Asymptotic Behavior of Discrete Fractional Systems

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Abstract This work comprehensively describes and extends the results on asymptotic properties of linear discrete time-varying fractional order systems with Caputo and Riemann-Liouville forward and backward difference operators. In our considerations we take into account various definitions from the literature of fractional difference operators and we compare the dynamic properties of the corresponding systems. These equations are studied by converting them to the corresponding Volterra convolution equations. The main results are: explicit formulas for solutions, results on asymptotic stability, rates of growth or decay of solutions and solution separation. The work also formulates a number of open questions that may be the subject of future research.

1 Introduction

Continuous fractional calculus has a long history and is nearly as old as the integer-order calculus. Nowadays fractional calculus is studied both for its theoretical interest as well as its use in applications. In spite of the existence of a substantial mathematical theory of continuous fractional calculus, there was no similar development of discrete fractional calculus until very recently [16]. Over the past decade, there has been an

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P. Kulczycki et al. (eds.), *Fractional Dynamical Systems: Methods, Algorithms and Applications*, Studies in Systems, Decision and Control 402,
https://doi.org/10.1007/978-3-030-89972-1_5

increased interest in developing discrete fractional calculus and dynamical models described by discrete fractional calculus (see [17, 23] and the reference therein).

In this work we investigate discrete linear fractional systems with variable coefficients. We consider forward and backward equations with Caputo and Riemann-Liouville operators. For backward equations, we distinguish two types of Caputo operators and two types of Riemann-Liouville operators depending on whether the sum of the fractional order that appears in the definition of the operator includes the initial condition or not. The first works on backward operators were based on the definition of the sum containing the initial condition (see [18]), however later, the operator based on the definition of the sum without the initial condition was introduced, justifying it by the greater similarity of such operators to the case with continuous time (see [15]).

The main aim of this work is to collect the existing results on asymptotic properties of the considered equations, their extension, supplementation, and comparison. The basic research method used in this work is to transform the fractional order equations into the appropriate convolution-type Volterra equations. The work is organized as follows: The next section is devoted to fractional-order differences and the relationships between them. In Sect. 3, we define the equations considered in the work, as well as the initial value problem and the existence and uniqueness of its solution. In Sect. 4 we present for each of the fractional order equations considered, two Volterra equations that are equivalent to them. Section 5 is devoted to multidimensional systems with constant coefficients. The main results of this section are explicit formulas for solutions and conditions for stability. In Sect. 6 we present results on the stability and the rate of growth or decay for one-dimensional equations. The results of separation of solutions of the Volterra equations are discussed in Sect. 7. Finally, Sect. 8 contains conclusions, summaries and directions for further research. In the remainder of this section, we introduce notation and definitions of fractional sums.

Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set $\{0, 1, 2, \dots\}$ of natural numbers including 0, and by $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$ we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \dots\}$ and for a function $x\colon \mathbb{N}_a \to \mathbb{R}^d$ we define $\Delta x\colon \mathbb{N}_a \to \mathbb{R}^d$ by $(\Delta x)(t) = x(t+1) - x(t)$ and $\nabla x\colon \mathbb{N}_{a+1} \to \mathbb{R}^d$ by $(\nabla x)(t) = x(t) - x(t-1)$.

The Euler Gamma function $\Gamma\colon \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{R}$ is defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$$

For $x \in \mathbb{R}$ we write as usual $\lceil x \rceil := \min\{k \in \mathbb{Z}\colon k \geq x\}$ and $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \leq x\}$. A reader who is familiar with fractional difference equations may skip the next definition, see e.g. [17, 27].

Definition 1 *(Basic notions of fractional calculus)* Let $s, \nu \in \mathbb{R}$.
(a) *Falling factorial power* $(s)^{(\nu)}$: If $s+1, s+1-\nu \notin \mathbb{Z}_{\leq 0}$

$$(s)^{(\nu)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\nu)}.$$

(b) *Rising factorial power* $(s)^{\overline{(\nu)}}$: If $s,\ s+\nu \notin \mathbb{Z}_{\leq 0}$

$$(s)^{\overline{(\nu)}} := \frac{\Gamma(s+\nu)}{\Gamma(s)}.$$

(c) *Binomial coefficient* $\binom{s}{\nu}$: If $s+1,\ \nu+1,\ s+1-\nu \notin \mathbb{Z}_{\leq 0}$

$$\binom{s}{\nu} := \frac{(s)^{(\nu)}}{\Gamma(\nu+1)} = \frac{\Gamma(s+1)}{\Gamma(\nu+1)\Gamma(s+1-\nu)}.$$

We recall notation for fractional sums (see e.g. [17]) and relate it to various notions from the literature in a remark afterwards.

