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A PARALLEL EXTRAGRADIENT SUBGRADIENT ALGORITHM FOR FINDING A SOLUTION OF BILEVEL SYSTEM OF PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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This paper is dedicated to the memory of Professor Hang-Chin Lai.

ABSTRACT. In this paper, we introduce a bilevel system of variational inequality problems and establish a strong convergence theorem for finding a solution of the considered problem under reasonable assumptions on the problem data. A numerical example is given to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Throughout this article, we assume that C is a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ with an inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and the induced norm $\| \cdot \|$. Let $I = \{1, 2, ..., N\}$ be a finite indexing set. Consider the following *bilevel system of variational inequality problems* (shortly, $\mathrm{BSVIP(A_i, B, C)}$)

$$\text{find } x^* \in \Omega := \bigcap_{i=1}^{N} \mathrm{SVIP}(A_i, C) \text{ such that } \langle B(x^*), y - x^* \rangle \geq 0, \ \forall y \in \Omega, \tag{1.1}$$

where B and $\{A_i\}_{i \in I}$ are finite family of nonlinear mappings from C to $\mathbb{H}$ and $\mathrm{SVIP}(A_i, C)$ denotes the nonempty solution set of the variational inequality problem defined as:

$$\langle A_i(x^*), y - x^* \rangle \geq 0, \ \forall y \in C. \tag{1.2}$$

The formulation of the $\mathrm{BSVIP(A_i, B, C)}$ is inspired in a bid to provide a single framework within which a wide variety of mathematical problems such as the system of variational inequality problems, bilevel optimization problems, bilevel Nash equilibrium problems, bilevel variational inequality problems.

Over the past few years, bilevel problems have become an increasingly popular way of modelling various important practical problems in many different fields, such as transportation (network design, optimal pricing), management (network facility location, coordination of multi-divisional firms), engineering (optimal design, optimal chemical equilibria), economics (Stackelberg games, principal-agent problem, taxation, policy decisions), etc. see, for example, [3, 6, 7, 11, 15, 20, 22, 33].

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Let us assume that problem (1.2) has a solution and let us denote by $\mathrm{SOL}(\mathrm{A_i}, \mathrm{B}, \mathrm{C})$ the solution set of problem (1.1). In the case $N = 1$, we see that, the $\mathrm{BSVIP}(\mathrm{A_i}, \mathrm{B}, \mathrm{C})$ reduces to the bilevel variational inequality problem, introduced by Kalashnikov and Kalashnikova [16] and developed by Anh et al. [4] (see, also [2, 5, 12, 30]) as

$$\text{(1.3)} \qquad \text{find } x^* \in \mathrm{VIP}(\mathrm{A}, \mathrm{C}) \text{ such that } \langle B(x^*), y - x^* \rangle \geq 0, \ \forall y \in \mathrm{VIP}(\mathrm{A}, \mathrm{C}),$$

where A and B are nonlinear mappings from C to $\mathbb{H}$ and $\mathrm{VIP}(\mathrm{A}, \mathrm{C})$ denotes the nonempty solution set of the following variational inequality problem:

$$\text{(1.4)} \qquad \langle A(x^*), y - x^* \rangle \geq 0, \ \forall y \in C.$$

Observe that when B is strongly monotone, then problem (1.3) has at most one solution and in such instance, it is well-known that this problem becomes

$$\text{find } x^* \in \mathrm{VIP}(\mathrm{A}, \mathrm{C}) \text{ such that } x^* = \mathrm{P}_{\mathrm{VIP}(\mathrm{A},\mathrm{C})} \left(x^* - \lambda B(x^*) \right),$$

where, $\lambda > 0$ and $\mathrm{P}_{\mathrm{VIP}(\mathrm{A},\mathrm{C})}$ is a metric projection onto $\mathrm{VIP}(\mathrm{A}, \mathrm{C})$. Another consequence of $\mathrm{BSVIP}(\mathrm{A_i}, \mathrm{B}, \mathrm{C})$ is the bilevel optimization problem, widely studied in [10, 25, 27, 28]. In this case, for each $i \in I$, we take finite family of mappings $\{\psi_i\}_{i\in I}, \{\hat{\psi}\}$ from C to $\mathbb{R}$ and relate as

$$\langle A_i(x^*), y - x^* \rangle = \psi_i(x^*) - \psi_i(y) \text{ and } \langle B(x^*), y - x^* \rangle = \hat{\psi}(x^*) - \hat{\psi}(y), \forall y \in C.$$

Furthermore, when $B \equiv 0$, we recover the system of variational inequalities (1.2) due to Pang [24], developed by Konnov [19–21]. This problem has been applied extensively to model traffic equilibrium problems, spartial equilibrium problems, Nash equilibrium problems and general equilibrium programming problems, see, e.g, [1, 8, 9, 17, 21, 34].

The first projected subgradient algorithm for finding an approximate solution of the bilevel variational inequality problems was introduced in 2014 by Anh et al. [5]. To improve the practical benefits of this method, Anh [2] initiated the following strongly convergent extragradient subgradient algorithm for solving (1.3)

(1.5)

$$\begin{cases} \text{choose initial point } x_1 \in C, \text{ and for all } n \in \mathbb{N}, \text{ compute update } x_{n+1} \text{ via} \\ y_n = \mathrm{P_C}\left(x_n - \rho_n A(x_n)\right), \ T_n = \{u \in \mathbb{H} : \langle x_n - \rho_n A(x_n) - y_n, u - y_n \rangle \leq 0\}, \\ z_n = \mathrm{P_{T_n}}\left(x_n - \rho_n A(y_n)\right), \ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n - \gamma_n \delta B(z_n). \end{cases}$$

The advantage of algorithm (1.5) over Algorithm 1 of Anh et al. [5], is that, the second projection can be found in a close form thereby lower computational cost and so could be used in instances of problems with large datasets.

In this article, our sole aim is to develop a parallel extragradient subgradient algorithmic approach for solving the bilevel system of variational inequality problem (1.1). In section 2, we recall some elementary concepts and lemmas which are used in the proof of our main results. In section 3, we list some conditions often imposed on $\{A_i\}_{i\in I}$ and $\{B\}$ and combine ideas from parallel approximation method introduced by Kim and Dinh [18] with the approaches of Anh [2] to propose a parallel extragradient subgradient algorithm for finding the unique approximate solution of $\mathrm{BSVIP}(\mathrm{A_i}, \mathrm{B}, \mathrm{C})$ (see, [31] for more detail parallel extragradient algorithms). We prove that the sequence $\{x_n\}$ generated by the proposed algorithm converges strongly to the unique element of $\mathrm{SOL}(\mathrm{A_i}, \mathrm{B}, \mathrm{C})$. The last section is devoted to present a numerical example to illustrate the convergence of the proposed algorithm.

