New Geometrically Uniform Codes Using QAM which Improve the Performance/Complexity Tradeoff of 2D codes

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***Abstract*— Given a signal set whose Geometrically Uniform (GU) partition allows binary isometric labelings, a GU code $C(\mathrm{g},C)$ is determined by a binary label code C and a binary isometric labeling g. Two such codes are defined to be equivalent if they have the same transmission rate and the same transfer function $T(D)$. It is then shown that all binary isometric labelings for the given GU partition are equally good so that it is enough to search for good codes using a single fixed isometric labeling. It is also shown that the set of codes with the same number of states is partitioned into subsets of equivalent codes. This allows a fast search algorithm which rejects all but one of codes that are equivalent to each other. New 128-, 256-, and 512-state codes using QAM constellations are given to improve the performance/complexity tradeoff of 2D trellis codes.**

I. Introduction

This work is focused on the search for good GU codes [1] with Quadrature Amplitude Modulation (QAM) for an Additive White Gaussian Noise (AWGN) channel. The work has been motivated initially as an effort to confirm a conclusion of Forney and Ungerboeck in [2]. They compared effective coding gains versus complexity for 1D, 2D Ungerboeck codes [3], and multidimensional Wei codes [4] accepted for V.32 and V.34 (Fig. 6, p. 356, [2]) and made a conclusion that (up to that time) no one has improved on the performance vs. complexity tradeoff of the original 1D and 2D trellis codes of Ungerboeck. A familiar comparison is given in [5]. The performance/complexity curve for 2D Ungerboeck codes in [2] and [5] shows that the 128- and 256-state codes may be not the best ones for those numbers of states since, intuitively, the curve must look smoother due to very nice symmetry properties of the QAM constellation, the 8-way GU partition, and the class of linear label codes. This stimulates a new search for codes that might give a better performance.

To this end, however, we must solve two unsolved problems. The first problem is associated with the method we use to generate the subclass of codes that we have to search for good codes. Normally, if a GU signal set has a GU partition, which allows binary isometric labelings, then one isometric labeling that seems to be good for coding is chosen. This isometric labeling is, in general, constructed according to Ungerboeck's rules of "mapping by set partitioning" [3]. Since there are many binary isometric labelings for a given GU partition (we will show in this paper how many they are), then a question arises whether the selected isometric labeling is best.

The second problem is associated with the performance measure we use to compare different codes in the search for good codes. So far, we compute the free distance d_{free} and the number of codewords at free distance N_{free} for each code and choose the best code that has the smallest N_{free} between the codes that have the largest d_{free}. However, codes with the largest d_{free} and smallest N_{free} can be proved to be optimal only at large signal-to-noise power ratio (SNR). To search for really a good GU code, we have to compare codes in terms of relevant performance measures, like the first event error probability or the bit error probability. Thus, to be sure that particular codes are the best ones, in terms P_e of and P_b, for the given GU partition we have a) to do an exhaustive search over all possible label codes and all isometric labelings and b) to compare codes in terms of the given performance measure. However, such an exhaustive code search has been always proved to be time-consuming and impractical.

In this paper we show that it is possible to carry out an exhaustive search for codes with a large state size. In Sec. II, we show that, for the given GU partition, all isometric labelings are generated from a fixed one by the action of the group of nonsingular matrices. This tells us the number of isometric labelings associated with the given GU partition. This also allows us to combine two linear transforms, one by the convolutional encoding and one by the isometric labeling as it is done in Section III. To this end we introduce a definition of equivalent codes and show that the code equivalence allows reducing the number of codes to be generated and to be evaluated in the search. We show that, given a GU partition, all isometric labelings are equally good for coding and, hence, this gives a proof of Ungerboeck's conjecture associated with his famous three rules of "mapping by set partitioning." In Sec. IV, we give a code search algorithm for good GU codes based on the numerical computation of P_e. The new algorithm is more accurate and less time-consuming compared to code search algorithm reported so far. The result of an exhaustive search is new 128-, 256-, and 512 codes with QAM that have better performance than 2D Ungerboeck codes.