Definition 2 *(Fractional sum)* Let $a \in \mathbb{R}$, $\nu \in (0,\ 1)$ and $x : \mathbb{N}_a \to \mathbb{R}^d$. The function $\Delta_a^{-\nu} x : \mathbb{N}_{a+\nu} \to \mathbb{R}^d$ defined by

$$(\Delta_a^{-\nu} x)(t) := \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-k-1)^{(\nu-1)} x(k)$$

is called *ν-th fractional sum of x*.

Remark 1 (a) The ν-th fractional sum of $x : \mathbb{N}_a \to \mathbb{R}^d$ satisfies

$$\begin{aligned}
(\Delta_a^{-\nu} x)(t) &= \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-k-\nu+1)^{\overline{(\nu-1)}} \\
&= \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} \frac{\Gamma(t-k)}{\Gamma(t-k-\nu+1)} x(k) \\
&= \sum_{k=a}^{t-\nu} \binom{t-k-1}{t-k-\nu} x(k) \\
&= \sum_{k=a}^{t-\nu} (-1)^{t-k-\nu} \binom{-\nu}{t-k-\nu} x(k), \qquad t \in \mathbb{N}_{a+\nu},
\end{aligned}$$

or equivalently,

$$(\Delta_a^{-\nu} x)(n+a+\nu) = \sum_{k=a}^{n+a} (-1)^{n+a-k} \binom{-\nu}{n+a-k} x(k), \qquad n \in \mathbb{N}_a.$$

(b) Some authors (see e.g. [10, 18]) define the ν-th fractional sum $\overline{\Delta}_a^{-\nu} x : \mathbb{N}_a \to \mathbb{R}^d$ of $x : \mathbb{N}_a \to \mathbb{R}^d$ as follows

$$
\begin{aligned}
(\overline{\Delta}_a^{-\nu} x)(t) &:= \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t} (t-k+1)^{\overline{(\nu-1)}} x(k) \\
&= \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t} \frac{\Gamma(t-k+\nu)}{\Gamma(t-k+1)} x(k) \\
&= \sum_{k=a}^{t} \binom{t-k+\nu-1}{t-k} x(k) \\
&= \sum_{k=a}^{t} (-1)^{t-k} \binom{-\nu}{t-k} x(k), \qquad t \in \mathbb{N}_a.
\end{aligned}
$$

It is easy to check that

$$
(\Delta_a^{-\nu} x)(t+\nu) = (\overline{\Delta}_a^{-\nu} x)(t), \qquad t \in \mathbb{N}_a. \tag{1}
$$

(c) Some authors (see e.g. [14, 25]) exclude $x(a)$ from the definition of the fractional sum and define the ν-th fractional sum $\widetilde{\Delta}_a^{-\nu} x \colon \mathbb{N}_{a+1} \to \mathbb{R}^d$ of $x \colon \mathbb{N}_a \to \mathbb{R}^d$ as follows

$$
\begin{aligned}
(\widetilde{\Delta}_a^{-\nu} x)(t) &:= \frac{1}{\Gamma(\nu)} \sum_{k=a+1}^{t} (t-k+1)^{\overline{(\nu-1)}} x(k) \\
&= \frac{1}{\Gamma(\nu)} \sum_{k=a+1}^{t} \frac{\Gamma(t-k+\nu)}{\Gamma(t-k+1)} x(k) \\
&= \sum_{k=a+1}^{t} \binom{t-k+\nu-1}{t-k} x(k) \\
&= \sum_{k=a+1}^{t} (-1)^{t-k} \binom{-\nu}{t-k} x(k), \qquad t \in \mathbb{N}_{a+1}.
\end{aligned}
$$

(d) An extensive discussion about relationships between the fractional sums $\overline{\Delta}_a^{-\nu}$ and $\widetilde{\Delta}_a^{-\nu}$ is presented in [2], where it has been shown [2, Lemma 3.1] that

$$
(\overline{\Delta}_{a+1}^{-\nu} \, x|_{\mathbb{N}_{a+1}})(t) = (\widetilde{\Delta}_a^{-\nu} x)(t), \qquad t \in \mathbb{N}_{a+1},
$$

and

$$
(\overline{\Delta}_a^{-\nu} x)(t) = \binom{t-a+\alpha-1}{t-a} x(a) + (\widetilde{\Delta}_a^{-\nu} x)(t), \qquad t \in \mathbb{N}_a,
$$

where $x|_{\mathbb{N}_{a+1}}$ is the restriction of $x \colon \mathbb{N}_a \to \mathbb{R}^d$ to the set $\mathbb{N}_{a+1}$.

2 Fractional Differences

In this section we provide definitions of fractional differences and discuss the relationships between them.