2. Preliminaries

Now, let C be a nonempty closed convex subset of $\mathbb{H}$. Recall that the metric projection onto C is the mapping $\mathrm{P_C} : \mathbb{H} \to C$ which assigns each $x \in \mathbb{H}$ to the unique point $\mathrm{P_C}(x)$ in C satisfying

$$\|x - \mathrm{P_C}(x)\| = \min\{\|x - y\| : y \in C\}.$$

For example, if $H = \{y \in \mathbb{H} : \langle w, y - y^0\rangle \leq 0\}$ for some $w, y^0 \in \mathbb{H}$, then for any $x \in \mathbb{H}$, we have

$$P_H(x) = \begin{cases} x - \frac{\langle w, x-y^0\rangle}{\|w\|^2} w & \text{if } x \notin H \\ x & \text{if } x \in H. \end{cases}$$

The following properties of metric projection is quite well-known.

Lemma 2.1 (Section 3, [14]). *Given $x \in \mathbb{H}$ and $z \in C$. Then, we have*

(a) $\|\mathrm{P_C}(x) - \mathrm{P_C}(y)\|^2 \leq \langle \mathrm{P_C}(x) - \mathrm{P_C}(y), x - y\rangle, \ \forall\, y \in C$;
(b) $z = \mathrm{P_C}(x) \iff \langle x - z, z - y\rangle \geq 0, \ \forall\, y \in C$;
(c) $\|\mathrm{P_C}(x) - y\|^2 \leq \|x - y\|^2 - \|x - \mathrm{P_C}(x)\|^2, \ \forall\, y \in C.$

Definition 2.2 ([2, 4]). *A mapping $\phi : \mathbb{H} \to \mathbb{H}$ with domain $D(\phi)$ in $\mathbb{H}$ is said to be:*

(a) β-strongly monotone *on $D(\phi)$, if there exists $\beta > 0$ such that*

$$\langle \phi(x) - \phi(y), x - y\rangle \geq \beta\|x - y\|^2, \ \forall\, x, y \in D(\phi);$$

(b) monotone *on $D(\phi)$, if*

$$\langle \phi(x) - \phi(y), x - y\rangle \geq 0, \ \forall\, x, y \in D(\phi);$$

(c) pseudomonotone *on $D(\phi)$, if*

$$\langle \phi(x), y - x\rangle \geq 0 \ \Rightarrow \ \langle \phi(y), y - x\rangle \geq 0, \ \forall\, x, y \in D(\phi);$$

(d) L-Lipschitz continuous *on $D(\phi)$, if $\|\phi(x) - \phi(y)\| \leq L\|x - y\|, \ \forall\, x, y \in D(\phi)$.*

We need the following basic results in our convergence analysis.

Lemma 2.3 ([32]). *Let $\{\gamma_n\}_{n=1}^\infty$ be a sequences in $(0, 1)$ and $\{\delta_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$ satisfying $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^\infty |\gamma_n\delta_n| < \infty$. If $\{a_n\}_{n=1}^\infty$ is a sequence of nonnegative real number such that $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$, $\forall n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4 ([23]). *Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that it does not decrease at infinity, in the sense that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$a_{n_i} < a_{n_i+1}, \ \forall\, i \geq 0.$$

Then there exists an increasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} \tau(n) = \infty$ and the following properties are satisfiied by all (sufficiently large) numbers $n \in \mathbb{N}$,

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}.$$

In fact, $\tau(n) = \max\{k \le n : a_k < a_{k+1}\}$.

Lemma 2.5 ([2]). *Let* $\phi : \mathbb{H} \to \mathbb{H}$ *be a* β*- strongly monotone and L-Lipschitz continuous on* $\mathbb{H}$ *with* $0 < \theta < 1,\ 0 \le \sigma \le 1 - \theta$ *and* $0 < \mu < \frac{2\beta}{L^2}$. *Then for all* $x, y \in \mathbb{H}$, *we have*

$$\left\| \big((1-\sigma)x - \theta\mu\phi(x)\big) - \big((1-\sigma)y - \theta\mu\phi(y)\big) \right\| \le (1 - \sigma - \theta\tau)\left\| x - y \right\|,$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.

3. Main results

In this section, a parallel extragradient subgradient algorithm for finding a unique approximate solution of bilevel system of variational inequalities is introduced and its convergence is investigated.

We first impose the following conditions on A_i and B for each $i \in I$ in consistent with Anh [2] and Anh et al. [4].

Condition A

(A1) $B : \mathbb{H} \to \mathbb{H}$ is β- strongly monotone and L-Lipschitz continuous on H.
(A2) $A_i : \mathbb{H} \to \mathbb{H}$ is pseudomonotone and γ-Lipschitz continuous on $\mathbb{H}$.
(A3) $\limsup_{n\to\infty} \langle A_i(x_n), y - y_n \rangle \le \langle A_i(x^*), y - y^* \rangle$ for every sequence $\{x_n\}, \{y_n\} \subset \mathbb{H}$ converging weakly to x^* and y^*, respectively.

The following algorithm give us a way to find a solution of $\text{BSVIP}(A_i, B, C)$.

Algorithm 1 (Parallel extragradient subgradient algorithm for bilevel system of variational inequalities).
Initialization: Given an initial choices $x_0 \in C$ and $0 < \mu < \frac{2\beta}{L^2}$. Choose parameters $\{\rho_n^i\} \subset [\rho, \hat{\rho})$ for some $\rho, \hat{\rho} \in (0, \frac{1}{\gamma})$; $\{\alpha_n^i\} \subset [\alpha, \hat{\alpha}) \subset (0, 1]$ for each $i \in I$ such that $\sum_{i=1}^N \alpha_n^i = 1$. Take $\{\delta_n\}$ in $(0,1)$ such that $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^\infty \delta_n = \infty$ and $0 \le \gamma_n \le 1 - \delta_n$ with $\lim_{n\to\infty} \gamma_n = \zeta < 1$.
Iterative steps: Assume that x_n is known for $n \in \mathbb{N} \cup \{0\}$, then compute the update x_{n+1} by the following rule:
Step 1: Compute the following projections in parallel

$$y_n^i = \mathrm{P_C}\left(x_n - \rho_n^i A_i(x_n)\right),$$
$$v_n^i = \mathrm{P}_{\mathrm{T}_n^i}\left(x_n - \rho_n^i A_i(y_n^i)\right),$$

where $T_n^i = \left\{u \in \mathbb{H} : \langle x_n - \rho_n^i A(x_n) - y_n^i, u - y_n^i \rangle \le 0\right\}$.
Step 2: Set $z_n = \sum_{i=1}^N \alpha_n^i v_n^i$.
Step 3: Update

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n - \delta_n \mu B(z_n). \tag{3.1}$$

Stopping criterion: If $x_{n+1} = x_n$, then x_n is a solution of the BSVIP (1.1) and the iterative process stops, otherwise, put $n := n + 1$ and go back to **Step 1**.

