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Well-posedness and regularity for solutions of caputo stochastic fractional differential equations in L^p spaces

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ABSTRACT

In the first part of this paper, we establish the well-posedness for solutions of Caputo stochastic fractional differential equations (for short Caputo SFDE) of order $\alpha \in \left(\frac{1}{2}, 1\right)$ in L^p spaces with $p \geq 2$ whose coefficients satisfy a standard Lipschitz condition. More precisely, we first show a result on the existence and uniqueness of solutions, next we show the continuous dependence of solutions on the initial values and on the fractional exponent α. The second part of this paper is devoted to studying the regularity in time for solutions of Caputo SFDE. As a consequence, we obtain that a solution of Caputo SFDE has a δ-Hölder continuous version for any $\delta \in \left(0, \alpha - \frac{1}{2}\right)$. The main ingredient in the proof is to use a temporally weighted norm and the Burkholder-Davis-Gundy inequality.

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1. Introduction

In this paper, we consider a Caputo stochastic fractional differential equation of order $\alpha \in \left(\frac{1}{2}, 1\right)$ on the interval $[0, T]$ of the following form

$$^{C}D^{\alpha}_{0+}X(t) = b(t, X(t)) + \sigma(t, X(t))\,\frac{dW_t}{dt}, \tag{1}$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable and $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$.

The stochastic differential equations involving Caputo fractional time derivative operator as in (1) give good models to investigate the memory and hereditary properties of various branches of physical and biological sciences more precisely (see [1–3]). The main reason for this fact is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. For more details we refer the reader to the monographs [2, 4, 5] and the references therein.

According to the authors' knowledge, the main achieved results for Equation (1) are limited to problem of the existence and uniqueness of strong solutions [6–8] and mild

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solutions [9] in L^2 spaces. A proof of coincidence of strong and mild solution of (1) in L^2 spaces under some natural assumptions on the coefficients has recently been proved in [10]. An Euler-Maruyama type scheme for Caputo stochastic fractional differential equations and the convergence rate of this scheme have been established in [11]. Very recently, several results on well-posedness and continuity on fractional exponent α for solutions of (1) in L^2 spaces have been developed in [12]. However, the well-posedness as well as the regularity of solutions of Equation (1) in L^p spaces with $p \geq 2$ were not investigated systematically in any article. Thus, in this paper, we are trying to fill this gap. Precisely, our first aim in this paper is to establish the well-posedness for the solutions of Caputo stochastic fractional differential equations with generalized Lipschitz-type coefficients driven by general multiplicative noise in L^p spaces. Secondly, we are interested in the Hölder continuity of the solutions for Equation (1) in L^p spaces.

The paper is organized as follows: In Section 2, we introduce briefly about Caputo SFDE and state the main results of the paper. The first main result is about global existence and uniqueness of solutions (Theorem 1(i)), continuous dependence of solutions on the initial values (Theorem 1(ii)) and on the fractional exponent (Theorem 1(iii)). The second main result is about regularity in time of solutions (Theorem 2). The proof of the first main result and the second main result are given in Sections 3 and 4, respectively.

2. Preliminaries and main results

2.1. Caputo fractional stochastic differential equations and standing assumptions

For $p \geq 2, t \in [0, \infty)$, let $\mathfrak{X}_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all $\mathcal{F}_t$-measurable, p^{th} integrable functions $f = (f_1, f_2, \ldots, f_d)^{\mathrm{T}} : \Omega \to \mathbb{R}^d$ with

$$\|f\|_p := \left(\sum_{i=1}^{d} \mathbb{E}(|f_i|^p) \right)^{\frac{1}{p}}.$$

A measurable process $X : [0, T] \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{F}$-adapted if $X(t) \in \mathfrak{X}_t^p$ for all $t \geq 0$. For each $\eta \in \mathfrak{X}_0^p$, a $\mathbb{F}$-adapted process X is called a solution of (1) with the initial condition $X(0) = \eta$ if $X(0) = \eta$ and the following equality holds on $\mathfrak{X}_t^p$ for $t \in (0, T]$

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-\tau)^{\alpha-1} b(\tau, X(\tau))\, d\tau + \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, X(\tau))\, dW_\tau \right), \qquad (2)$$

where $\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1} \exp(-\tau)\, d\tau$ is the Gamma function, see [7, p. 209].

2.2. Well-posedness and regularity of solutions

In this article, we assume that the coefficients b and σ of (1) satisfy the following conditions:

(H1) Global Lipschitz continuity in $\mathbb{R}^d$ of the drift and diffusion term: For all $x, y \in \mathbb{R}^d$, $t \in [0, T]$, there exists $\mathrm{L} > 0$ such that

$$|b(t,x)-b(t,y)|_p+|\sigma(t,x)-\sigma(t,y)|_p\le L|x-y|_p.$$

(H2) *Essential boundedness in time for drift and diffusion term*: $\sigma(\cdot,0)$ is essentially bounded, i.e.

$$\operatorname*{esssup}_{\tau\in[0,T]}|b(\tau,0)|_p<M,\quad \operatorname*{esssup}_{\tau\in[0,T]}|\sigma(\tau,0)|_p<M.$$

Note that the contents of assumptions (H1) and (H2) are independent on the choice of the norm on $\mathbb{R}^d$. However, for a convenience in several estimates below, we equip $\mathbb{R}^d$ with the p norm, i.e. for any vector $x=(x_1,...,x_d)^{\mathrm{T}}\in\mathbb{R}^d$, the p norm $|x|_p$ of x is defined by $|x|_p:=(\sum_{i=1}^d|x_i|^p)^{\frac{1}{p}}$.