Definition 3 *(Caputo and Riemann-Liouville forward and backward differences)* Let $\alpha \in (0, 1)$, $a \in \mathbb{R}$ and $x\colon \mathbb{N}_a \to \mathbb{R}^d$.
(a) *Caputo forward difference* ${}_{\mathrm{C}}\Delta_a^\alpha := \Delta_a^{-(1-\alpha)} \circ \Delta$:

$$ {}_{\mathrm{C}}\Delta_a^\alpha x\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}^d, \qquad t \mapsto ({}_{\mathrm{C}}\Delta_a^\alpha x)(t) = (\Delta_a^{-(1-\alpha)}(\Delta x))(t). \tag{2} $$

(b) *Riemann-Liouville forward difference* ${}_{\text{R-L}}\Delta_a^\alpha := \Delta \circ \Delta_a^{-(1-\alpha)}$:

$$ {}_{\text{R-L}}\Delta_a^\alpha x\colon \mathbb{N}_{a+1-\alpha} \to \mathbb{R}^d, \qquad t \mapsto ({}_{\text{R-L}}\Delta_a^\alpha x)(t) = (\Delta(\Delta_a^{-(1-\alpha)} x))(t). \tag{3} $$

(c) *Caputo backward difference* ${}_{\mathrm{C}}\nabla_a^\alpha := \Delta_{a+1}^{-(1-\alpha)} \circ \nabla$:

$$ {}_{\mathrm{C}}\nabla_a^\alpha x\colon \mathbb{N}_{a+2-\alpha} \to \mathbb{R}^d, \qquad t \mapsto ({}_{\mathrm{C}}\nabla_a^\alpha x)(t) = (\Delta_{a+1}^{-(1-\alpha)}(\nabla x))(t). \tag{4} $$

(d) *Riemann-Liouville backward difference* ${}_{\text{R-L}}\nabla_a^\alpha := \nabla \circ \Delta_a^{-(1-\alpha)}$:

$$ {}_{\text{R-L}}\nabla_a^\alpha x\colon \mathbb{N}_{a+2-\alpha} \to \mathbb{R}^d, \qquad t \mapsto ({}_{\text{R-L}}\nabla_a^\alpha x)(t) = (\nabla(\Delta_a^{-(1-\alpha)} x))(t). \tag{5} $$

(e) The Caputo and Riemann-Liouville forward and backward differences with the fractional sum $\overline{\Delta}_a^{-\nu}$ instead of $\Delta_a^{-\nu}$ are defined as

$$ {}_{\mathrm{C}}\overline{\Delta}_a^\alpha x\colon \mathbb{N}_a \to \mathbb{R}^d, \quad t \mapsto ({}_{\mathrm{C}}\overline{\Delta}_a^\alpha x)(t) = (\overline{\Delta}_a^{-(1-\alpha)}(\Delta x))(t), \tag{6} $$
$$ {}_{\text{R-L}}\overline{\Delta}_a^\alpha x\colon \mathbb{N}_a \to \mathbb{R}^d, \quad t \mapsto ({}_{\text{R-L}}\overline{\Delta}_a^\alpha x)(t) = (\Delta(\overline{\Delta}_a^{-(1-\alpha)} x))(t), \tag{7} $$
$$ {}_{\mathrm{C}}\overline{\nabla}_a^\alpha x\colon \mathbb{N}_{a+1} \to \mathbb{R}^d, \quad t \mapsto ({}_{\mathrm{C}}\overline{\nabla}_a^\alpha x)(t) = (\overline{\Delta}_{a+1}^{-(1-\alpha)}(\nabla x))(t), \tag{8} $$
$$ {}_{\text{R-L}}\overline{\nabla}_a^\alpha x\colon \mathbb{N}_{a+1} \to \mathbb{R}^d, \quad t \mapsto ({}_{\text{R-L}}\overline{\nabla}_a^\alpha x)(t) = (\nabla(\overline{\Delta}_a^{-(1-\alpha)} x))(t). \tag{9} $$

(f) The Caputo and Riemann-Liouville backward differences with $\widetilde{\Delta}_a^{-\nu}$ instead of $\Delta_a^{-\nu}$ are

$$ {}_{\mathrm{C}}\widetilde{\nabla}_a^\alpha x\colon \mathbb{N}_{a+2} \to \mathbb{R}^d, \quad t \mapsto ({}_{\mathrm{C}}\widetilde{\nabla}_a^\alpha x)(t) = (\widetilde{\Delta}_{a+1}^{-(1-\alpha)}(\nabla x))(t), $$
$$ {}_{\text{R-L}}\widetilde{\nabla}_a^\alpha x\colon \mathbb{N}_{a+2} \to \mathbb{R}^d, \quad t \mapsto ({}_{\text{R-L}}\widetilde{\nabla}_a^\alpha x)(t) = (\nabla(\widetilde{\Delta}_a^{-(1-\alpha)} x))(t). $$

If $a = 0$ we write ${}_{\mathrm{C}}\Delta^\alpha$, ${}_{\text{R-L}}\Delta^\alpha$, ${}_{\mathrm{C}}\nabla^\alpha$, ${}_{\text{R-L}}\nabla^\alpha$, as well as ${}_{\mathrm{C}}\overline{\Delta}^\alpha$, ${}_{\text{R-L}}\overline{\Delta}^\alpha$, ${}_{\mathrm{C}}\overline{\nabla}^\alpha$, ${}_{\text{R-L}}\overline{\nabla}^\alpha$, and ${}_{\mathrm{C}}\widetilde{\nabla}^\alpha$, ${}_{\text{R-L}}\widetilde{\nabla}^\alpha$, respectively.