Remark 3.1. *At iteration* n, *if* $y_n^i = x_n$ *then* x_n *is an element of* $\text{SVIP}(A_i, C)$. *Similarly, if* $z_n \in \Omega$ *and* $B(z_n) = z_n$, *then* z_n *is a unique element of* $\text{VIP}(B, \Omega)$. *Therefore, in our convergence theorem, we will assume that this does not occur* *after finitely many iterations, so that Algorithm 1 generates an infinite sequence satisfying $y_n^i \neq x_n$ for some $i \in I$ and $B(z_n) \neq z_n$ for all $n \in \mathbb{N}$.*

We need the following lemma to prove the convergence of Algorithm 1.

Lemma 3.2. *Assume that A_i satisfies condition (A2), for each $i \in I$ and $x^* \in \Omega = \bigcap_{i=1}^N \mathrm{SVIP}(A_i, C)$. Let $\{x_n\}_{n=1}^\infty$ be a sequence generated by Algorithm 1, then we have*

$$\begin{aligned}\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 &- \sum_{i=1}^N \alpha_n^i (1-\gamma\rho_n^i)\|x_n - y_n^i\|^2 \\ &- \sum_{i=1}^N \alpha_n^i (1-\gamma\rho_n^i)\|y_n^i - v_n^i\|^2.\end{aligned}$$

Proof. Let $x^* \in \Omega$, this implies that $x^* \in \mathrm{SVIP}(A_i, C)$ for each $i \in I$.
Since $y_n^i = \mathrm{P_C}\left(x_n - \rho_n^i A_i(x_n)\right)$, then by Lemma 2.1 (b), we have

$$\langle x_n - \rho_n^i A_i(x_n) - y_n^i, z - y_n^i\rangle \le 0,\ \forall\ z \in C. \tag{3.2}$$

By definition, we have

$$T_n^i = \left\{u \in \mathbb{H} : \langle x_n - \rho_n^i A(x_n) - y_n^i, u - y_n^i\rangle \le 0\right\},\ \text{for each } i \in I. \tag{3.3}$$

It follows from (3.3) and (3.2) that $C \subset T_n^i$ for each $i \in I$.
Using $v_n^i = \mathrm{P_{T_n^i}}\left(x_n - \rho_n^i A_i(y_n^i)\right)$ and by Lemma 2.1 (c), we have

$$\begin{aligned}\|v_n^i - x^*\|^2 =& \|\,\mathrm{P_{T_n^i}}\left(x_n - \rho_n^i A_i(y_n^i)\right) - x^*\|^2 \\ \le& \|x_n - \rho_n^i A_i(y_n^i) - x^*\|^2 - \|x_n - \rho_n^i A_i(y_n^i)) - v_n^i\|^2 \\ =& \|x_n - x^*\|^2 - 2\rho_n^i\langle x_n - x^*, A_i(y_n^i)\rangle \\ &+ (\rho_n^i)^2\|A_i(y_n^i)\|^2 - \|x_n - v_n^i\|^2 \\ &+ 2\rho_n^i\langle x_n - v_n^i, A_i(y_n^i)\rangle - (\rho_n^i)^2\|A_i(y_n^i)\|^2 \\ =& \|x_n - x^*\|^2 - \|x_n - v_n^i\|^2 + 2\rho_n^i\langle x^* - v_n^i, A_i(y_n^i)\rangle \\ =& \|x_n - x^*\|^2 - \|x_n - v_n^i\|^2 - 2\rho_n^i\langle y_n^i - x^*, A_i(y_n^i)\rangle \\ &+ 2\rho_n^i\langle y_n^i - v_n^i, A_i(y_n^i)\rangle.\end{aligned} \tag{3.4}$$

By pseudomonotonity of A_i for each $i \in I$, we have that

$$\langle y_n^i - x^*, A(y_n^i)\rangle \ge 0,\ \forall\ i \in I. \tag{3.5}$$

It folows from (3.4) and (3.5) that

$$
\begin{aligned}
\|v_n^i - x^*\|^2 \leq & \|x_n - x^*\|^2 - \|x_n - v_n^i\|^2 + 2\rho_n^i \langle y_n^i - v_n^i, A_i(y_n^i)\rangle \\
= & \|x_n - x^*\|^2 + 2\rho_n^i \langle y_n^i - v_n^i, A_i(y_n^i)\rangle - \|x_n - y_n^i + y_n^i - v_n^i\|^2 \\
= & \|x_n - x^*\|^2 + 2\rho_n^i \langle y_n^i - v_n^i, A_i(y_n^i)\rangle - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& - 2\langle y_n^i - v_n^i, x_n - y_n^i\rangle \\
= & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& + 2\langle y_n^i - v_n^i, \rho_n^i A_i(y_n^i) - x_n + y_n^i\rangle \\
= & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& + 2\langle x_n - \rho_n^i A_i(x_n) - y_n^i, v_n^i - y_n^i\rangle \\
& + 2\rho_n^i \langle A_i(x_n) - A_i(y_n^i), v_n^i - y_n^i\rangle \\
\leq & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \qquad (3.6) \\
& + 2\rho_n^i \langle A_i(x_n) - A_i(y_n^i), v_n^i - y_n^i\rangle \\
\leq & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& + 2\rho_n^i \|A_i(x_n) - A_i(y_n^i)\| \|v_n^i - y_n^i\| \\
\leq & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& + 2\gamma\rho_n^i \|x_n - y_n^i\| \|v_n^i - y_n^i\| \\
\leq & \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 - \|y_n^i - v_n^i\|^2 \\
& + \gamma\rho_n^i \big(\|x_n - y_n^i\|^2 + \|y_n^i - v_n^i\|^2\big) \\
= & \|x_n - x^*\|^2 - \big(1 - \gamma\rho_n^i\big) \|x_n - y_n^i\|^2 \\
& - \big(1 - \gamma\rho_n^i\big) \|y_n^i - v_n^i\|^2.
\end{aligned}
$$