The first main result in this article is the well-posedness of solutions of Caputo SFDE.

Theorem 1. *(Well-posedness of solutions of Caputo SFDE). Suppose that (H1) and (H2) hold. Then the following statements hold:*

i. *Existence and uniqueness for solutions of Caputo SFDE: for any $\eta\in\mathfrak{X}_0^p$, the initial value problem (1) with the initial condition $X(0)=\eta$ has a unique solution on $[0,T]$ denoted by $\varphi_\alpha(\cdot,\eta)$.*
ii. *Continuous dependence on the initial values for solutions of Caputo SFDE: for any $\zeta,\eta\in\mathfrak{X}_0^p$, the solution $\varphi_\alpha(\cdot,\eta)$ depends Lipschitz continuously on η, i.e. there exists $L_1>0$ such that*
$$\|\varphi_\alpha(t,\zeta)-\varphi_\alpha(t,\eta)\|_p\le L_1\|\zeta-\eta\|_p\quad\text{for all } t\in[0,T].$$
iii. *Continuous dependence on the fractional exponent α for solutions of Caputo SFDE: The solution $\varphi_\alpha(\cdot,\eta)$ depends continuously on α, i.e.*
$$\lim_{\hat\alpha\to\alpha}\operatorname*{esssup}_{t\in[0,T]}\|\varphi_\alpha(t,\eta)-\varphi_{\hat\alpha}(t,\eta)\|_p=0.$$

The second main result in this article is the regularity of solutions of Caputo SFDE.

Theorem 2. *(The regularity of solutions). Suppose that (H1) and (H2) hold. Then, there exists $D>0$ depending on α,p,L,M,T such that*

$$\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p\le D\,|t-s|^{\alpha-\frac{1}{2}},\quad\text{for all } s,t\in[0,T].\tag{3}$$

Corollary 3. *For all $\delta\in\left(0,\alpha-\frac{1}{2}\right)$, there exists a modification Y of X with δ-Hölder continuous paths, i.e.*

$$\mathbb{P}(X_t=Y_t)=1\quad\text{for all } t\in[0,T].$$

Proof. Thanks to (3) and using Kolmogorov test, see e.g. [13, Theorem, p. 51], $X(t)$ has an δ-Hölder continuous modification for any $\delta \in \left(0, \alpha - \frac{1}{2}\right)$. □

3. Well-posedness of solutions

3.1. Existence and uniqueness of solutions

Our aim in this subsection is to prove the result on existence and uniqueness of solutions to the Equation (1). For this purpose, let $\mathbb{H}^p([0,T])$ be the space of all the processes X which are measurable, $\mathbb{F}_T$-adapted, where $\mathbb{F}_T := (\mathcal{F}_t)_{t\in[0,T]}$ and satisfies that

$$\|X\|_{\mathbb{H}^p} := \operatorname*{esssup}_{t\in[0,T]} \|X(t)\|_p < \infty.$$

Obviously, $(\mathbb{H}^p([0,T]), \|\cdot\|_{\mathbb{H}^p})$ is a Banach space. For any $\eta \in \mathfrak{X}_0^p$, we define an operator $\mathcal{T}_\eta : \mathbb{H}^p([0,T]) \to \mathbb{H}^p([0,T])$ by $\mathcal{T}_\eta\xi(0) := \eta$ and the following equality holds for $t \in (0,T]$

$$\mathcal{T}_\eta\xi(t) := \eta + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))\,d\tau + \int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))\,dW_\tau\right). \tag{4}$$

The following lemma is devoted to show the well-defined property of this operator. In the proof of this result and also several results below, we use the following elementary inequality

$$|x+y|_p^p \le 2^{p-1}(|x|_p^p + |y|_p^p) \quad \text{for all} \quad x, y \in \mathbb{R}^d. \tag{5}$$

Lemma 4. *For any $\eta \in \mathfrak{X}_0^p$, the operator $\mathcal{T}_\eta$ is well-defined.*

Proof. Let $\xi \in \mathbb{H}^p([0,T])$ be arbitrary. From the definition of $\mathcal{T}_\eta\xi$ as in (11) and the inequality (13), we have that for all $t \in [0,T]$

$$\begin{aligned} \|\mathcal{T}_\eta\xi(t)\|_p^p \le \quad & 2^{p-1}\|\eta\|_p^p + \frac{2^{2p-2}}{\Gamma^p(\alpha)}\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|_p^p \\ & + \frac{2^{2p-2}}{\Gamma^p(\alpha)}\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|_p^p. \end{aligned} \tag{6}$$

By the Hölder inequality, we obtain that

$$\begin{aligned}
&\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau \right\|_p^p \\
&\le \sum_{i=1}^d \mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1} |b_i(\tau,\xi(\tau))| d\tau \right)^p \\
&\le \sum_{i=1}^d \mathbb{E}\left(\left(\int_0^t (t-\tau)^{\frac{(\alpha-1)p}{p-1}} d\tau \right)^{p-1} \int_0^t |b_i(\tau,\xi(\tau))|^p d\tau \right) \\
&\le \frac{T^{(\alpha p-1)}(p-1)^{p-1}}{(\alpha p-1)^{p-1}} \int_0^t \|b(\tau,\xi(\tau))\|_p^p d\tau.
\end{aligned} \tag{7}$$