Remark 2 Using (1) in Remark 1(d), we get for $\alpha \in (0, 1)$, $a \in \mathbb{R}$ and $x\colon \mathbb{N}_a \to \mathbb{R}^d$ the following relationships between the fractional differences (2)–(5) and (6)–(9)

$$
\begin{aligned}
({}_{\mathrm{C}}\overline{\Delta}_a^{\alpha} x)(t) &= \ ({}_{\mathrm{C}}\Delta_a^{\alpha} x)(t+1-\alpha), \quad && t \in \mathbb{N}_a, \\
({}_{\text{R-L}}\overline{\Delta}_a^{\alpha} x)(t) &= \ ({}_{\text{R-L}}\Delta_a^{\alpha} x)(t+1-\alpha), \quad && t \in \mathbb{N}_a, \\
({}_{\mathrm{C}}\overline{\nabla}_a^{\alpha} x)(t) &= \ ({}_{\mathrm{C}}\nabla_{a+1}^{\alpha} x)(t+1-\alpha), \quad && t \in \mathbb{N}_{a+1}, \\
({}_{\text{R-L}}\overline{\nabla}_a^{\alpha} x)(t) &= ({}_{\text{R-L}}\nabla_{a+1}^{\alpha} x)(t+1-\alpha), \quad && t \in \mathbb{N}_{a+1}.
\end{aligned}
$$

The following two lemmas enable us to rewrite fractional difference equations as Volterra convolution equations in the next section.

Lemma 1 (Sum representations of fractional operators)
Let $\alpha \in (0, 1)$ *and* $x : \mathbb{N}_a \to \mathbb{R}^d$. *Then*

$$
\begin{aligned}
({}_{\mathrm{C}}\overline{\Delta}_a^{\alpha} x)(n) = &\sum_{k=a+1}^{n+1} (-1)^{n+1-k} \binom{\alpha}{n+1-k} x(k) \\
&- (-1)^{n-a} \binom{\alpha-1}{n-a} x(a), \quad n \in \mathbb{N}_a,
\end{aligned}
\tag{10}
$$

$$
({}_{\text{R-L}}\overline{\Delta}_a^{\alpha} x)(n) = \sum_{k=a}^{n+1} (-1)^{n+1-k} \binom{\alpha}{n+1-k} x(k), \quad n \in \mathbb{N}_a, \tag{11}
$$

$$
\begin{aligned}
({}_{\mathrm{C}}\overline{\nabla}_a^{\alpha} x)(n) = &\sum_{k=a+1}^{n} (-1)^{n-k} \binom{\alpha}{n-k} x(k) \\
&- (-1)^{n-a-1} \binom{\alpha-1}{n-a-1} x(a), \quad n \in \mathbb{N}_{a+1},
\end{aligned}
\tag{12}
$$

$$
({}_{\text{R-L}}\overline{\nabla}_a^{\alpha} x)(n) = \sum_{k=a}^{n} (-1)^{n-k} \binom{\alpha}{n-k} x(k), \quad n \in \mathbb{N}_{a+1}, \tag{13}
$$

$$
\begin{aligned}
({}_{\mathrm{C}}\widetilde{\nabla}_a^{\alpha} x)(n) = &\sum_{k=a+2}^{n} (-1)^{n-k} \binom{\alpha}{n-k} x(k) \\
&- (-1)^{n-a-2} \binom{\alpha-1}{n-a-2} x(a+1), \quad n \in \mathbb{N}_{a+2},
\end{aligned}
\tag{14}
$$

$$
({}_{\text{R-L}}\widetilde{\nabla}_a^{\alpha} x)(n) = \sum_{k=a+1}^{n} (-1)^{n-k} \binom{\alpha}{n-k} x(k), \quad n \in \mathbb{N}_{a+2}. \tag{15}
$$

Proof We prove only (10), the proofs of (11)–(15) are similar. Using the facts (see [27, pp. 158, 164]) that

$$\binom{s}{\ell} = (-1)^{\ell}\binom{\ell - s - 1}{\ell} \quad \text{and} \quad \binom{s}{\ell} - \binom{s-1}{\ell-1} = \binom{s-1}{\ell}, \qquad s \in \mathbb{R}, \ell \in \mathbb{N},$$