Since inequality (3.6) holds for each $i \in I$ and $z_n = \sum_{i=1}^N \alpha_n^i v_n^i$, we obtain

$$
\begin{aligned}
\|z_n - x^*\|^2 = & \|\sum_{i=1}^N \alpha_n^i v_n^i - x^*\|^2 \\
= & \|\sum_{i=1}^N \alpha_n^i (v_n^i - x^*)\|^2 \qquad (3.7) \\
= & \sum_{i=1}^N \alpha_n^i \|v_n^i - x^*\|^2 - \frac{1}{2}\sum_{i=1}^N \sum_{t=1}^N \alpha_n^i \alpha_n^t \|v_n^i - v_n^t\|^2.
\end{aligned}
$$

Substituting equation (3.6) in the above equation, we have that

$$
\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \sum_{i=1}^N \alpha_n^i (1 - \gamma\rho_n^i) \|x_n - y_n^i\|^2 - \sum_{i=1}^N \alpha_n^i (1 - \gamma\rho_n^i) \|y_n^i - v_n^i\|^2. \qquad (3.8)
$$

Owing to the hypothesis on $\rho_n^i,\ i \in I$ and γ we have that $(1-\gamma\rho_n^i) > 0$ for each $i \in I$. Therefore, equation (3.8) reduces to

$$\|z_n - x^*\| \leq \|x_n - x^*\|,\ \forall n \in \mathbb{N}. \tag{3.9}$$

□

Next, by using the above lemma, we prove the following theorem for solving the BSVIP (1.1).

Theorem 3.3. *For each $i \in I$, let A_i and B satisfy conditions $(A1)-(A3)$ such that $\Omega = \bigcap_{i=1}^N \mathrm{SVIP}(A_i, C) \neq \emptyset$. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by Algorithm 1 converges strongly to the unique element $x^* \in \mathrm{SOL}(A_i, B, C)$.*

Proof. For each $i \in I$, let $x^* \in \Omega$. Using Algorithm 1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| =& \|\gamma_n x_n + (1-\gamma_n) z_n - \delta_n \mu B(z_n) - x^*\| \\
=& \|(1-\gamma_n) z_n - \delta_n \mu B(z_n) - (1-\gamma_n) x^* + \delta_n \mu B(x^*) \\
& + \gamma_n (x_n - x^*) - \delta_n \mu B(x^*)\| \\
\leq& \|((1-\gamma_n) z_n - \delta_n \mu B(z_n)) - ((1-\gamma_n) x^* - \delta_n \mu B(x^*))\| \\
& + \gamma_n \|x_n - x^*\| + \delta_n \mu \|B(x^*)\|.
\end{aligned}$$

Using Lemma 2.5 and inequality (3.9) we have

$$\begin{aligned}
\|x_{n+1} - x^*\| \leq& (1-\gamma_n - \delta_n \tau)\|z_n - x^*\| + \gamma_n \|x_n - x^*\| + \delta_n \mu \|B(x^*) \\
\leq& (1-\gamma_n - \delta_n \tau)\|x_n - x^*\| + \gamma_n \|x_n - x^*\| + \delta_n \mu \|B(x^*)\| \\
=& (1-\delta_n \tau)\|x_n - x^*\| + \delta_n \mu \|B(x^*)\| \\
=& (1-\delta_n \tau)\|x_n - x^*\| + \delta_n \tau \frac{\mu \|B(x^*)\|}{\tau} \\
\leq& \max\left\{\|x_n - x^*\|, \frac{\mu \|B(x^*)\|}{\tau}\right\} \\
& \vdots \\
\leq& \max\left\{\|x_0 - x^*\|, \frac{\mu \|B(x^*)\|}{\tau}\right\},
\end{aligned}$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.
Hence, $\{x_n\}_{n=1}^\infty$, $\{B(x_n)\}_{n=1}^\infty$, and $\{z_n\}_{n=1}^\infty$ are bounded sequences.

Further, using Algorithm 1, Lemma 2.5 again and the inequality

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, x - y\rangle,\ \forall\ x, y \in \mathbb{H},$$

we get

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 =&\|\gamma_n x_n+(1-\gamma_n)z_n-\delta_n\mu B(z_n)-x^*\|^2\\
=&\|((1-\gamma_n)z_n-\delta_n\mu B(z_n))-((1-\gamma_n)x^*-\delta_n\mu B(x^*))\\
&+\gamma_n(x_n-x^*)-\delta_n\mu B(x^*)\|^2\\
\leq&\|((1-\gamma_n)z_n-\delta_n\mu B(z_n))-((1-\gamma_n)x^*-\delta_n\mu B(x^*))\\
&+\gamma_n(x_n-x^*)\|^2-2\delta_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\
\leq&\big[\|(1-\gamma_n)z_n-\delta_n\mu B(z_n))-((1-\gamma_n)x^*-\delta_n\mu B(x^*))\|\\
&+\gamma_n\|x_n-x^*\|\big]^2-2\delta_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\
\leq&\big[(1-\gamma_n-\delta_n\tau)\|z_n-x^*\|+\gamma_n\|x_n-x^*\|\big]^2\\
&-2\delta_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\
\leq&(1-\gamma_n-\delta_n\tau)\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\
&-2\delta_n\mu\langle B(x^*),x_{n+1}-x^*\rangle.
\end{aligned}
$$

This means that

$$\|x_{n+1}-x^*\|^2\leq(1-\delta_n\tau)\|x_n-x^*\|^2+2\delta_n\mu\langle B(x^*),x^*-x_{n+1}\rangle. \tag{3.10}$$

Now, we show that x_n converges strongly to the unique solution x^* of the BSVIP (1.1), in that purpose, it is enough to show that

$$\lim_{n\to\infty}\|x_n-x^*\|=0.$$

Indeed, we consider two possibilities on the sequence $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ with respect to relation (3.10).

CASE 1. We assume that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ is decreasing for $n\geq n_0$.

Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, then the limit of $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ exists. In this case, it follows from equation (3.10) and (3.9) that

$$
\begin{aligned}
0\leq&\|x_n-x^*\|^2-\|z_n-x^*\|^2\\
\leq&-\frac{\delta_n\tau}{1-\gamma_n}\|z_n-x^*\|^2-\frac{2\delta_n\mu}{1-\gamma_n}\langle B(x^*),x^*-x_{n+1}\rangle\\
&+\frac{1}{1-\gamma_n}\left(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\right).
\end{aligned}
$$

Owing to the hypotheses on $\{\gamma_n\}$ and $\{\delta_n\}$, we get

$$\lim_{n\to\infty}\left(\|x_n-x^*\|^2-\|z_n-x^*\|^2\right)=0. \tag{3.11}$$

Since $0<\rho_n^i<\hat{\rho}$ for each $i\in I$, we get from from (3.8) that

$$
\begin{aligned}
\sum_{i=1}^{N}\alpha_n^i(1-\hat{\rho})\|x_n-y_n^i\|^2\leq&\sum_{i=1}^{N}\alpha_n^i(1-\rho_n^i\gamma)\|x_n-y_n^i\|^2\\
\leq&\|x_n-x^*\|^2-\|z_n-x^*\|^2,
\end{aligned} \tag{3.12}
$$

and

$$\sum_{i=1}^{N} \alpha_n^i (1-\hat{\rho})\|y_n^i - v_n^i\|^2 \leq \sum_{i=1}^{N} \alpha_n^i (1-\rho_n^i \gamma)\|y_n^i - v_n^i\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2. \tag{3.13}$$

Therefore, combining with (3.11), we deduce from inequalities (3.12) and (3.13) that

$$\lim_{n\to\infty}\|x_n - y_n^i\| = 0 \text{ and } \lim_{n\to\infty}\|y_n^i - v_n^i\| = 0 \text{ for each } i \in I. \tag{3.14}$$

Further, since $\{x_n\}_{n=1}^{\infty}$ is bounded and $\mathbb{H}$ is reflexive, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$x_{n_k} \rightharpoonup \theta \text{ as } k \to \infty \text{ and} \tag{3.15}$$

$$\limsup_{n\to\infty}\langle B(x^*), x^* - x_{n+1}\rangle = \lim_{k\to\infty}\langle B(x^*), x^* - x_{n_k+1}\rangle. \tag{3.16}$$

It follows from (3.14), (3.15) and the boundedness of $\{y_{n_k}^i\}_{k=1}^{\infty}$ for each $i \in I$ that

$$y_{n_k}^i \rightharpoonup \theta \text{ as } \ k \to \infty \text{ for each } i \in I. \tag{3.17}$$

Now, since C is closed and convex, then it is weakly closed and this implies that $\theta \in C$.

Next, we will show that $\theta \in \Omega$. Indeed, observe that (3.15) and (3.16) we have

$$\limsup_{n\to\infty}\langle B(x^*), x^* - x_{n+1}\rangle = \lim_{k\to\infty}\langle B(x^*), x^* - x_{n_k+1}\rangle = \langle B(x^*), x^* - \theta\rangle. \tag{3.18}$$

Based on definition of $y_n^i = \mathrm{P}_C\left(x_n - \rho_n^i A_i(x_n)\right)$ and by using Lemma 2.1 (b), we get

$$\langle x_n - \rho_n^i A_i(x_n) - y_n^i, x - y_n^i\rangle \leq 0, \ \forall\ x \in C. \tag{3.19}$$

By utilizing the pseudomonotonicity of A_i for each $i \in I$ with the fact that $\rho_{n_k}^i \geq \rho > 0$, we obtain

$$\begin{aligned}
\langle A_i(x), x_{n_k} - x\rangle &\leq \langle A_i(x_{n_k}), x_{n_k} - x\rangle \\
&= \langle A_i(x_{n_k}), x_{n_k} - y_{n_k}^i\rangle + \frac{1}{\rho_{n_k}^i}\langle x_{n_k} - y_{n_k}^i, y_{n_k}^i - x\rangle \\
&\quad + \frac{1}{\rho_{n_k}^i}\langle x_{n_k} - \rho_{n_k}^i A_i(x_{n_k}) - y_{n_k}^i, x - y_{n_k}^i\rangle \\
&\leq \langle A_i(x_{n_k}), x_{n_k} - y_{n_k}^i\rangle + \frac{1}{\rho_{n_k}^i}\langle x_{n_k} - y_{n_k}^i, y_{n_k}^i - x\rangle \\
&\leq \|A_i(x_{n_k})\|\|x_{n_k} - y_{n_k}^i\| + \frac{1}{\rho_{n_k}^i}\|x_{n_k} - y_{n_k}^i\|\|y_{n_k}^i - x\| \\
&\leq \|A_i(x_{n_k})\|\|x_{n_k} - y_{n_k}^i\| + \frac{1}{\rho}\|x_{n_k} - y_{n_k}^i\|\|y_{n_k}^i - x\|.
\end{aligned}$$

It follows from the boundedness of $\{A_i(x_{n_k})\}_{k=1}^{\infty}$, $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y^i_{n_k}\}_{k=1}^{\infty}$ for each $i \in I$ and equation (3.14), (3.17), (3.15) that

$$0 \leq \limsup_{k\to\infty}\langle A_i(x), x - x_{n_k}\rangle \leq \langle A_i(x), x - \theta\rangle, \forall x \in C \text{ and each } i \in I. \quad (3.20)$$

Let $x_t = (1-t)\theta + tx$, $t \in [0,1]$. This and relation (3.20) imply that

$$\begin{aligned}0 &\leq \langle A_i(x_t), x_t - \theta\rangle \leq \langle A_i(x_t), (1-t)\theta + tx - \theta\rangle \\ &= t\langle A_i(x_t), x - \theta\rangle.\end{aligned}$$

Hence, by Cauchy- Schwarz inequality for all $t \in (0,1]$, we obtain

$$\begin{aligned}0 &\leq \langle A_i(x_t), x - \theta\rangle \\ &= \langle A_i(x_t) - A_i(\theta), x - \theta\rangle + \langle A_i(\theta), x - \theta\rangle \\ &\leq \gamma\|x_t - \theta\|\|x - \theta\| + \langle A_i(\theta), x - \theta\rangle \\ &= \gamma\|(1-t)\theta + tx - \theta\|\|x - \theta\| + \langle A_i(\theta), x - \theta\rangle \\ &= \gamma t\|x - \theta\|\|x - \theta\| + \langle A_i(\theta), x - \theta\rangle, \text{ for each } i \in I.\end{aligned}$$

Taking limit as $t \to 0^+$, we get

$$\langle A_i(\theta), x - \theta\rangle \geq 0, \ \forall x \in C \text{ and for each } i \in I. \text{ This implies that } \theta \in \Omega.$$

Thus, we deduce that

$$\langle B(x^*), \theta - x^*\rangle \geq 0. \quad (3.21)$$

Therefore from equation (3.21) and (3.18), we obtain

$$\limsup_{n\to\infty}\langle B(x^*), x^* - x_{n+1}\rangle \leq 0. \quad (3.22)$$

Note that inequality (3.10) can be expressed as

$$\|x_{n+1} - x^*\|^2 \leq (1 - \delta_n\tau)\|x_n - x^*\|^2 + \delta_n\tau\zeta_n, \quad (3.23)$$

where $\zeta_n = \frac{2\mu}{\tau}\langle B(x^*), x^* - x_{n+1}\rangle$.