From (H1), we derive

$$\begin{aligned}
|b(\tau,\xi(\tau))|_p^p &\le 2^{p-1}\left(|b(\tau,\xi(\tau)) - b(\tau,0)|_p^p + |b(\tau,0)|_p^p \right) \\
&\le 2^{p-1} L^p |\xi(\tau)|_p^p + 2^{p-1} |b(\tau,0)|_p^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^t \|b(\tau,\xi(\tau))\|_p^p d\tau &\le 2^{p-1} L^p \left(\operatorname*{esssup}_{\tau\in[0,T]} ||\xi(\tau)||_p \right)^p \int_0^t 1 d\tau + 2^{p-1} \int_0^t |b(\tau,0)|_p^p \, d\tau \\
&\le 2^{p-1} L^p T ||\xi||_{\mathbb{H}^p}^p + 2^{p-1} \int_0^T |b(\tau,0)|_p^p \, d\tau
\end{aligned}$$

which together with (7) implies that

$$\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau \right\|_p^p \le \frac{T^{(\alpha p-1)}(2p-2)^{p-1}}{(\alpha p-1)^{p-1}} \left(L^p T \|\xi\|_{\mathbb{H}^p}^p + \int_0^T |b(\tau,0)|_p^p \, d\tau \right). \tag{8}$$

Now, using the Burkholder-Davis-Gundy and the Hölder inequalities, we obtain

$$\begin{aligned}
&\left\| \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau,\xi(\tau)) dW_\tau \right\|_p^p \\
&\le \sum_{i=1}^d \mathbb{E}\left| \int_0^t (t-\tau)^{\alpha-1} \sigma_i(\tau,\xi(\tau)) dW_\tau \right|^p \\
&\le \sum_{i=1}^d C_p \mathbb{E}\left| \int_0^t (t-\tau)^{2\alpha-2} \middle| \sigma_i(\tau,\xi(\tau))|^2 d\tau \right|^{\frac{p}{2}} \\
&\le \sum_{i=1}^d C_p \mathbb{E} \int_0^t (t-\tau)^{2\alpha-2} |\sigma_i(\tau,\xi(\tau))|^p d\tau \left(\int_0^t (t-\tau)^{2\alpha-2} d\tau \right)^{\frac{p-2}{2}} \\
&\le C_p \left(\frac{T^{2\alpha-1}}{2\alpha-1} \right)^{\frac{p-2}{2}} \int_0^t (t-\tau)^{2\alpha-2} \|\sigma(\tau,\xi(\tau))\|_p^p d\tau,
\end{aligned}$$

where $C_p = \left(\frac{p^{p+1}}{2(p-1)^{p-1}} \right)^{\frac{p}{2}}$. From (H1) and (H2), we also have

$$|\sigma(\tau,\xi(\tau))|_p^p \le 2^{p-1}L^p|\xi(\tau)|_p^p + 2^{p-1}|\sigma(\tau,0)|_p^p \le 2^{p-1}L^p|\xi(\tau)|_p^p + 2^{p-1}M^p.$$

Therefore, for all $t \in [0, T]$ we have

$$\begin{aligned}
&\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau))\|_p^p d\tau \\
&\le 2^{p-1}L^p \int_0^t (t-\tau)^{2\alpha-2}\left(\operatorname*{esssup}_{\tau\in[0,T]}\|\xi(\tau)\|_p\right)^p d\tau + 2^{p-1}M^p\int_0^t (t-\tau)^{2\alpha-2}\,d\tau \\
&\le \frac{2^{p-1}T^{2\alpha-1}}{2\alpha-1}\left(L^p\|\xi\|_{\mathbb{H}_p}^p + M^p\right).
\end{aligned}$$

This together with (6), (8) and (H2) implies that $\|\mathcal{T}_\eta\xi\|_{\mathbb{H}^p} < \infty$. Hence, the map $\mathcal{T}_\eta$ is well-defined.

To prove existence and uniqueness of solutions, we will show that the operator $\mathcal{T}_\eta$ defined as above is contractive under a suitable weighted norm (cf. [14, Remark 2.1] for the same method to prove the existence and uniqueness of solutions of stochastic differential equations). Here, the weight function is the Mittag-Leffler function $E_{2\alpha-1}(\cdot)$ defined as follows

$$E_{2\alpha-1}(t) := \sum_{k=0}^{\infty}\frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad \text{for all } t \in \mathbb{R}.$$

For more details on the Mittag-Leffler functions we refer the reader to the book [2, p. 16].

We are now in a position to prove Theorem 1(i).