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II. THE NUMBER OF ISOMETRIC LABELINGS AND THE PROBLEM OF EXHAUSTIVE CODE SEARCH

A. Geometrical Uniformity (GU) and GU Partition

A *signal set* S is a set of discrete points in an N-dimensional Euclidean space $\mathbf{R}^N$. An *isometry* u of $\mathbf{R}^N$ is a mapping of $\mathbf{R}^N$ onto itself that preserves the *Euclidean distance* as $\| u(\mathbf{x}) - u(\mathbf{y}) \|^2 = \| \mathbf{x} - \mathbf{y} \|^2$, where $u(\mathbf{x})$ denotes the image of $\mathbf{x}$ under the transformation u. All isometries of $\mathbf{R}^N$ can be derived from the three "primitive" transformations: *translation* (in a certain direction), *rotation* (about a certain line or axis), and *reflection* (relative to a certain hyperplane). For a given isometry u of $\mathbf{R}^N$ and a set $S \in \mathbf{R}^N$, $u(S) = \{u(\mathbf{x}) : \mathbf{x} \in S\}$ denotes the image of S under u. A *symmetry* of S is an isometry u that leaves S invariant, that is $u(S) = S$. All symmetries of S form a group $\Gamma(S)$, with respect to composition operation, called the *symmetry group* of S. Then a signal set S is *geometrically uniform* (GU) if, given any two points $\mathbf{x}$ and $\mathbf{y}$ in S, there exists a symmetry u of S such that $u(\mathbf{x}) = \mathbf{y}$. We can also say that S is GU if it is generated from a point $\mathbf{x} \in S$ under the action of its symmetry group. The subgroup of $\Gamma(S)$ that is minimally sufficient to generate S from its arbitrary point is called the *generating group* of S.

Let G be a generating group of S. We are interested in a smallest normal subgroup N of G such that a) the quotient group G/N is isomorphic to Z_2^k for some integer k and b) the minimum squared distance (MSD) of subsets in the partition of S induced by G/N is maximal. If we can find such an N, then G/N induces a 2^k-way GU partition S/S_0 which allows a binary isometric labeling $m : \mathbf{Z}_2{}^k \to S/(S_0$, where $S_0 = N(s_0)$. Such a 2^k-way GU partition $S_0 = N(s_0)$ then gives a binary partition chain of S with symmetry $g_i \in G$ characterizing the partition at level $i, 1 \le i \le k$. These symmetries are normally selected such that the MSD of subsets in the partition increases with the partition level (Ungerboeck's rules for set partitioning). An ordered subset $\boldsymbol{g} = \{g_1, g_2, \ldots, g_k\}$ of G represents the binary partition chain of S if any coset X of S_0 in S can be expressed as $X = g_1^{z_1} g_2^{z_2} \cdots g_k^{z_k}(S_0)$, where $g_i^0 = e$ is the identity mapping, and $g_i^1 = g_i$. The binary vector $\mathbf{z} = (z_k, \cdots, z_2, z_1)$ is then called the *binary isometric label* of the coset X. The symmetry g_i defines the partition at level i so that the subset

$$S_{k-i+1} = \{g_i^{z_i} g_{i+1}^{z_{i+1}} \cdots g_k^{z_k}(S_0) : (z_i, z_{i+1}, \ldots, z_k) \in Z_2{}^{k-i+1}\}$$

is refined by g_i into two subsets S_{k-i} and $g_i(S_{k-i})$. Let d_i^2 be the MSD of the subset S_{k-i}. Then the partition chain $\boldsymbol{g} = \{g_1, g_2, \ldots, g_k\}$ gives the distance chain $d_1^2 / d_2^2 / \ldots / d_k^2$.

