in \Omega, x \in X, g(\omega, x) = y\}, \\ Gr(F) &= \{(\omega, x, y) \mid \omega \in \Omega, x \in X, f(\omega, x) = g(\omega, x) = y\}. \end{aligned}$$

It is clear that

$$Gr(F) = Gr(f) \cap Gr(g).$$

Hence, $Gr(F) \in (\mathcal{F} \times \mathcal{B}(X)) \times \mathcal{B}(Y) = \mathcal{F} \times \mathcal{B}(X \times Y)$.

By Theorem 1.4, there exists a measurable mapping $\xi\colon \Omega \to X \times Y$ such that $\xi(\omega) \in F(\omega)$ a.s. Let $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega))$. We have

$$f(\omega, \xi_1(\omega)) = g(\omega, \xi_1(\omega)) = \xi_2(\omega) \quad \text{a.s.}$$

Since ξ is measurable, $\xi_1\colon \Omega \to X$ is also measurable. Thus ξ_1 is a random solution of the random equation $f(\omega, x) = g(\omega, x)$.

Now, assume that for almost all ω the equation $f(\omega, x) = g(\omega, x)$ has a unique solution and ξ, η are two random solutions. From this it follows that $\xi(\omega) = \eta(\omega)$ a.s. and we are done. □

Corollary 2.4. *Let X, Y be Polish spaces and $f, g\colon \Omega \times X \to Y$ continuous random operators. Then the random equation $f(\omega, x) = g(\omega, x)$ has a random solution if and only if it has a solution for almost all ω.*

Moreover, if for almost all ω the equation $f(\omega, x) = g(\omega, x)$ has a unique solution, then the random equation $f(\omega, x) = g(\omega, x)$ has a random unique solution.

Proof. By Theorem 1.2, f and g are measurable random operators. Hence the claims follows from Theorem 2.3. □

Thus, every theorem concerning the solvability of deterministic operator equations produces a theorem on random operator equations. As an illustration, we have the following theorem.

Theorem 2.5. (i) *Let h be a continuous random operator on a separable Hilbert space X satisfying the Lipschitz property, i.e. there exists a mapping $L\colon \Omega \to (0, \infty)$ such that for all $x_1, x_2 \in X, \omega \in \Omega$*

$$\|h(\omega, x_1) - h(\omega, x_2)\| \le L(\omega)\|x_1 - x_2\|.$$

Assume that $k(\omega)$ is a positive real-valued random variable such that

$$L(\omega) < k(\omega) \quad a.s.$$

Then for any X-valued random variable η, the random equation $h(\omega, x) + k(\omega)x = \eta(\omega)$ has a random unique solution.

(ii) *Let X be a separable Banach space and let $L(X)$ be the Banach space of linear continuous operators from X into X. Suppose that $A\colon \Omega \to L(X)$ is a mapping such that for each $x \in X$, the mapping $\omega \mapsto A(\omega)x$ is an X-valued random variable and $\lambda(\omega)$ is a real-valued random variable satisfying*

$$\|A(\omega)\| < \lambda(\omega) \quad a.s.$$

Then for any X-valued random variable η, the random equation

$$(A(\omega) - \lambda(\omega)I)x = \eta(\omega)$$

has a random unique solution which is denoted by $(A(\omega) - \lambda(\omega)I)^{-1}\eta$.