Proof of Theorem 1 (i). Choose and fix a positive constant γ such that

$$\gamma > \kappa 2^{p-1}\Gamma(2\alpha-1), \tag{9}$$

where

$$\kappa = \frac{2^{p-1}L^p}{\Gamma^p(\alpha)}\left(\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}} + \frac{T^{(p-2)\alpha+1}}{\left(\frac{(p-2)\alpha+1}{p-1}\right)^{p-1}}\right). \tag{10}$$

On the space $\mathbb{H}^p([0,T])$, we define a weighted norm $\|\cdot\|_\gamma$ as below

$$\|X\|_\gamma := \operatorname*{esssup}_{t\in[0,T]}\left(\frac{\|X(t)\|_p^p}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\right)^{\frac{1}{p}} \quad \text{for all } X \in \mathbb{H}^p([0,T]), \tag{11}$$

Obviously, two norms $\|\cdot\|_{\mathbb{H}^p}$ and $\|\cdot\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^p([0,T]), \|\cdot\|_\gamma)$ is also a Banach space. Choose and fix $\eta \in \mathfrak{X}_0^p$. By virtue of Lemma 4, the operator $\mathcal{T}_\eta$ is well-defined. Now, we will prove that the map $\mathcal{T}_\eta$ is contractive with respect to the norm $\|\cdot\|_\gamma$. For this purpose, let $\xi, \hat{\xi} \in \mathbb{H}^p([0,T])$ be arbitrary. From (4) and the inequality (5), we derive the following inequality for all $t \in [0,T]$:

$$\|\mathcal{T}_\eta\xi(t)-\mathcal{T}_\eta\hat{\xi}(t)\|_p^p \le \frac{2^{p-1}}{\Gamma^p(\alpha)}\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau))-b(\tau,\hat{\xi}(\tau)))d\tau\right\|_p^p + \frac{2^{p-1}}{\Gamma^p(\alpha)}\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau))-\sigma(\tau,\hat{\xi}(\tau)))\,dW_\tau\right\|_p^p.$$

Using the Hölder inequality and (H1), we obtain

$$\begin{aligned}&\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau))-b(\tau,\hat{\xi}(\tau)))d\tau\right\|_p^p\\ &\le \sum_{i=1}^d \mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1}|b_i(\tau,\xi(\tau))-b_i(\tau,\hat{\xi}(\tau))|d\tau\right)^p\\ &\le \sum_{i=1}^d \mathbb{E}\left(\left(\int_0^t (t-\tau)^{\frac{(\alpha-1)(p-2)}{p-1}}d\tau\right)^{p-1}\left(\int_0^t (t-\tau)^{2\alpha-2}|b_i(\tau,\xi(\tau))-b_i(\tau,\hat{\xi}(\tau))|^p d\tau\right)\right)\\ &\le \frac{L^pT^{(\alpha p-2\alpha+1)}(p-1)^{p-1}}{(\alpha p-2\alpha+1)^{p-1}}\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)-\hat{\xi}(\tau)\|_p^p d\tau.\end{aligned}$$

On the other hand, by the Burkholder-Davis-Gundy inequality and (H1), we have

$$\begin{aligned}&\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau))-\sigma(\tau,\hat{\xi}(\tau)))dW_\tau\right\|_p^p\\ &=\sum_{i=1}^d \mathbb{E}\left|\int_0^t (t-\tau)^{\alpha-1}(\sigma_i(\tau,\xi(\tau))-\sigma_i(\tau,\hat{\xi}(\tau)))dW_\tau\right|^p\\ &\le \sum_{i=1}^d C_p\mathbb{E}\left|\int_0^t (t-\tau)^{2\alpha-2}|\sigma_i(\tau,\xi(\tau))-\sigma_i\left(\tau,\hat{\xi}(\tau)\right)|^2 d\tau\right|^{\frac{p}{2}}\\ &\le \sum_{i=1}^d C_p\mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}|\sigma_i(\tau,\xi(\tau))-\sigma_i\left(\tau,\hat{\xi}(\tau)\right)|^p d\tau\left(\int_0^t (t-\tau)^{2\alpha-2}d\tau\right)^{\frac{p-2}{2}}\\ &\le L^pC_p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)-\hat{\xi}(\tau)\|_p^p d\tau.\end{aligned}$$

Thus, for all $t\in[0,T]$ we have

$$\begin{aligned}&\|\mathcal{T}_\eta\xi(t)-\mathcal{T}_\eta\hat{\xi}(t)\|_p^p\\ &\quad\le \kappa\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)-\hat{\xi}(\tau)\|_p^p d\tau,\end{aligned}$$

where κ is given as in (10). This estimate with the definition of $\|\cdot\|_\gamma$ as in (11) implies that

$$
\begin{aligned}
&\frac{\|\mathcal{T}_\eta\xi(t)-\mathcal{T}_\eta\hat{\xi}(t)\|_p^p}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\\
&\leq \frac{\kappa\int_0^t(t-\tau)^{2\alpha-2}\frac{\|\xi(\tau)-\hat{\xi}(\tau)\|_p^p}{E_{2\alpha-1}(\gamma\tau^{2\alpha-1})}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\\
&\leq \kappa\left(\operatorname*{ess\ sup}_{\tau\in[0,T]}\left(\frac{\|\xi(\tau)-\hat{\xi}(\tau)\|_p^p}{E_{2\alpha-1}(\gamma\tau^{2\alpha-1})}\right)^{\frac{1}{p}}\right)^p\frac{\int_0^t(t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\\
&\leq \frac{\kappa\Gamma(2\alpha-1)}{\gamma}\|\xi-\hat{\xi}\|_\gamma^p.
\end{aligned}
$$

where in the final step, we used the inequality

$$\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^t(t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau\leq E_{2\alpha-1}(\gamma t^{2\alpha-1})$$

see [6, Lemma 5]. Consequently,

$$\|\mathcal{T}_\eta\xi-\mathcal{T}_\eta\hat{\xi}\|_\gamma\leq\left(\frac{\kappa\Gamma(2\alpha-1)}{\gamma}\right)^{\frac{1}{p}}\|\xi-\hat{\xi}\|_\gamma.$$