B. The number of binary isometric labelings

Here $< u_1, u_2, \ldots, u_k >$ denotes the group generated by $u_1, u_2, \ldots, u_k$. We define a partition chain of $\{G/N\}$ as follows.

Definition 2.1 We call an ordered subset $\boldsymbol{g} = \{g_1, g_2, \ldots, g_k\}$ of G a partition chain of $\{G/N\}$ if it is a minimal subset of G such that $< g_k, \ldots, g_2, g_1 > N = \{G/N\}$. Each partition chain $\mathbf{g}$ of $\{G/N\}$ determines a binary isometric labeling $m_g : \mathbf{Z}_2{}^k \to S/S_0$ with $m_g(\mathbf{z}) = g_k^{z_k} \ldots g_1^{z_1}(S_0) \triangleq \mathbf{g}^{\mathbf{z}}(S_0)$ for each $\mathbf{z} \in \mathbf{Z}_2{}^k$.

Lemma 1. If $\mathbf{g}$ and $\mathbf{f}$ are two different partition chains of $\{G/N\}$, then m_f is a linear transform of m_g. Namely, there is a non-singular $k \times k$ matrix K_f defined in $GF(2)$ such that, for each $\mathbf{x} \in \mathbf{Z}_2{}^k$, $\mathbf{g}^{\mathbf{x}} N = \mathbf{f}^{\mathbf{x} K_f} N$.

Let us denote by $\mathbf{K}_k$ the set of all nonsingular $k \times k$ matrices defined over $GF(2)$ and let us fix a partition chain $\mathbf{g}$. By Lemma 1, for each $\mathbf{f}$, there is $K_f \in \mathbf{K}_k$ that transforms m_g into m_f. Let $\mathbf{F}$ be the set of all partition chains of $\{G/N\}$.

Lemma 2: There is a one-to-one correspondence of $\mathbf{F}$ and $\mathbf{K}_k$.

Definition 2.1 and Lemma 2 imply that the number of isometric labelings is equal to the cardinality of $\mathbf{K}_k$, denoted by $|\mathbf{K}_k|$.

Lemma 3: $|\mathbf{K}_k| = 2^{k-1}(2^k - 1)|\mathbf{K}_{k-1}|$ with $|\mathbf{K}_0| = 1$, for $k \ge 1$.

The number of binary isometric labelings (the number of mappings) grows very fast when k increases. This makes it impossible to do an exhaustive search for good GU codes when one considers all possible mappings.

We notice that, if we fix a point $s_0 \in S$ and a partition chain $\mathbf{g}$ of $\{G/N\}$, then a subset X in the partition $G(s_0)/N(s_0)$ has a label $\mathbf{x}$ if $m_g(\mathbf{x}) = \mathbf{g}^{\mathbf{x}}(S_0) = X$. The same subset has another $\mathbf{x}K_f$ on the basis of another partition chain $\mathbf{f}$ (Lemma 1). Thus, we can define a binary isometric labeling α_f associated with $\mathbf{f}$ as $\alpha_f : \mathbf{Z}_2{}^k \to \mathbf{Z}_2{}^k$ where $\alpha_f(\mathbf{x}) = \mathbf{x}K_f$ for each $\mathbf{x} \in \mathbf{Z}_2{}^k$. Let $\alpha = \{\alpha_f\}$ denotes the set of all binary isometric labelings.

Theorem 1: Given a 2^k-way GU partition $\{G/N\}$, α forms a group which is isomorphic to $\mathbf{K}_k$.

Theorem 1 says that all isometric labelings can be generated from an initial isometric labeling. Moreover, the set of all isometric labelings has an algebraic structure.

III. LINEAR COMBINATION AND THE PROOF OF UNGERBOECK'S CONJECTURE THAT "MAPPING BY SET PARTITIONING" GIVES RISE TO BEST TCM CODES

The original encoder scheme of Ungerboeck signal space codes consists of a binary convolutional encoder and a memoryless linear mapper, which is essentially a binary isometric labeling [1]. In this section we show that the combination of two linear transformations, one as a convolutional encoder and one as a non-singular matrix K_f, forms a new encoder which gives rise to the same class of signal space codes.