Proof. (i) The random equation under consideration is of the form $f(\omega, x) = g(\omega, x)$ where f, g are the random operators given by $f(\omega, x) = h(\omega, x) + k(\omega)x$, $g(\omega, x) = \eta(\omega)$. Clearly, f, g are continuous random operators. By the Lipschitz property of h, we have for all $x_1, x_2 \in X$

$$\begin{aligned}
&\big\langle f(\omega, x_1) - f(\omega, x_2), x_1 - x_2 \big\rangle \\
&= \big\langle h(\omega, x_1) - h(\omega, x_2), x_1 - x_2 \big\rangle + k(\omega) \cdot \|x_1 - x_2\|^2 \\
&\geq k(\omega) \cdot \|x_1 - x_2\|^2 - \|h(\omega, x_1) - h(\omega, x_2)\| \cdot \|x_1 - x_2\| \\
&\geq [k(\omega) - L(\omega)] \cdot \|x_1 - x_2\|^2 = m(\omega) \cdot \|x_1 - x_2\|^2 \quad \text{a.s.},
\end{aligned}$$

where $m(\omega) = k(\omega) - L(\omega) > 0$. Hence there is a set D of probability one such that for each $\omega \in D$ the mapping $f(\omega, \cdot)$ is strongly monotone. By the deterministic result due to Browder [8, Theorem 1], there exists a unique element $u(\omega) \in X$ such that $f(\omega, u(\omega)) = \eta(\omega)$. Hence, the equation $f(\omega, x) = g(\omega, x)$ has a unique solution for almost all ω. By Corollary 2.4 the random equation $h(\omega, x) + k(\omega)x = \eta(\omega)$ has a random unique solution.

(ii) The random equation under consideration is of the form $f(\omega, x) = g(\omega, x)$, where f, g are the random operators given by

$$f(\omega, x) = A(\omega)x - \lambda(\omega)x, \quad g(\omega, x) = \eta(\omega).$$

Clearly, f, g are continuous random operators. By assumption and the well-known deterministic result, for almost all ω there exists a unique element $u(\omega) \in X$ such that $f(\omega, u(\omega)) = \eta(\omega)$. Hence the equation $f(\omega, x) = g(\omega, x)$ has a unique solution for almost all ω. By Corollary 2.4 the random equation $(A(\omega) - \lambda(\omega)I)x = \eta(\omega)$ has a random unique solution. □

Now we extend Theorem 2.3 to the case of multivalued random operators.

Definition 2.6. Let $S, T : \Omega \times X \to C(Y)$ be multivalued random operators. Consider the random equation of the form

$$S(\omega, x) \cap T(\omega, x) \neq \emptyset. \tag{2.2}$$

We say that the random equation (2.2) admits a deterministic solution for almost all ω if there is a set D of probability one such that for each $\omega \in D$ there exists $u(\omega) \in X$ such that

$$S(\omega, u(\omega)) \cap T(\omega, u(\omega)) \neq \emptyset.$$

An X-valued random variable $\xi : \Omega \to X$ is said to be a random solution of the equation (2.2) if

$$S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega)) \neq \emptyset \quad \text{a.s.}$$

The following theorem gives a sufficient condition under which the existence of a deterministic solution for almost all ω is equivalent to the existence of a random solution.

Theorem 2.7. *Let X and Y be Polish spaces and let $S, T : \Omega \times X \to C(Y)$ be measurable multivalued random operators. Then the random equation $S(\omega, x) \cap T(\omega, x) \neq \emptyset$ has a random solution if and only if it has a solution for almost all ω.*

More generally, let $T_n : \Omega \times X \to C(Y)$ be measurable multivalued random operators ($n = 1, 2, \ldots$). Then the random equation $\bigcap_{n=1}^{\infty} T_n(\omega, x) \neq \emptyset$ has a random solution if and only if it has a solution for almost all ω.

Proof. It suffices to prove the part "if". Suppose that equation (2.2) has a solution for almost all ω. Without lost of generality, we suppose that equation (2.2) has a solution for any ω. Let $F : \Omega \to 2^{X \times Y}$ be a mapping defined by

$$F(\omega) = \{(x, y) \mid x \in X, y \in S(\omega, x) \cap T(\omega, x)\}.$$

Since equation (2.2) has a solution for any ω, the mapping F has non-empty values for all ω, so F is a multivalued mapping. We shall show that F has a measurable graph.