By (9), we have $\frac{\kappa\Gamma(2\alpha-1)}{\gamma}<1$ and therefore the operator $\mathcal{T}_\eta$ is a contractive map on $(\mathbb{H}^p([0,T]),\|\cdot\|_\gamma)$. Using the Banach fixed point theorem, there exists a unique fixed point of this map in $\mathbb{H}^p([0,T])$. This fixed point is also the unique solution of (1) with the initial condition $X(0)=\eta$. The proof of this theorem is complete. □

Remark 5. (i) Results about the existence and uniqueness of solutions of Caputo SFDE in case $p=2$ have been shown in [6].

(ii) In [8], the author considered the following form in L^p

$$X(t)=X(0)+\int_0^t(t-\tau)^{-\alpha_1}b(t,\tau,X(\tau))\,d\tau+\int_0^t(t-\tau)^{-\alpha_2}\sigma(t,\tau,X(\tau))\,dW_\tau, \tag{12}$$

where $X(0)\in\mathbb{R}^d,\sigma:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^m, b:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ are Borel measurable functions and $\alpha_1\in(0,1),\alpha_2\in\left(0,\frac{1}{2}\right)$. He obtained the existence and uniqueness of solutions with non-Lipschitz coefficients in L^p with $p>\max\left(\frac{1}{1-\alpha_1},\frac{2}{1-2\alpha_2}\right)$. In this paper, we prove that the result still holds for all $\mathrm{p}>2$ and random initial condition $X(0)=\eta\in\mathfrak{X}_0^p$.

3.2. The continuous dependence of the solutions on the initial values

Our next result is to evaluate the distance between two different solutions. As a consequence, we obtain the Lipschitz continuity dependence of solutions on the initial values.

Proof of Theorem 1 (ii). Choose and fix $\zeta\in\mathfrak{X}_0^p$. Let $\eta\in\mathfrak{X}_0^p$ arbitrarily. Since $\varphi_\alpha(\cdot,\eta)$ and $\varphi_\alpha(\cdot,\zeta)$ are solutions of (1) it follows that

$$\varphi_\alpha(t,\eta)-\varphi_\alpha(t,\zeta)=\eta-\zeta+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{b(\tau,\varphi_\alpha(\tau,\eta))-b(\tau,\varphi_\alpha(\tau,\zeta))}{(t-\tau)^{1-\alpha}}\,d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\sigma(\tau,\varphi_\alpha(\tau,\eta))-\sigma(\tau,\varphi_\alpha(\tau,\zeta))}{(t-\tau)^{1-\alpha}}\,dW_\tau.$$

Hence, using (5), the Hölder and the Burkholder-Davis-Gundy inequalities and (H1) (see proof of Theorem 1(i)), we obtain

$$||\varphi_\alpha(t,\eta)-\varphi_\alpha(t,\zeta)||_p^p\le 2^{p-1}\kappa\int_0^t(t-\tau)^{2\alpha-2}||\varphi_\alpha(\tau,\eta)-\varphi_\alpha(\tau,\zeta)||_p^p\,d\tau$$
$$+2^{p-1}||\eta-\zeta||_p^p.$$

Applying the Gronwall inequality for fractional differential equations, see [15, Lemma 7.1.1] or [16, Corollary 2], we arrive at

$$\|\varphi_\alpha(t,\eta)-\varphi_\alpha(t,\zeta)\|_p^p\le 2^{p-1}\,E_{2\alpha-1}\big(2^{p-1}\kappa\,\Gamma(2\alpha-1)t^{2\alpha-1}\big)\,\|\eta-\zeta\|_p^p.$$

Hence,

$$\lim_{\eta\to\zeta}\|\varphi_\alpha(t,\eta)-\varphi_\alpha(t,\zeta)\|_p=0.$$

The proof is complete. □

Remark 6. A result about the continuity dependence of solutions of SFDE on the initial values in case $p=2$ has been shown in [6].

3.3. The continuous dependence of the solutions on the fractional exponent α

In this part, we shall prove the continuous dependence on α of the solution.