A. Description of the encoder

Codewords of a signal space GU code $\boldsymbol{C}(\mathbf{g}, C)$ are elements of the signal sequence set $\{\mathbf{s} \in \mathbf{m}_g(\mathbf{c}), \mathbf{c} \in C\}$, where C is a subgroup of the label space $\mathbf{Z}_2{}^Z$ and $\mathbf{m}_g : \mathbf{Z}_2{}^Z \to (S/S_0)^Z$ is the sequence extension of the isometric labeling m_g [1]. Each pair of a binary label code C and a binary isometric labeling m_g determines a binary signal space code $\boldsymbol{C}(\mathbf{g}, C)$, which is a generalized coset code and, hence, is GU [1].

In this paper, we use the structure of the encoder in Fig. 1. Let $\mathbf{a} = \gamma(\boldsymbol{x}, \sigma)$ be a combination of convolutional encoder's state σ and input $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$. The encoder's output label $\mathbf{b} = (b_1, \ldots, b_k)$ then is determined by $\mathbf{a}$ and the encoder's coefficient matrix $H = [h_{ij}], h_{ij} \in \{0,1\}, 1 \le i \le m + \nu, 1 \le j \le k$. The label $\mathbf{b}$ is transformed to a label $\mathbf{c}$ which specifies a subset according to labeling of α. By Theorem 1, α is generated from an arbitrary but fixed α_g by the action of the group $\mathbf{K}_k$.

Two first blocks form an encoder for the label code C. Two last blocks form a mapper using an arbitrary partition chain $\mathbf{f}$. We assume that the 2^k-way partition of the signal set S provides us with a required MSD between signal points in each subset. We require that H is a full-rank matrix so that the MSD between signals in a subset determines the MSD of parallel transitions.

For a fixed ν, a class of 2^{ν}-state codes then is generated by using all possible H and all **f**. An exhaustive search for good GU codes is generally impossible since the number of codes to generate and to analyze is very large. Next, we show that the totality of codes is partitioned into equivalent codes and the search space is reduced.

B. Equivalent codes

It is known from the theory of signal space codes that the transfer function $T(D)$ of a code gives distance spectra of the totality of paths, which start from the state 0 and return to it. Important parameters d_{free} and N_{free} and upper bound on P_e can be derived from $T(D)$ [7].

Definition 3.1: Two signal space GU codes of the same transmission rate are equivalent if they have the same transfer function.

The transfer function of a signal space GU code is obtained by using the error-state diagram [7]. Due to the one-to-one correspondence between labels $\mathbf{c}$ and signal subsets $\mathbf{g}^{\mathbf{c}}(S_0)$, we assign to the branch connecting two states the corresponding label $\mathbf{c}$ through a function of the form $Mult(\mathbf{g},\mathbf{c})D^{d^2_{\min}(\mathbf{g},\mathbf{c})}$, where $d^2_{\min}(\mathbf{g},\mathbf{c})$ is the MSD between the subset $\mathbf{g}^{\mathbf{c}}(S_0)$ and S_0, and $Mult(\mathbf{g},\mathbf{c})$ is the number of points in $\mathbf{g}^{\mathbf{c}}(S_0)$ that are in the distance $d_{\min}(\mathbf{g},\mathbf{c})$ from a point in S_0. Computation of $T(D)$ thus requires the knowledge of distance profile (DP) of the employed GU partition which is defined as follows.

Definition 3.2 For a partition chain $\boldsymbol{g}$ and a signal point $s_0 \in S$, the distance profile of the GU partition of S induced by G/N is $DP(\mathbf{g},s_0) = \sum_{\mathbf{x}\in(\mathbf{Z}_2)^k} Mult(\mathbf{g},\mathbf{x})D^{d^2_{\min}(\mathbf{g},\mathbf{x})}$

The geometrical uniformity of S implies that the choice of the initial signal point $s_0 \in S$ as well as the partition chain $\boldsymbol{g}$ can be arbitrary [1]. Since all the binary isometric labelings of the same GU partition give rise to the same DP, it is reasonable to introduce the following definition.