We have

$$Gr(S) = \{(\omega, x, y) \mid \omega \in \Omega, x \in X, y \in S(\omega, x)\},$$
$$Gr(T) = \{(\omega, x, y) \mid \omega \in \Omega, x \in X, y \in T(\omega, x)\},$$
$$Gr(F) = \{(\omega, x, y) \mid \omega \in \Omega, x \in X, y \in S(\omega, x) \cap T(\omega, x)\}.$$

It is clear that

$$Gr(F) = Gr(S) \cap Gr(T).$$

By Theorem 1.3, S and T have measurable graphs, i.e. $Gr(S), Gr(T) \in (\mathcal{F} \times \mathcal{B}(X)) \times \mathcal{B}(Y)$. Hence, $Gr(F) \in (\mathcal{F} \times \mathcal{B}(X)) \times \mathcal{B}(Y) = \mathcal{F} \times \mathcal{B}(X \times Y)$.

By Theorem 1.4, there exists a measurable function $\xi\colon \Omega \to X \times Y$ such that $\xi(\omega) \in F(\omega)$ a.s. Let $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega))$. We have

$$\xi_2(\omega) \in S(\omega, \xi_1(\omega)) \cap T(\omega, \xi_1(\omega)) \quad \text{a.s.}$$

Since ξ is measurable, $\xi_1\colon \Omega \to X$ is also measurable. Thus ξ_1 is a random solution of the equation $S(\omega, x) \cap T(\omega, x) \neq \emptyset$.

A similar argument can be used for the general random equation

$$\bigcap_{n=1}^{\infty} T_n(\omega, x) \neq \emptyset.$$

□

The above theorem shows that the measurability of S, T together with the existence of the deterministic solution for almost all ω implies the existence of a random solution. The converse is not true as the following simple example illustrates.

Example 2.8. Let $\Omega = \{0, 1\}$, $\mathcal{F} = \{\emptyset, \Omega\}$, $X = [0, 1], Y = [2, 3]$ and let $T\colon \Omega \times X \to C(Y)$ be a mapping defined by $T(0, x) = T(1, x) = Y$ for any $x \in X$. Let D be a non-Borel subset of X. We define $S\colon \Omega \times X \to C(Y)$ by

$$S(0, x) = S(1, x) = \begin{cases} Y & \text{if } x \in D, \\ \{2\} & \text{if } x \in \overline{D}, \end{cases}$$

where $\overline{D} = X \setminus D$. It is easy to check that S and T are multivalued random operators. Let $B = (2, 3)$. Because

$$S^{-1}(B) = \{(\omega, x) \mid S(\omega, x) \cap B \neq \emptyset\} = \Omega \times D \notin \mathcal{F} \times \mathcal{B}(X),$$

S is not measurable. However, the X-valued random variable ξ defined by $\xi(\omega) = c$ for any ω, where c is an arbitrary element of X, is a random solution of the random equation $S(\omega, x) \cap T(\omega, x) \neq \emptyset$.

Corollary 2.9. *Let X and Y be Polish spaces and $T_n: \Omega \times X \to C(Y)$ continuous multivalued random operators ($n = 1, 2, \ldots$). Then the random equation $\bigcap_{n=1}^{\infty} T_n(\omega, x) \neq \emptyset$ has a random solution if and only if it has a solution for almost all ω.*

Proof. By Theorem 2.7, it suffices to show that if $T: \Omega \times X \to C(Y)$ is a continuous multivalued random operator, then T is a measurable multivalued random operator. By Theorem 1.3, to prove the measurability of T, we prove the measurability of the mapping $(\omega, x) \mapsto d(y, T(\omega, x))$ for each $y \in Y$. Define $\varphi_y: \Omega \times X \to R$ by $\varphi_y(\omega, x) = d(y, T(\omega, x))$. By the continuity of the mapping $x \mapsto T(\omega, x)$ it follows that $\varphi_y(\omega, x)$ is continuous w.r.t. x. We now prove the measurability of $\varphi_y(\omega, x)$ w.r.t. ω. Indeed, for each fixed x, $T(\omega, x)$ is measurable, so $\omega \mapsto d(y, T(\omega, x))$ is measurable by Theorem 1.3. By Theorem 1.2, φ_y is measurable. This means that $(\omega, x) \mapsto d(y, T(\omega, x))$ is measurable for each $y \in Y$ and we are done. □