Using Lemma 2.3, inequalities (3.22) and (3.23), we can conclude that $x_n \to x^* \in \mathrm{SOL}(A_i, B, C)$ as $n \to \infty$.

Case 2. Suppose that the sequence $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is increasing. That is, there exists a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$\|x_{n_m} - x^*\| \leq \|x_{n_{m+1}} - x^*\|, \ \forall m \in \mathbb{N}.$$

By Lemma 2.4, there exists a nondecreasing sequence $\{\tau(n)\}_{n=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}\tau(n) = \infty$ and for sufficiently large $n \in \mathbb{N}$ we have

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \text{ and } \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \ . \quad (3.24)$$

Now since $\{x_{\tau(n)}\}_{n=1}^{\infty}$ and $\{z_{\tau(n)}\}_{n=1}^{\infty}$ are bounded, then repeating the steps in Case 1, we deduce that

$$\lim_{n\to\infty}\left(\|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2\right) = 0. \quad (3.25)$$

Consequently,

$$\lim_{n\to\infty}\|x_{\tau(n)} - y^i_{\tau(n)}\| = \lim_{n\to\infty}\|y^i_{\tau(n)} - v^i_{\tau(n)}\| = 0 \text{ for each } i \in I. \quad (3.26)$$

Using (3.25) and triangle inequality, we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.27}$$

Since $\{x_{\tau(n)}\}_{n=1}^{\infty}$ is bounded, following similar steps in Case 1, we have that

$$\limsup_{n\to\infty}\langle B(x^*), x^* - x_{\tau(n)+1}\rangle \le 0. \tag{3.28}$$

By combining (3.10) and (3.24), we get

(3.29)

$$\begin{aligned}\|x_{\tau(n)+1} - x^*\|^2 &\le (1-\delta_{\tau(n)}\tau)\|x_{\tau(n)} - x^*\|^2 + 2\delta_{\tau(n)}\mu\langle B(x^*), x^* - x_{\tau(n)+1}\rangle\\ &\le (1-\delta_{\tau(n)}\tau)\|x_{\tau(n)+1} - x^*\|^2 + 2\delta_{\tau(n)}\mu\langle B(x^*), x^* - x_{\tau(n)+1}\rangle.\end{aligned}$$

It follows from (3.24) and (3.29) that

$$\|x_n - x^*\|^2 \le \|x_{\tau(n)+1} - x^*\|^2 \le \frac{2\mu}{\tau}\langle B(x^*), x^* - x_{\tau(n)+1}\rangle. \tag{3.30}$$

By taking limsup in (3.30), using (3.28) as $n\to\infty$, we have $\limsup_{n\to\infty}\|x_n - x^*\|^2 \le 0$. Hence, in both cases, $x_n \to x^* \in \mathrm{SOL}(A_i, B, C)$. This completes the proof. □

Remark 3.4. *When $N = 1$, our results in this paper immediately reduce to the results in Anh et al. [4], Anh et al. [5], and Anh [2] for solving bilevel variational inequalities* (1.3).

When $B(x) = x - x^g$ (x^g is given, may play the role of a guessed solution) then $\mathrm{BSVIP}(A_i, B, C)$ becomes

find $x^* \in \Omega = \cap_{i=1}^{N}\mathrm{SVIP}(A_i, C)$ such that $\langle x^* - x^g, x - x^*\rangle \ge 0,\ \forall x \in \Omega.$

It is equivalent to the following problem

find $x^* \in \Omega = \cap_{i=1}^{N}\mathrm{SVIP}(A_i, C)$ such that $\|x^* - x^g\| \le \|x - x^g\|,\ \forall x \in \Omega.$

This problem can be considered as a generalization of problems studied by Dinh and Muu [13], Konnov [19, 20]. When $x^g = 0$, it reduces to the problem of finding the minimum-norm solution in Ω which studied quite intensively in literature, see, for example [26, 29].

In this case, we get the following corollary.

Corollary 3.5. *Let $x^g \in C$, suppose that $A_i, i = 1, ..., N$ satisfy conditions $(A2)-(A3)$ such that $\Omega = \bigcap_{i=1}^{N}\mathrm{SVIP}(A_i, C) \ne \emptyset$. Let $\{\rho_n^i\} \subset [\rho, \hat\rho)$ for some $\rho, \hat\rho \in (0, \frac{1}{\gamma})$; $\{\alpha_n^i\} \subset [\alpha, \hat\alpha) \subset (0, 1]$ for each $i \in I$ such that $\sum_{i=1}^{N}\alpha_n^i = 1$. Take $\{\delta_n\}$ in $(0, 1)$ such that $\lim_{n\to\infty}\delta_n = 0$, $\sum_{n=1}^{\infty}\delta_n = \infty$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x^0 = x^g;\\ y_n^i = \mathrm{P_C}\left(x_n - \rho_n^i A_i(x_n)\right);\\ T_n^i = \left\{u \in \mathbb{H} : \langle x_n - \rho_n^i A(x_n) - y_n^i, u - y_n^i\rangle \le 0\right\},\\ v_n^i = \mathrm{P}_{\mathrm{T}_n^i}\left(x_n - \rho_n^i A_i(y_n^i)\right), i = 1, 2, ..., N;\\ z_n = \sum_{i=1}^{N}\alpha_n^i v_n^i;\\ x_{n+1} = \delta_n x^g + (1-\delta_n) z_n. \end{cases}$$

Then $\{x_n\}$ converges strongly to $P_\Omega(x^g)$.