Definition 3.3: Binary isometric labelings m_g and m_f are equivalent if $d^2_{\min}(\mathbf{g},\mathbf{z}) = d^2_{\min}(\mathbf{f},\mathbf{z})$ and $Mult(\mathbf{g},\mathbf{z}) = Mult(\mathbf{f},\mathbf{z})$ for all $\mathbf{z} \in \mathbf{Z}_2{}^k$.

Theorem 2: If m_g and m_f are equivalent, then $\boldsymbol{C}(\mathbf{g},C)$ is equivalent to $\boldsymbol{C}(\mathbf{f},C)$.

That is we need not to do search for m_f if m_f is equivalent to m_g. We can further reduce the search space by exploiting the encoder structure of Fig. 2. Two linear transforms in Fig. 2 can be combined in one, since $H' = HK_f^{-1}$ is also a full-rank matrix.

Theorem 3: Let $\mathbf{g}$ and $\mathbf{f}$ be two different partition chains. For any given $\boldsymbol{C}(\mathbf{f},C)$ there exists an equivalent code $\boldsymbol{C}(\mathbf{g},C')$.

Thus all isometric labelings are equally good for coding. The Ungerboeck's conjecture now is proved. The Ungerboeck's "mapping by set partitioning" rules give rise to a binary isometric labeling of the GU partition of the GU signal set [1]. Codes found in exhaustive search over all label codes C using this fixed isometric labeling are at least as good as codes using other isometric labelings.

Although Theorem 2 becomes weak now at the presence of Theorem 3, the code equivalence allows us to further reduce the number of C to be generated. We have the following theorem.

Theorem 4: If m_f is equivalent to m_g, then $\boldsymbol{C}(\mathbf{g},C)$ is equivalent to $\boldsymbol{C}(\mathbf{g},C')$ where C' has the same memory part as C has but with another linear combiner $H' = HK_f^{-1}$.

Finally we note on the reversibility of the encoder for the label code. Obviously, $T(D)$ does not change if we reverse all branches in the flow graph. Consequently, if C' is a label code whose error-state diagram is an reversed version of the error-state diagram of C, then $\boldsymbol{C}(\mathbf{g},C')$ is equivalent to $\boldsymbol{C}(\mathbf{g},C)$. In the literature, the convolutional code C' appears to have H' as a bit-reversed version of the generator matrix H for code C.

IV. FAST CODE SEARCH ALGORITHM

A. Computation of Transfer Function Upper Bounds

The code equivalence has been defined in the term of transfer function $T(D)$ of the codes. To employ the code equivalence in the code search in order to reduce the search space, in comparing different codes we can use the transfer function upper bound on the first event error probability [7]

$$P_e \le \sum_{d=d_{free}} n(d)D^{d^2} = T(D) \qquad (1)$$

where $n(d)$ is the number of paths diverging from the all-zero path at Euclidean distance d^2 over the unmerged segment and $D = \exp(-1/4N_0)$. The average signal energy is set to 1.