and Definition 3, we get

$$\begin{aligned}
({}_{\mathrm{C}}\overline{\Delta}_a^{\alpha} x)(n) &= (\overline{\Delta}_a^{-(1-\alpha)} \circ \Delta x)(n) = \sum_{k=a}^{n} \binom{n-k-\alpha}{n-k} \Delta x(k) \\
&= \sum_{k=a}^{n} \binom{n-k-\alpha}{n-k} (x(k+1) - x(k)) \\
&= \sum_{k=a}^{n} \binom{n-k-\alpha}{n-k} x(k+1) - \sum_{k=a}^{n} \binom{n-k-\alpha}{n-k} x(k) \\
&= \sum_{k=a+1}^{n+1} \binom{n-k+1-\alpha}{n-k+1} x(k) - \sum_{k=a}^{n} \binom{n-k-\alpha}{n-k} x(k) \\
&= \sum_{k=a+1}^{n} \left(\binom{n-k+1-\alpha}{n-k+1} - \binom{n-k-\alpha}{n-k} \right) x(k) + x(n+1) \\
&\quad - \binom{n-a-\alpha}{n-a} x(a) \\
&= \sum_{k=a+1}^{n} \binom{n-k-\alpha}{n-k+1} x(k) + x(n+1) - \binom{n-a-\alpha}{n-a} x(a) \\
&= \sum_{k=a+1}^{n+1} \binom{n-k-\alpha}{n-k+1} x(k) - \binom{n-a-\alpha}{n-a} x(a) \\
&= \sum_{k=a+1}^{n+1} (-1)^{n-k+1} \binom{\alpha}{n-k+1} x(k) - (-1)^{n-a} \binom{\alpha-1}{n-a} x(a),
\end{aligned}$$

which proves (10). □

The following lemma provides fractional forward and backward Taylor difference formulas.

Lemma 2 (Taylor formula)
Let $\alpha \in (0,1)$, $a \in \mathbb{R}$ and $x \colon \mathbb{N}_a \to \mathbb{R}^d$. Then for each $n \in \mathbb{N}_a$

$$x(n+1) = x(a) + \sum_{k=a}^{n} (-1)^{n-k} \binom{-\alpha}{n-k} ({}_{\mathrm{C}}\overline{\Delta}_a^{\alpha} x)(n), \tag{16}$$

$$x(n) = x(a) + \sum_{k=a}^{n} (-1)^{n-k} \binom{-\alpha}{n-k} ({}_{\mathrm{C}}\overline{\nabla}_a^{\alpha} x)(k). \tag{17}$$

Proof See [4, Theorems 35.8 and 36.4]. □

3 Linear Fractional Forward and Backward Difference Equations

Let $A\colon \mathbb{N}_0 \to \mathbb{R}^{d\times d}$, $x\colon \mathbb{N}_0 \to \mathbb{R}^d$, and $\alpha \in (0,1)$. We investigate linear time-varying

$$\text{forward } (\widehat{\Delta}^{\alpha} x)(n) = A(n)x(n) \qquad \text{and} \qquad \text{backward } (\widehat{\nabla}^{\alpha} x)(n) = A(n)x(n)$$

fractional difference equations with $\widehat{\Delta}^{\alpha} \in \{{}_{\mathrm{C}}\overline{\Delta}^{\alpha}, {}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha}\}$, $\widehat{\nabla}^{\alpha} \in \{{}_{\mathrm{C}}\overline{\nabla}^{\alpha}, {}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}, {}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}, {}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}\}$. According to Definition 3(e)–(f), the equations are defined on $\mathbb{N}_0$, $\mathbb{N}_1$ or $\mathbb{N}_2$. More precisely,

$$({}_{\mathrm{C}}\overline{\Delta}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_0, \tag{18}$$

$$({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_0, \tag{19}$$

$$({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_1, \tag{20}$$

$$({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_1, \tag{21}$$

$$({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_2, \tag{22}$$

$$({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n), \qquad n \in \mathbb{N}_2. \tag{23}$$

In order to define initial value problems, note that by Lemma 1,

$$\begin{aligned}
({}_{\mathrm{C}}\overline{\Delta}^{\alpha} x)(0) &= x(1) - x(0), & ({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha} x)(0) &= x(1) - \alpha x(0),\\
({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(1) &= x(1) - x(0), & ({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(1) &= x(1) - \alpha x(0),\\
({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(2) &= x(2) - x(1), & ({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(2) &= x(2) - \alpha x(1),
\end{aligned}$$

and hence for the forward fractional difference equations (18)–(19) an initial value $x(0) \in \mathbb{R}^d$ determines $x(1)$ (and also $x(n)$ for $n \in \mathbb{N}$). However for the backward fractional difference equations (20)–(21) and (22)–(23)

$$
\begin{aligned}
({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(1) = A(1)x(1) \quad &\Leftrightarrow \quad (I - A(1))x(1) = x(0),\\
({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(1) = A(1)x(1) \quad &\Leftrightarrow \quad (I - A(1))x(1) = \alpha x(0),\\
({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(2) = A(2)x(2) \quad &\Leftrightarrow \quad (I - A(2))x(2) = x(1),\\
({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(2) = A(2)x(2) \quad &\Leftrightarrow \quad (I - A(2))x(2) = \alpha x(1).
\end{aligned}
$$

As can be seen, $\mathrm{Im}(I - A(n))$ plays a role for the existence of solutions $x : \mathbb{N}_0 \to \mathbb{R}$. We now define initial value problems and remark on initial conditions before we show the unique existence of solutions to initial value problems under a sufficient condition on A.