3 Applications to random fixed point theorems

Let X be a separable metric space and C a nonempty complete subset of X, let $f: \Omega \times C \to X$ be a random operator and $T: \Omega \times C \to 2^X$ a multivalued random operator. Recall that

(i) an X-valued random variable ξ is said to be a random fixed point of f if $f(\omega, \xi(\omega)) = \xi(\omega)$ a.s.,

(ii) an X-valued random variable ξ is said to be a random fixed point of T if $\xi(\omega) \in T(\omega, \xi(\omega))$ a.s.,

(iii) an X-valued random variable ξ is called a random coincidence point of the pair (f, T) if $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ a.s.

As a concequence of Theorem 2.3 and Theorem 2.7 we get the following random fixed point theorem.

Theorem 3.1. *Let X be a Polish space, $f: \Omega \times C \to X$ a measurable random operator and $T: \Omega \times C \to C(X)$ a measurable multivalued random operator.*

(i) *f has a random fixed point if and only if for almost all ω the mapping $f(\omega, \cdot)$ has a fixed point.*

(ii) *T has a random fixed point if and only if for almost all ω the mapping $T(\omega, \cdot)$ has a fixed point.*

(iii) *The pair of random operators* (f, T) *has a random coincidence point if and only if for almost all* ω *the pair of mappings* $(f(\omega,\cdot), T(\omega,\cdot))$ *has a coincidence point (i.e. there exists* $u(\omega)$ *such that* $f(\omega, u(\omega)) \in T(\omega, u(\omega))$*).*

(iv) *Let* $S\colon \Omega \times C \to C(X)$ *be another measurable multivalued random operator. Then the pair* (S, T) *has a common random fixed point if and only if for almost all* ω *the pair of mappings* $(S(\omega,\cdot), T(\omega,\cdot))$ *has a common fixed point.*

Proof. (i) Use Theorem 2.3 for the random equation $f(\omega, x) = g(\omega, x)$, where $g(\omega, x) = x$.

(ii) Use Theorem 2.7 for the random equation $T(\omega, x) \cap S(\omega, x) \neq \emptyset$, where $S(\omega, x) = \{x\}$.

(iii) Use Theorem 2.7 for the random equation $T(\omega, x) \cap S(\omega, x) \neq \emptyset$, where $S(\omega, x) = \{f(\omega, x)\}$.

(iv) Use Theorem 2.7 for the random equation

$$T(\omega, x) \cap S(\omega, x) \cap R(\omega, x) \neq \emptyset,$$

where $R(\omega, x) = \{x\}$. □

Remark.

- Claim 1 extends [27, Lemma 3.1], which plays a crucial role in the proof of its main results, where it is assumed that f is a continuous random operator satisfying the so-called condition (A).
- Claim 2 removes some assumptions on T in Theorem 3.1, Theorem 3.2 and Theorem 3.3 of [4].
- Claim 3 extends and improves Theorem 3.1, Theorem 3.3 and Theorem 3.12 in [29], which contains most of the known random fixed point theorems as special cases (see [29, Remark 3.16]).

In view of Theorem 3.1 every fixed point theorem for deterministic mappings or multivalued deterministic mappings gives rise to some random fixed point theorems for random operators or multivalued random operators, respectively. As illustrations we have the following theorems.