Given a GU partition of the signal constellation induced by a binary partition chain by $\boldsymbol{g}$, we use the DP defined above to form a $2^{\nu} \times 2^{\nu}$ transition matrix $A = [a_{ij}]$ as follows. If there is a state transition $(i-1) \to (j-1)$, we let $a_{ij} = Mult(\mathbf{g},\mathbf{c}_{ij})D^{d^2_{\min}(\mathbf{g},\mathbf{c}_{ij})}$ for $D = \exp(-SNR/4E_s)$ with E_s being the average energy of the given signal set and $SNR = E_s/N_0$ being the given SNR. Otherwise, let $a_{ij} = 0$. Since the contribution to P_e of the parallel transitions (error events of length $L=1$) can be evaluated separately, using the knowledge on the subset S_0, we let $a_{11} = 0$ so that, for a while, we consider only error events of length $L \ge 2$. Define by $\boldsymbol{a}^{(1)}$ the first row of A. Let $\boldsymbol{a}^{(\ell)} = \boldsymbol{a}^{(\ell-1)}A$ for $\ell \ge 2$. Denote by $a_j^{(\ell)}$ the *j*-th element of $\boldsymbol{a}^{(\ell)}$. It is easy to see that while $a_1^{(\ell)} = 0$ all potential error paths that depart from the state 0 and come to the state $(j-1)$ for $2 \le j \le 2^{\nu}$ exactly after ℓ discrete time epochs accumulate an amount $a_j^{(\ell)}$ to contribute to P_e. Suppose at $\ell = \ell_1 \ge 2$ we have $a_1^{(\ell)} > 0$ for the first time, meaning that there are some paths that remerge back to the state 0 to form error events of length ℓ_1. It is easy to be seen that $a_1^{(\ell_1)}$ equals to the sum of all PEP of error events of length ℓ_1. Let $P_e(\ell_1) = a_1^{\ell_1}$ to store this sum. The computation then proceeds by setting $a_1^{(\ell_1)} = 0$ and recursively $\boldsymbol{a}^{(\ell)} = \boldsymbol{a}^{(\ell-1)}A$, with $\ell \ge \ell_1$, for the next $\ell = \ell_2 \ge \ell_1$ to obtain $P_e(\ell_2) = a_1^{\ell_2} > 0$. The algorithm then is composed of two loops.

1. Inner loop: Set $\ell = 2$. While $a_1^{\ell-1} = 0$, do $\boldsymbol{a}^{(\ell)} = \boldsymbol{a}^{(\ell-1)}A$ and increase ℓ by one. Update $P(e) = P(e) + a_1^{\ell}$.
2. Outer loop: While $\ell \le L_{\max}$, repeat the inner loop after setting $a_1^{\ell} = 0$.

If we initially set P_e to the value of the contribution of parallel transitions and carry out the computation up to a large enough $L_{\max}$, the final value of P_e should be a good approximation of the transfer function at the given SNR.

B. Fast Search for GU Codes with QAM

For the linear combiner we utilize the rate- $(k-1)/k$ binary systematic convolutional encoder with feedback [3] . The output bits z_i^j , $j=1,\dots,k$, of the encoder are computed from input bits $x_i^j, j=1,2,\dots,k-1$, as

$$z_i^j = x_i^{j-1},\ \ 2\le j\le k;\quad z_i^1=\sum_{j=1}^{k-1}\sum_{t=1}^{\nu+1} h_i^j x_{i-t+1}^j + \sum_{t=2}^{\nu} h_t^0 z_{i-t+1}^1 \qquad (2)$$

where i is the time index and ν is the memory length of the encoder. Following Ungerboeck and many authors, we use encoders with the following property

$$h_{\nu+1}^0 = h_1^0 = 1,\ \ h_{\nu+1}^j = h_1^j = 0 \quad j=1,2,\dots,k-1 \qquad (3)$$

This ensures that c_i^0 is the same for transitions that diverge from a state (or remerge to the same state). Then adjacent transitions are assigned with signals taken from subset of a large MSD. In consequence, a large free distance can be obtained. Moreover, with condition (3) it is easy to show that one can apply all results of previous sections to the code search.