Definition 4 *(Initial value problem)*
Let $x_0 \in \mathbb{R}^d$. Then $x : \mathbb{N}_0 \to \mathbb{R}$ with $x(0) = x_0$ is called *solution of the initial value problem* (18), (19), (20), (21), (22) or (23) *with initial value* x_0, if it satisfies the corresponding equation and in case of Eq. (22) it additionally satisfies $x(1) = x(0)$ and in case of (23) it additionally satisfies $(I - A(1))x(1) = x(0)$.

Remark 3 *(Initial value problems)*
Formally the initial value problems for (22)–(23) may be defined on $\mathbb{N}_1$ as a problem of finding, for a given $x_1 \in \mathbb{R}^d$, a sequence $x : \mathbb{N}_1 \to \mathbb{R}^d$ such that $x(1) = x_1$ and the corresponding equations are satisfied for all $n \in \mathbb{N}_2$. We choose the formulation of the initial value problem as in Definition 4 because this way all solutions of equations (22)–(23) are defined on the same set $\mathbb{N}_0$ and also in earlier papers on these equations (see [15, 22]) initial value problems are defined as in Definition 4.

In this paper we assume for the backward fractional difference equations (20)–(21) and (22)–(23) the condition

$$
\det(I - A(n)) \neq 0 \quad \text{for each } n \in \mathbb{N}_0, \tag{24}
$$

which is sufficient for unique existence of solutions.

Theorem 1 (Existence and uniqueness of solutions to initial value problems)
Let $A : \mathbb{N}_0 \to \mathbb{R}^{d\times d}$, $\alpha \in (0, 1)$ *and* $x_0 \in \mathbb{R}^d$.

(a) For each of the forward fractional difference equations (18)–(19) *there exists a unique solution* $x : \mathbb{N}_0 \to \mathbb{R}^d$ *with initial value* x_0, *denoted by* $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0)$ *and* $\varphi_{\mathrm{R-L}}^{\Delta}(\cdot, x_0)$, *respectively.*

(b) Assume (24). *Then for each of the backward fractional difference equations* (20)–(21) *and* (22)–(23) *there exists a unique solution* $x : \mathbb{N}_0 \to \mathbb{R}^d$ *with initial value* x_0, *denoted by* $\varphi_{\mathrm{C}}^{\overline{\nabla}}(\cdot, x_0)$, $\varphi_{\mathrm{R-L}}^{\overline{\nabla}}(\cdot, x_0)$ *and* $\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(\cdot, x_0)$, $\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(\cdot, x_0)$, *respectively.*

Proof (a) Using (10) and (11) of Lemma 1, it follows that $x(n + 1)$ in Eqs. (18) and (19) is recursively defined from $x(0), \ldots, x(n)$ for $n \in \mathbb{N}_0$.

(b) Under the assumption (24) and using (12)–(13), it follows that $x(n + 1)$ in Eqs. (20)–(21) is recursively defined from $x(0), \ldots, x(n)$ for $n \in \mathbb{N}_0$. Similarly, using (14)–(15), $x(n + 1)$ in (22)–(23) is recursively defined from $x(1), \ldots, x(n)$ for $n \in \mathbb{N}_1$ and defining $x(0)$ according to Definition 4 shows unique existence. $\square$

4 Solution Representation with Volterra Convolution Sums

In this section we show that every solution of each of the forward equations (18)–(19) can be equivalently rewritten as a Volterra convolution equation

$$x(n+1) = \sum_{k=0}^{n} a(n-k)x(k) + g(k)$$

with appropriate sequences $a\colon \mathbb{N}_0 \to \mathbb{R}^{d\times d}$ and $g\colon \mathbb{N}_0 \to \mathbb{R}^d$. The backward equations (20)–(23) can be written as

$$(I - A(n))x(n+1) = \sum_{k=0}^{n} a(n-k)x(k) + g(k).$$

We use the convention $\sum\limits_{k=p}^{q} s(k) = 0$ for $q < p$.