Proof of Theorem 1 (iii). Let $\alpha,\hat{\alpha}\in\left(\frac{1}{2},1\right)$ be arbitrarily but fixed. Choose and fix $\eta\in\mathfrak{X}_0^p$. Since $\varphi_\alpha(\cdot,\eta)$ and $\varphi_{\hat{\alpha}}(\cdot,\eta)$ are solutions of (1) it follows that

$$\begin{aligned}\varphi_\alpha(t,\eta)&-\varphi_{\hat{\alpha}}(t,\eta)\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(b(\tau,\varphi_\alpha(\tau,\eta))-b(\tau,\varphi_{\hat{\alpha}}(\tau,\eta)))\,d\tau\\
&\quad+\int_0^t\left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat{\alpha}-1}}{\Gamma(\hat{\alpha})}\right)b(\tau,\varphi_{\hat{\alpha}}(\tau,\eta))\,d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(\sigma(\tau,\varphi_\alpha(\tau,\eta))-\sigma(\tau,\varphi_{\hat{\alpha}}(\tau,\eta)))\,dW_\tau\\
&\quad+\int_0^t\left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat{\alpha}-1}}{\Gamma(\hat{\alpha})}\right)\sigma(\tau,\varphi_{\hat{\alpha}}(\tau,\eta))\,dW_\tau.\end{aligned}$$

Using the inequality (5), the Hölder and the Burkholder-Davis-Gundy inequalities and (H1) (see proof of Theorem 1(i)), we obtain

$$
\begin{aligned}
&\|\varphi_\alpha(t,\eta)-\varphi_{\hat\alpha}(t,\eta)\|_p^p\\
&\quad\le 2^{p-1}\kappa\int_0^t (t-\tau)^{2\alpha-2}\|\varphi_\alpha(\tau,\eta)-\varphi_{\hat\alpha}(\tau,\eta)\|_p^p\,d\tau\\
&\qquad+2^{2p-2}\left\|\int_0^t\left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat\alpha-1}}{\Gamma(\hat\alpha)}\right)b(\tau,\varphi_{\hat\alpha}(\tau,\eta))\,d\tau\right\|_p^p\\
&\qquad+2^{2p-2}\left\|\int_0^t\left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat\alpha-1}}{\Gamma(\hat\alpha)}\right)\sigma(\tau,\varphi_{\hat\alpha}(\tau,\eta))\,dW_\tau\right\|_p^p.
\end{aligned}
$$

Now, let

$$
g(t,\tau,\alpha,\hat\alpha):=\left|\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat\alpha-1}}{\Gamma(\hat\alpha)}\right|.
$$

Applying the Hölder inequality, (5), (H1) and (H2), we obtain that

$$
\begin{aligned}
&\left\|\int_0^t\left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-\tau)^{\hat\alpha-1}}{\Gamma(\hat\alpha)}\right)b(\tau,\varphi_{\hat\alpha}(\tau,\eta))\,d\tau\right\|_p^p\\
&\quad\le\sum_{i=1}^d\mathbb{E}\left(\int_0^t g(t,\tau,\alpha,\hat\alpha)|b_i(\tau,\varphi_{\hat\alpha}(\tau,\eta))|d\tau\right)^p\\
&\quad\le\sum_{i=1}^d\mathbb{E}\left(\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^{\frac{p}{p-1}}d\tau\right)^{p-1}\int_0^t|b_i(\tau,\varphi_{\hat\alpha}(\tau,\eta))|^p\,d\tau\right)\\
&\quad\le\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2d\tau\right)^{\frac{p}{2}}\left(\int_0^t 1\,d\tau\right)^{\frac{p-2}{2}}\int_0^t\|b(\tau,\varphi_{\hat\alpha}(\tau,\eta))\|_p^p\,d\tau\\
&\quad\le\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2d\tau\right)^{\frac{p}{2}}T^{\frac{p-2}{2}}\int_0^t 2^{p-1}\left(L^p\|\varphi_{\hat\alpha}(\tau,\eta)\|_p^p+\|b(\tau,0)\|_p^p\right)d\tau\\
&\quad\le\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2d\tau\right)^{\frac{p}{2}}T^{\frac{p}{2}}2^{p-1}(L^p\operatorname*{esssup}_{\tau\in[0,T]}\|\varphi_{\hat\alpha}(\tau,\eta)\|_p^p+M^p).
\end{aligned}
$$

On the other hand, using the Burkholder-Davis-Gundy inequality, (H1) and (H2), we have

$$\begin{aligned}
&\left\| \int_0^t \left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-\tau)^{\hat{\alpha}-1}}{\Gamma(\hat{\alpha})} \right) \sigma(\tau, \varphi_{\hat{\alpha}}(\tau,\eta))\, dW_\tau \right\|_p^p \\
&\leq \sum_{i=1}^d \mathbb{E} \left| \int_0^t g(t,\tau,\alpha,\hat{\alpha}) |\sigma_i(\tau, \varphi_{\hat{\alpha}}(\tau,\eta))|\, dW_\tau \right|^p \\
&\leq \sum_{i=1}^d C_p\, \mathbb{E} \left| \int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2 |\sigma_i(\tau, \varphi_{\hat{\alpha}}(\tau,\eta))|^2\, d\tau \right|^{\frac{p}{2}} \\
&\leq \sum_{i=1}^d C_p\, \mathbb{E} \left[\left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2 |\sigma_i(\tau, \varphi_{\hat{\alpha}}(\tau,\eta))|^p\, d\tau \right)^{\frac{2}{p}} \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2\, d\tau \right)^{\frac{p-2}{p}} \right]^{\frac{p}{2}} \\
&= C_p \int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2 \|\sigma(\tau, \varphi_{\hat{\alpha}}(\tau,\eta))\|_p^p\, d\tau \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2\, d\tau \right)^{\frac{p-2}{2}} \\
&\leq C_p \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2\, d\tau \right)^{\frac{p}{2}} 2^{p-1} (L^p \operatorname*{esssup}_{\tau\in[0,T]} \|\varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p + M^p).
\end{aligned}$$