Proof. By setting $B(x) = x - x^g$, it can be checked that B is 1-Lipschitz continuous and 1-strongly monotone on $\mathbb{H}$. By choosing $\gamma_n = 0$ and $\mu = 1$, we get Corollary 3.5 imtemidiately from Algorithm 1 and Theorem 3.1. □

4. A Numerical Example

In this section, we illustrate Algorithm 1 by a class of Problems $\mathrm{BSVIP(A_i, B, C)}$). Let $\mathbb{H} = \mathbb{R}^5$ with the standard inner product and the Euclidean norm. Let C be a polyhedron given by the following formula:

$$C = \left\{ x = (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 : \begin{array}{ll} x_1 + x_2 - x_3 - x_4 + 2x_5 & \geq -1 \\ 2x_1 - x_2 - x_3 + x_4 + x_5 & \geq -1 \\ x_1 - x_2 - x_3 + x_4 + x_5 & \geq -1 \end{array} \right\}$$

Let $A_i : \mathbb{R}^5 \to \mathbb{R}^5, i = 1, 2, 3$ be operators defined by the following formula:

$$A_1(x) = (\sin\|x\| + 2)a^1, A_2(x) = (\cos\|x\| + 2)a^2, A_3(x) = (\sin 2\|x\| + 3)a^3,$$

where $a^1 = (1, 1, -1, -1, 2)^T, a^2 = (2, -1, -1, 1, 1)^T, a^3 = (1, -1, -1, 1, 1)^T$. Since, for all $x, y \in \mathbb{R}^5$ such that $\langle A_1(x), y - x\rangle \geq 0$ implies $\langle A_1(y), x - y\rangle \leq 0$, A_1 is pseudomonotone on $\mathbb{R}^5$. Morover, for $\bar{x} = (3\pi/2, 0, 0, 0, 0)^T$ and $\bar{y} = (0, 0, 0, 0, 0)^T$, we have $\bar{x}, \bar{y} \in C$ and $\langle A_1(\bar{x}) - A_1(\bar{y}), \bar{x} - \bar{y}\rangle = -3\pi/2 < 0$, so A_1 is not monotone on $\mathbb{R}^5$. Similarly, we have that A_2, A_3 are pseudomonotone on $\mathbb{R}^5$ but they are not monotone on $\mathbb{R}^5$. Furthermore, forall $x, y \in \mathbb{R}^5$, we have

$$\begin{aligned} \|A_1(x) - A_1(y)\| &= \|a^1\| |\sin\|x\| - \sin\|y\|| \\ &\leq 2\sqrt{5} |\|x\| - \|y\|| \\ &\leq 2\sqrt{5}\|x - y\|. \end{aligned}$$

So, A_1 satisfies the Lipschitz condition on $\mathbb{R}^5$ with constant $\gamma = 2\sqrt{5}$. Similarly, we have that A_2, A_3 are $2\sqrt{5}$-Lipschitz continuous on $\mathbb{R}^5$.

Now, let $t > 1$, consider a mapping $B : \mathbb{R}^5 \to \mathbb{R}^5$ given by:

$$B(x) = (tx_1 + tx_2 + \sin x_1, -tx_1 + tx_2 + \sin x_2, (t-1)x_3, (t-1)x_4, (t-1)x_5)^T.$$

Then, we have

$$\begin{aligned} \langle B(x) - B(y), x - y\rangle &= [t(x_1 - y_1 + x_2 - y_2) + \sin x_1 - \sin y_1](x_1 - y_1) \\ &\quad + [t(-x_1 + y_1 + x_2 - y_2) + \sin x_2 - \sin y_2](x_2 - y_2) \\ &\quad + (t-1)(x_3 - y_3)^2 + (t-1)(x_4 - y_4)^2 + (t-1)(x_5 - y_5)^2 \\ &\geq t[(x_1 - y_1)^2 + (x_2 - y_2)^2)] - (x_1 - y_1)^2 - (x_2 - y_2)^2 \\ &\quad + (x_3 - y_3)^2 + (x_4 - y_4)^2 + (x_5 - y_5)^2 \\ &= (t-1)\|x - y\|^2. \end{aligned}$$

Therefore, B is strongly monotone with constant $\beta = t - 1$. Next, we show that B satisfies the Lipschitz on $\mathbb{R}^5$ with constant $L = \sqrt{2t^2 + 4t + 1}$. Indeed,

$$\begin{aligned} \|B(x) - B(y)\|^2 &= [t(x_1 - y_1 + x_2 - y_2) + \sin x_1 - \sin y_1]^2 \\ &\quad + [t(-x_1 + y_1 + x_2 - y_2) + \sin x_2 - \sin y_2]^2 \end{aligned}$$

$$
\begin{aligned}
&+ (t-1)^2[(x_3-y_3)^2 + (x_4-y_4)^2 + (x_5-y_5)^2] \\
&\le 2t^2[(x_1-y_1)^2 + (x_2-y_2)^2] \\
&\quad + 2t[(x_1-y_1)^2 + 2|x_1-y_1|.|x_2-y_2| + (x_2-y_2)^2] + (x_1-y_1)^2 \\
&\quad + (x_2-y_2)^2 + (t-1)^2[(x_3-y_3)^2 + (x_4-y_4)^2 + (x_5-y_5)^2] \\
&\le (2t^2+4t+1)\|x-y\|^2.
\end{aligned}
$$

So B is L-Lipschitz on $\mathbb{R}^5$.

We implement Algorithm 1 for this problem when $t = 5$ in Matlab R2014a running on a Laptop with Intel(R) Core(TM) i5-3230M CPU @ 2.60GHz, 4 GB RAM.

To compute y_n^i in Step 1, we use the quadratic-program solver from the Matlab optimization, while v_i^n can be computed by explicit formula since T_n^i are half spaces. We choose $\mu = 1.99\frac{\beta}{L^2}$, $\gamma_n = 0.05$, $\delta_n = \frac{0.5}{n+2}$, $\rho_n^i = 0.9\frac{1}{\gamma}$, $\alpha_n^i = 1/3$ for all n and for all $i = 1, 2, 3$. To terminate the Algorithm, we use the stopping criteria: $Tol = \|x^{k+1} - x^k\| < \epsilon$ for some given tolerance ϵ.

The computation results on Algorithm 1 for this problem are reported in Table 1, with some starting points $x^0 = a, x^0 = b$ or $x^0 = c$ with $a = (10, 20, 40, 50, 60)^T$, $b = (1, 1, 1, 1, 1)^T$, $c = (0, 0, 0, 0, 0)^T$ and the tolerance $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$.