Theorem 2 (Volterra sum representation of solutions)
Let $A\colon \mathbb{N}_0 \to \mathbb{R}^{d\times d}$, $\alpha \in (0,1)$, $x_0 \in \mathbb{R}^d$ and $x\colon \mathbb{N}_0 \to \mathbb{R}^d$ with $x(0) = x_0$. Then

(a) $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0) = x$ is equivalent to each of the two statements (25) or (26)

$$x(n) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{-\alpha}{n-1-k} A(k)x(k) + x(0), \quad n \in \mathbb{N}_1, \tag{25}$$

$$\begin{aligned} x(n) = A(n-1)x(n-1) &- \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k) \\ &+ (-1)^{n-1} \binom{\alpha - 1}{n-1} x(0), \quad n \in \mathbb{N}_1. \end{aligned} \tag{26}$$

(b) $\varphi_{\mathrm{R-L}}^{\Delta}(\cdot, x_0) = x$ is equivalent to each of the two statements (27) or (28)

$$x(n) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{-\alpha}{n-1-k} A(k)x(k) + (-1)^n \binom{-\alpha}{n} x(0), \quad n \in \mathbb{N}_1, \tag{27}$$

$$x(n) = A(n-1)x(n-1) - \sum_{k=0}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k), \quad n \in \mathbb{N}_1. \tag{28}$$

We additionally assume (24). Then also
(c) $\varphi_{\mathrm{C}}^{\nabla}(\cdot, x_0) = x$ is equivalent to each of the two statements (29) or (30)

$$x(n) = (I - A(n))^{-1}\left(\sum_{k=1}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k) + x(0)\right), \quad n \in \mathbb{N}_1, \tag{29}$$

$$\begin{aligned} x(n) &= (I - A(n))^{-1} \\ &\cdot \left(-\sum_{k=1}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k) + (-1)^{n-1}\binom{\alpha-1}{n-1}x(0)\right), \quad n \in \mathbb{N}_1. \end{aligned} \tag{30}$$

(d) $\varphi^{\overline{\nabla}}_{\text{R-L}}(\cdot, x_0) = x$ *is equivalent to each of the two statements* (31) *or* (32)

$$\begin{aligned} x(n) &= (I - A(n))^{-1} \\ &\cdot \left(\sum_{k=1}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k) + (-1)^{n}\binom{-\alpha}{n}x(0)\right), \quad n \in \mathbb{N}_1, \end{aligned} \tag{31}$$

$$x(n) = -(I - A(n))^{-1}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k), \quad n \in \mathbb{N}_1. \tag{32}$$

(e) $\varphi^{\widetilde{\nabla}}_{\text{C}}(\cdot, x_0) = x$ *is equivalent to* $x(1) = x(0)$ *and each of the two statements* (33) *or* (34)

$$x(n) = (I - A(n))^{-1}\left(x(1) + \sum_{k=2}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k)\right), \quad n \in \mathbb{N}_2, \tag{33}$$

$$\begin{aligned} x(n) &= (I - A(n))^{-1} \\ &\cdot \left((-1)^{n-2}\binom{\alpha-1}{n-2}x(1) - \sum_{k=2}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k)\right), \quad n \in \mathbb{N}_2. \end{aligned} \tag{34}$$

(f) $\varphi^{\widetilde{\nabla}}_{\text{R-L}}(\cdot, x_0) = x$ *is equivalent to* $x(1) = (I - A(1))^{-1}x(0)$ *and each of the two statements* (35) *or* (36)

$$\begin{aligned} x(n) &= (I - A(n))^{-1} \\ &\cdot \left((-1)^{n-1}\binom{-\alpha}{n-1}x(1) + \sum_{k=2}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}A(k)x(k)\right), \quad n \in \mathbb{N}_2, \end{aligned} \tag{35}$$

$$x(n) = -(I - A(n))^{-1} \left(\sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k) \right), \quad n \in \mathbb{N}_2. \tag{36}$$

Proof (a) Using (16) in Lemma 2, Eq. (25) is equivalent to (18), i.e. $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0) = x$. Assuming $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0) = x$, Eq. (18) follows and with (10) in Lemma 1 for $a = 0$ we get

$$\sum_{k=1}^{n+1} (-1)^{n-k+1} \binom{\alpha}{n-k+1} x(k) - (-1)^n \binom{\alpha-1}{n} x(0) = A(n)x(n), \qquad n \in \mathbb{N}_0,$$

and therefore

$$x(n+1) + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{\alpha}{n-k+1} x(k) - (-1)^n \binom{\alpha-1}{n} x(0) = A(n)x(n)$$

or equivalently

$$x(n+1) = A(n)x(n) - \sum_{k=1}^{n} (-1)^{n-k+1} \binom{\alpha}{n-k+1} x(k) + (-1)^n \binom{\alpha-1}{n} x(0), \quad n \in \mathbb{N}_0,$$

which proves (26). Similarly (26) implies $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0) = x$.

(b)–(f) As in (a), the equivalences to (28), (30), (32), (34) and (36) follow with (11)–(15). Similarly as in (a), the equivalence to (27) is obtained from (17) and (20). The equivalence to (29) follows from [11, (Eq. (2.4)]. The equivalence to (31) is proved in [9, Eq. (3.4)]. Finally the equivalences to (33) and (35) are proved in [2, Eq. (5.9)] and [22, Eq. (2.6)]. □

Remark 4 *(Relation between $\varphi_{\mathrm{C}}^{\nabla}(\cdot, x_0)$ and $\varphi_{\mathrm{C}}^{\tilde{\nabla}}(\cdot, x_0)$ and between $\varphi_{\mathrm{R-L}}^{\nabla}(\cdot, x_0)$ and $\varphi_{\mathrm{R-L}}^{\tilde{\nabla}}(\cdot, x_0)$)*
Comparing (c) with (e) and (d) with (f) in Theorem 2, we observe the following relations.