Combining the above calculations and by the definition of $\|\cdot\|_\gamma$ yields the estimate

$$\begin{aligned}
&\frac{\|\varphi_\alpha(t,\eta) - \varphi_{\hat{\alpha}}(t,\eta)\|_p^p}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \\
&\leq \frac{\kappa 2^{p-1} \int_0^t (t-\tau)^{2\alpha-2} \frac{\|\varphi_\alpha(\tau,\eta) - \varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p}{E_{2\alpha-1}(\gamma\tau^{2\alpha-1})} E_{2\alpha-1}(\gamma\tau^{2\alpha-1}) d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \\
&\quad + 2^{3p-3} (L^p \operatorname*{esssup}_{\tau\in[0,T]} \|\varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p + M^p) \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2 d\tau \right)^{\frac{p}{2}} T^{\frac{p}{2}} \\
&\quad + 2^{3p-3} (L^p \operatorname*{esssup}_{\tau\in[0,T]} \|\varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p + M^p) C_p \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2\, d\tau \right)^{\frac{p}{2}} \\
&\leq \frac{\kappa 2^{p-1} \Gamma(2\alpha-1)}{\gamma} \|\varphi_\alpha(\cdot,\eta) - \varphi_{\hat{\alpha}}(\cdot,\eta)\|_\gamma^p \\
&\quad + 2^{3p-3} (L^p \operatorname*{esssup}_{\tau\in[0,T]} \|\varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p + M^p) \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2 d\tau \right)^{\frac{p}{2}} T^{\frac{p}{2}} \\
&\quad + 2^{3p-3} (L^p \operatorname*{esssup}_{\tau\in[0,T]} \|\varphi_{\hat{\alpha}}(\tau,\eta)\|_p^p + M^p) C_p \left(\int_0^t (g(t,\tau,\alpha,\hat{\alpha}))^2\, d\tau \right)^{\frac{p}{2}},
\end{aligned}$$

where in the finall step, we have used Lemma 5 in [6]. Thus, we have

$$\left(1-\frac{\kappa 2^{p-1}\Gamma(2\alpha-1)}{\gamma}\right)\|\varphi_\alpha(\cdot,\eta)-\varphi_{\hat\alpha}(\cdot,\eta)\|_\gamma^p$$
$$\leq 2^{3p-3}(L^p \operatorname*{esssup}_{\tau\in[0,T]}\|\varphi_{\hat\alpha}(\tau,\eta)\|_p^p+M^p)\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2 d\tau\right)^{\frac{p}{2}} T^{\frac{p}{2}}$$
$$+2^{3p-3}(L^p \operatorname*{esssup}_{\tau\in[0,T]}\|\varphi_{\hat\alpha}(\tau,\eta)\|_p^p+M^p)C_p\left(\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2\, d\tau\right)^{\frac{p}{2}}.$$

By (9) and $p\geq 2$, therefore to complete the proof it is sufficient to show that

$$\lim_{\hat\alpha\to\alpha}\sup_{t\in[0,T]}\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2\, d\tau=0.$$

Indeed, we have

$$\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2\, d\tau=\int_0^t\frac{(t-\tau)^{2\alpha-2}}{\Gamma^2(\alpha)}\, d\tau+\int_0^t\frac{(t-\tau)^{2\hat\alpha-2}}{\Gamma^2(\hat\alpha)}\, d\tau-2\int_0^t\frac{(t-\tau)^{\alpha+\hat\alpha-2}}{\Gamma(\alpha)\Gamma(\hat\alpha)}\, d\tau$$
$$=\frac{t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}+\frac{t^{2\hat\alpha-1}}{(2\hat\alpha-1)\Gamma^2(\hat\alpha)}-\frac{2t^{\alpha+\hat\alpha-1}}{(\alpha+\hat\alpha-1)\Gamma(\alpha)\Gamma(\hat\alpha)}.$$

Thus, we conclude that

$$\lim_{\hat\alpha\to\alpha}\sup_{t\in[0,T]}\int_0^t (g(t,\tau,\alpha,\hat\alpha))^2\, d\tau=0.$$

The proof is complete. □

Remark 7. To prove Theorem 1(i) and Theorem 1(ii), we only require the assumption (H1) and the following assumption (weaker than (H2)):

(H3) L^p-integrable in time for drift term and essential boundedness in time for diffusion term:

$$\int_0^T |b(\tau,0)|_p^p\, d\tau<\infty,\qquad \operatorname*{esssup}_{\tau\in[0,T]}|\sigma(\tau,0)|_p<M.$$

4. Regularity of solutions

This section is devoted to proving the regularity of solutions to Caputo SFDE.

Proof of Theorem 2. Choose and fix $t,s\in[0,T]$ with $t>s$.