From the computed results reported in Tables 1, we can see that the computational

x^0	ϵ	Elapsed Time(s)	Iter.(n)	x_n
a	10^{-3}	7.5192	249	$(-0.0109, 0.0825, 0.3815, -0.1157, -0.3854)^T$
	10^{-5}	22.7293	770	$(-0.0064, 0.0841, 0.3317, -0.1341, -0.4299)^T$
b	10^{-3}	2.5896	75	$(-0.0191, 0.0799, 0.3445, -0.1233, -0.3892)^T$
	10^{-5}	22.5421	752	$(-0.0064, 0.0841, 0.3314, -0.1342, -0.4299)^T$
c	10^{-3}	1.0452	33.0	$(-0.0263, 0.0647, 0.3085, -0.1428, -0.3914)^T$
	10^{-5}	22.1989	752	$(-0.0064, 0.0841, 0.3314, -0.1342, -0.4299)^T$

TABLE 1. Results computed with Algorithm 1.

time and the number of iterations depend very much on the tolerance ϵ.

Conclusion

In this paper, we have demonstrated that it is possible to model system of variational inequalities and bilevel variational inequality problems within a single framework and improve algorithmic procedures for finding a solution of bilevel system of variational inequalities. We constructed a parallel extragradient subgradient algorithm for finding an approximate solution of bilevel system of variational inequalities and implemented a MATLAB version of our Algorithm. We leave for future work the splitting bilevel algorithm which would greatly improve the practical benefit of this method.

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References

[1] S. Al-Homidan, M. Alshahrani and Q. H. Ansari, *System of nonsmooth variational inequalities with applications*, Optimization **64** (2013), 1211–1218.

[2] T. V. Anh, *A strongly convergent subgradient extragradient Halpern method for solving a class of bilevel pseudomonotone variational inequalities*, Vietnam J. Math. **45** (2017), 317–332.

[3] P. K. Anh, T. V. Anh and L. D. Muu, *On bilevel split pseudomonotone variational inequality problems with applications*, Acta Math. Vietnam **42** (2017), 413–429.

[4] P. N. Anh, J. K. Kim and L. D. Muu, *An extragradient algorithm for solving bilevel pseudomonotone variational inequalities*, J. Glob. Optim. **52** (2012), 627–639.

[5] T. T. H. Anh, L. B. Long and T. V. Anh, *A projection method for bilevel variational inequalities*, Inequal. Appl. **2014** (2014): 205.

[6] T. V. Anh and L. D. Muu, *A projection-fixed point method for a class of bilevel variational inequalities with split fixed point constraints*, Optimization **65** (2016), 1229–1243.

[7] J. F. Bard, *Practical Bilevel Optimization: Algorithms and Spplications*, Springer Science and Business Media: Berlin (Heidelberg), 2013.

[8] Y. Censor, A. Gibali and S. Reich, *A von Neumann alternating method for finding common solutions to variational inequalities*, Nonlinear Anal. Series A: TMA **75** (2012), 4596–4603.

[9] Y. Censor, A. Gibali, S. Reich and S. Sabach, *Common solutions to variational inequalities*, Set-Valued Var. Anal. **20** (2012), 229–247.

[10] K. Deb and A. Sinha, *An efficient and accurate solution methodology for bilevel multi-objective programming problems using a hybrid evolutionary-local-search algorithm*, Evol. Comput. **18** (2010), 403–449.

[11] S. Dempe, *Annotated bibliography on bilevel programming and mathematical programs with equilibrium constrints*, Optimization **52** (2003), 333–359.

[12] B. V. Dinh and L. D. Muu, *On penalty and gap function methods for bilevel equilibrium problems*, J. Appl. Math. **2011** (2011): Article ID 646452.

[13] B. V. Dinh and L. D. Muu, *Algorithms for a class of bilevel programs involving pseudomonotone variational inequalities*, Acta Math. Vietnam **38** (2013), 529–540.

[14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[15] H. Iiduka, *Fixed point optimization algorithm and its application to power control in CDMA date networks*, Math. Program, **133** (2012), 227–242.

[16] V. V. Kalashnikov and Kalashnikova, *Solving two-level variational inequality*, J. Glob. Optim. **17** (1991), 289–294.

[17] K. R. Kazmi, *Solving a system of split variational inequality problems*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **62** (2016), 71–82.

[18] D. S. Kim and B. V. Dinh, *Parallel extragradient algorithms for multiple set split equilibrium problems in Hilbert spaces*, Numer Algor. **77** (2018), 741–761.

[19] I. V. Konnov, *On systems of variational inequalities*, Russian Mathematics, **41** (1997), 79–88.

[20] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin, 2001.

[21] I. V. Konnov, *Splitting-type method for systems of variational inequalities*, Comp. Oper. Res. **33** (2006), 520–534.

[22] Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical Programs with Equilibrum Constraints*, Cambridge University Press, Cambridge,1996.

[23] P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set Valued Anal. **16** (2008), 899–912.

[24] J. S. Pang, *Asymmetric variational inequality problems over product sets: applications and iterative methods*, Math. Programming, **31** (1985), 206-219.

[25] S. Sabach and S. Shtern, *A first order method for solving convex bilevel optimization problems*, SIAM J. Optim. **27** (2007), 640–660.

[26] A. Sabharwal and L. C. Potter, *Convexly constrained linear inverse problem: iterative least-squares and regularization*, IEEE Trans. Signal Proc. **46** (1998), 2345–2352.

[27] Y. Shehu, P. T. Vuong and Zemkoho, *An inertial extrapolation method for convex simple bilevel optimization*, Optim. Methods Softw. (2019), 1–19.

[28] M. Solodov, *An explicit descent method for bilevel convex optimization*, J. Convex Anal. **14** (2007), 227–237.

[29] N. N. Tam, J. C. Yao and N. D. Yen, *Solution methods for pseudomonotone variational inequalities*, J. Optim. Theory Appl. **138** (2008), 253–273.

[30] Z. Wan and J. W. Chen, *On bilevel variational inequalities*, J. Oper. Res. Soc. China **1** (2013), 483–510.

[31] L. Wang, L. Yu and T. Li, *Parallel extragradient algorithms for a family of pseudomonotone equilibrium problems and fixed point problems of nonself-nonexpansive mappings in Hilbert space*, J. Nonlinear Funct. Anal. **2020** (2020): Article ID 13.

[32] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), 240–256.

[33] M. H. Xu, M. Li and C. C. Yang, *Neural networks for a class of bi-level variational inequalities*, J. Glob. Optim. **44** (2009), 535–552.

[34] Z. T. Yu and L. J. Lin, *On split system of variational problems*, J. Nonlinear Convex Anal. **16** (2015), 95–107.

[35] Z. T. Yu and L. J. Lin, *Variational inequality problems over split fixed point sets of strict pseudo-nonspreading mappings and quasi-nonexpansive mappings with applications*, Fixed Point Theory Appl. **2014** (2014): 198.

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