Since the rate-2/3 encoder for the label code C is fully described by the matrix $H=[\boldsymbol{h}_2,\boldsymbol{h}_1,\boldsymbol{h}_0]$, hereafter we use the notation $\boldsymbol{C}(\mathbf{g}, H)$ for the signal space code, where $\boldsymbol{g}$ defines, for the 8-way GU partition $\boldsymbol{Z}^2/2R\boldsymbol{Z}^2$, the binary GU partition chain $\boldsymbol{Z}^2/R\boldsymbol{Z}^2/2\boldsymbol{Z}^2/2R\boldsymbol{Z}^2$ with the distance chain 1/2/4/8 and, hence, defines the binary isometric labeling. Let $\{K_0, K_1, ..., K_7\}$ be a set of elements of $\mathbf{K}_k$ associated with 8 equivalent isometric labelings and given below. Then we have $\boldsymbol{C}(\mathbf{g}, H)$ equivalent to $\boldsymbol{C}(\mathbf{g}, HK_p^{-1})$ for $1\le p\le 7$. If $\boldsymbol{C}(\mathbf{g}, H)$ has been generated and the transfer function upper bound has been evaluated for its $P(e)$, we need not to generate $\boldsymbol{C}(\mathbf{g}, HK_p^{-1})$. The code search algorithm then consists of the following steps.

1. Generate a matrix H , go to the next step if H satisfies all conditions, that H is full rank, that, for $1\le p\le 7$, the matrix $H_p = HK_p$ has not been ever generated and checked, and that the reversed version of H has not been ever generated and checked. Otherwise, generate a new H until all possible code space has been searched.
2. Generate and evaluate $\boldsymbol{C}(\mathbf{g}, H)$. Compare with the previously generated codes to find the one with the minimum bound (1).

$$K_0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad K_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix}\quad K_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&1&0&1\end{bmatrix}\quad K_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$$

$$K_4=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\quad K_5=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&0&1&1\end{bmatrix}\quad K_6=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&1&1&1\end{bmatrix}\quad K_7=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\0&1&1&1\end{bmatrix}$$

Table 1 gives result of the new code search for 128, 256, and 512 states. Fig. 2 compares the upper bounds on first event error probability of newly found codes with Ungerboeck codes. It can be seen that newly found codes outperform Ungerboeck codes. Thus they improve the performance versus complexity tradeoff of 2D trellis codes with 128, 256, and 512 states.

V. Conclusions

In this work, the concept of equivalent isometric labelings and equivalent codes allowed us to reduce the search space. The exhaustive search using the transfer function upper bound on first event error probability yielded new 128-, 256-, and 512-state codes that improved the performance/complexity tradeoff of 2D trellis codes.

Table 1. New 2D GU codes with QAM

# states	Generators		
	h^2	h^1	h^0
128	056	106	223
256	064	306	527
512	0070	0516	1015

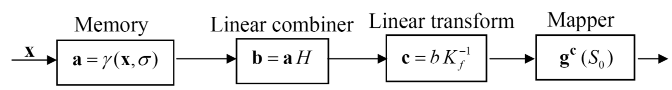

Fig.1 GU signal space encoder with a linear combiner

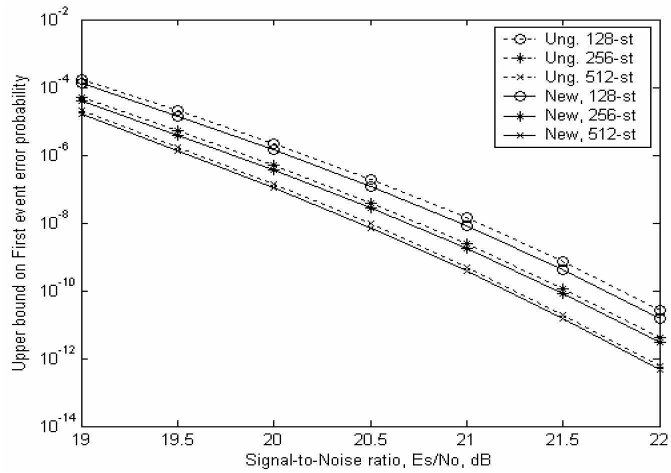

Fig. 2 Comparison of new codes with Ungerboeck codes, 64-QAM, 8-way partition

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