Using (5), we derive that

$$
\begin{aligned}
&\frac{\Gamma^p(\alpha)}{2^{2p-2}}\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p^p \\
&\le \left\|\int_s^t \frac{b(\tau,\varphi_\alpha(\tau,\eta))}{(t-\tau)^{1-\alpha}}\,d\tau\right\|_p^p+\left\|\int_s^t \frac{\sigma(\tau,\varphi_\alpha(\tau,\eta))}{(t-\tau)^{1-\alpha}}\,dW_\tau\right\|_p^p \\
&\quad+\left\|\int_0^s \left|\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right| b(\tau,\varphi_\alpha(\tau,\eta))\,d\tau\right\|_p^p \\
&\quad+\left\|\int_0^s \left|\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right| \sigma(\tau,\varphi_\alpha(\tau,\eta))\,dW_\tau\right\|_p^p.
\end{aligned}
$$

Now, applying the Hölder and the Burkholder-Davis-Gundy inequalities, we arrive at

$$
\begin{aligned}
&\frac{\Gamma^p(\alpha)}{2^{2p-2}}\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p^p \\
&\le \frac{(t-s)^{\alpha p-1}(p-1)^{p-1}}{(\alpha p-1)^{p-1}}\int_s^t \|b(\tau,\varphi_\alpha(\tau,\eta))\|_p^p d\tau \\
&\quad+C_p\int_s^t \frac{\|\sigma(\tau,\varphi_\alpha(\tau,\eta))\|_p^p}{(t-\tau)^{2-2\alpha}}\,d\tau\left(\int_s^t \frac{1}{(t-\tau)^{2-2\alpha}}\,d\tau\right)^{\frac{p-2}{2}} \\
&\quad+T^{\frac{p-2}{2}}\int_0^s \|b(\tau,\varphi_\alpha(\tau,\eta))\|_p^p\,d\tau\left(\int_0^s \left|\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right|^2 d\tau\right)^{\frac{p}{2}} \\
&\quad+C_p\int_0^s \left(\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right)^2 \|\sigma(\tau,\varphi_\alpha(\tau,\eta))\|_p^p\,d\tau \\
&\quad\times\left[\int_0^s \left(\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right)^2 d\tau\right]^{\frac{p-2}{2}}.
\end{aligned}
$$

On the other hand, since $\varphi_\alpha(\cdot,\eta)\in\mathbb{H}^p([0,T])$, there exists $M_1>0$ such that $\operatorname{ess\,sup}_{t\in[0,T]}\|\varphi_\alpha(t,\eta)\|_p^p\le M_1$. This together with (H1) and (H2) implies that

$$
\begin{aligned}
\|b(\tau,\varphi_\alpha(\tau,\eta))\|_p^p &\le 2^{p-1}(L^p\|\varphi_\alpha(\tau,\eta)\|_p^p+\|b(\tau,0)\|_p^p)\le 2^{p-1}(L^pM_1+M^p),\\
\|\sigma(\tau,\varphi_\alpha(\tau,\eta))\|_p^p &\le 2^{p-1}(L^p\|\varphi_\alpha(\tau,\eta)\|_p^p+\|\sigma(\tau,0)\|_p^p)\le 2^{p-1}(L^pM_1+M^p).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\int_0^s \left(\frac{1}{(t-\tau)^{1-\alpha}}-\frac{1}{(s-\tau)^{1-\alpha}}\right)^2 d\tau &\le \int_0^s \left(\frac{1}{(s-\tau)^{2-2\alpha}}-\frac{1}{(t-\tau)^{2-2\alpha}}\right) d\tau \\
&=\frac{s^{2\alpha-1}-t^{2\alpha-1}}{2\alpha-1}+\frac{(t-s)^{2\alpha-1}}{2\alpha-1} \\
&\le \frac{(t-s)^{2\alpha-1}}{2\alpha-1}.
\end{aligned}
$$

Combining the above calculations yields the following estimate

$$\frac{\Gamma^p(\alpha)}{2^{2p-2}}\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p^p$$
$$\le \frac{(2p-2)^{p-1}(L^pM_1+M^p)T^{\frac{p}{2}}}{(\alpha p-1)^{p-1}}(t-s)^{\frac{(2\alpha-1)p}{2}}+\frac{C_p2^{p-1}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}(t-s)^{\frac{(2\alpha-1)p}{2}}$$
$$+\frac{2^{p-1}(T)^{\frac{p}{2}}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}(t-s)^{\frac{(2\alpha-1)p}{2}}+\frac{C_p2^{p-1}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}(t-s)^{\frac{(2\alpha-1)p}{2}}.$$

Thus, we have

$$\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p \le D(t-s)^{\alpha-\frac{1}{2}},$$

where

$$D^p := \frac{2^{2p-2}}{\Gamma^p(\alpha)}\left(\frac{(2p-2)^{p-1}(L^pM_1+M^p)T^{\frac{p}{2}}}{(\alpha p-1)^{p-1}}+\frac{C_p2^{p-1}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}\right)$$
$$+\frac{2^{2p-2}}{\Gamma^p(\alpha)}\left(\frac{2^{p-1}(T)^{\frac{p}{2}}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}+\frac{C_p2^{p-1}(L^pM_1+M^p)}{(2\alpha-1)^{\frac{p}{2}}}\right),$$

which together the fact that $\alpha\in\left(\frac{1}{2},1\right)$ and $p\ge 2$ implies that

$$\lim_{s\to t}\|\varphi_\alpha(t,\eta)-\varphi_\alpha(s,\eta)\|_p = 0.$$

The proof is complete. □

Remark 8. When $p=2$, then result of Theorem 2 coincides with the result in [17] in the special case that $b(t,X(t)) := AX(t)+g(t)$ and $\sigma(t,X(t)) := f(t)$.

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