Theorem 3.2. *Let* X *be a Polish space and* $f\colon \Omega \times X \to X$ *a measurable random operator satisfying the following contractive condition:*

For all $\omega \in \Omega$ *and all* $x, y \in X$

$$d(f(\omega,x), f(\omega,y)) \le \lambda(\omega) \max \Big\{d(x,y), d(x, f(\omega,x)), d(y, f(\omega,y)), \frac{1}{2}[d(x, f(\omega,y)) + d(y, f(\omega,x))]\Big\} + \beta(\omega) \max\{d(x, f(\omega,x)), d(y, f(\omega,y)\} + \gamma(\omega)[d(x, f(\omega,y)) + d(y, f(\omega,x)],$$

where $\lambda, \beta, \gamma \colon \Omega \to (0,1)$ *are mappings such that* $\lambda + \beta + 2\gamma = 1$.

Then f *has a random unique fixed point.*

Proof. For each fixed ω, by Ciric [12, Theorem 2.1], $f(\omega,\cdot)$ has a unique fixed point. By Theorem 3.1, f has a random unique fixed point. □

Theorem 3.3. *Let* X *be a Polish space,* $f\colon \Omega \times X \to X$ *a random operator and let* $T\colon \Omega \times X \to CB(X)$ *be a multivalued random operator such that* $T(\omega, X) \subset f(\omega, X)$ *for all* ω *and that for all* $x, y \in X$

$$H(T(\omega,x), T(\omega,y)) \le \lambda(\omega) \max \Big\{d(f(\omega,x), f(\omega,y)), d(f(\omega,x), T(\omega,x)), d(f(\omega,y), T(\omega,y)), \frac{1}{2}[d(f(\omega,x), T(\omega,y)) + d(f(\omega,y), T(\omega,x))]\Big\},$$

where $\lambda\colon \Omega \to [0,1)$. *In addition, suppose that for each* ω *either*

(i) $f(\omega,\cdot)$ *and* $T(\omega,\cdot)$ *are continuous and compatible*

or

(ii) f *and* T *are measurable and* $T(\omega, X)$ *or* $f(\omega, X)$ *is complete.*

Then f *and* T *have a random coincidence point.*

Recall that the mappings $T\colon X \to CB(X)$ and $f\colon X \to X$ are compatible if for each sequence (x_n) in X satisfying $\lim f x_n \in \lim T x_n \in CB(X)$ we have $\lim H(f T x_n, T f x_n) = 0$.

Proof. By [18, Theorem 2] and [20, Theorem 1.4], for each ω, $f(\omega,\cdot)$ and $T(\omega,\cdot)$ have a coincidence point. By Theorem 3.1, f and T have a random coincidence point. □

Theorem 3.3 extends I. Beg and N. Shahzad [3, Theorem 5.1], which claims that if for all $x, y \in X, \omega \in \Omega$,

$$H(T(\omega, x), T(\omega, y)) \leq \lambda(\omega) d(f(\omega, x), f(\omega, y)),$$

where $\lambda\colon \Omega \to (0, 1)$ is measurable, $T(\omega, X) \subset f(\omega, X)$ and $f(\omega, \cdot)$, $T(\omega, \cdot)$ are continuous and compatible, then f and T have a random coincidence point.

Theorem 3.4. *Let X be a Polish space and let $S, T\colon \Omega \times X \to C(X)$ be measurable multivalued random operators. If*

$$H(S(\omega, x), T(\omega, y)) \leq \lambda(\omega) \max \Big\{ d(x, y), d(x, S(\omega, x)), d(y, T(\omega, y)), \frac{1}{2}[d(y, S(\omega, x)) + d(x, T(\omega, y))] \Big\}$$

for all $x, y \in X$, $\omega \in \Omega$, where $\lambda\colon \Omega \to (0, 1)$, then S and T have a common random fixed point.

Proof. By [34, Corollary 2.6], $S(\omega, \cdot)$ and $T(\omega, \cdot)$ have a common deterministic fixed point. By Theorem 3.1, S and T have a common random fixed point. □

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