(a) If $x : \mathbb{N}_0 \to \mathbb{R}^d$ satisfies (29) (or equivalently (30)), then

$$\widetilde{x} : \mathbb{N}_1 \to \mathbb{R}^d, \quad n \mapsto \widetilde{x}(n) := x(n-1)$$

satisfies (33) (or equivalently (34)) with A replaced by

$$\widetilde{A} : \mathbb{N}_2 \to \mathbb{R}^{d \times d}, \quad n \mapsto \widetilde{A}(n) := A(n-1).$$

Similarly, if $x : \mathbb{N}_0 \to \mathbb{R}^d$ satisfies (31) (or equivalently (32)), then $\widetilde{x} : \mathbb{N}_1 \to \mathbb{R}^d$, $\widetilde{x}(n) := x(n-1)$, satisfies (35) (or equivalently (36)) with A replaced by $\widetilde{A} : \mathbb{N}_2 \to \mathbb{R}^{d \times d}$, $\widetilde{A}(n) := A(n-1)$.

(b) Conversely, if $x : \mathbb{N}_1 \to \mathbb{R}^d$ satisfies (33) (or equivalently (34)) then

$$\overline{x}\colon \mathbb{N}_0 \to \mathbb{R}^d, \quad n \mapsto \overline{x}(n) := x(n+1)$$

satisfies (29) (or equivalently (30)) with A replaced by

$$\overline{A}\colon \mathbb{N}_1 \to \mathbb{R}^{d\times d}, \quad n \mapsto \overline{A}(n) := A(n+1).$$

Similarly, if $x\colon \mathbb{N}_1 \to \mathbb{R}^d$ satisfies (35) (or equivalently (36)), then $\overline{x}\colon \mathbb{N}_1 \to \mathbb{R}^d$, $\overline{x}(n) := x(n+1)$ satisfies (31) (or equivalently (32)) with A replaced by $\overline{A}\colon \mathbb{N}_1 \to \mathbb{R}^{d\times d}$, $\overline{A}(n) := A(n+1)$.

(c) From (a), (b) and Definition 4, it follows that for each $x_0 \in \mathbb{R}^d$

$$\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0) = \varphi_{\mathrm{C}}^{\overline{\nabla}}(n-1, x_0), \qquad n \in \mathbb{N}_1,$$

and

$$\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, (I - A(1))x_0) = \varphi_{\mathrm{R-L}}^{\overline{\nabla}}(n-1, x_0), \qquad n \in \mathbb{N}_1.$$

5 Linear Time-Invariant Fractional Systems

In this section we consider linear time-invariant fractional systems (18)–(23) with constant linear part $A(n) = A \in \mathbb{R}^{d\times d}$ for $n \in \mathbb{N}_0$. We prove and cite explicit solution formulas. For another formulation of the stability problem, see [12, 24, 26, 29, 30].

Theorem 3 (Solution representation for linear time-invariant fractional systems) *Let $x_0 \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and $\alpha \in (0,1)$.*

(a) The solutions of (18)–(19) with time-invariant A satisfy

$$\varphi_{\mathrm{C}}^{\Delta}(n, x_0) = \sum_{k=0}^{n} A^k \binom{n-k+k\alpha}{n-k} x_0 = \sum_{k=0}^{n} A^k (-1)^{n-k} \binom{-k\alpha - 1}{n-k} x_0, \qquad n \in \mathbb{N}_0,$$

$$\begin{aligned}\varphi_{\mathrm{R-L}}^{\Delta}(n, x_0) &= \sum_{k=0}^{n} A^k \binom{n-k+(k+1)\alpha - 1}{n-k} x_0 \\ &= \sum_{k=0}^{n} A^k (-1)^{n-k} \binom{-(k+1)\alpha}{n-k} x_0, \qquad n \in \mathbb{N}_0.\end{aligned}$$

(b) If all eigenvalues of A lie inside the unit circle, then the solutions of (20)–(21) and (22)–(23) with time-invariant A satisfy

$$\varphi_{\mathrm{C}}^{\overline{\nabla}}(n, x_0) = \left(I + (-1)^{n-1} \sum_{k=1}^{\infty} A^k \binom{-k\alpha - 1}{n-1}\right) x_0, \qquad n \in \mathbb{N}_1, \tag{37}$$

$$\varphi_{\mathrm{R-L}}^{\overline{\nabla}}(n, x_0) = \sum_{k=0}^{\infty} A^k \binom{-k\alpha - \alpha}{n} (I - A) x_0, \qquad n \in \mathbb{N}_1, \tag{38}$$

and

$$\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0) = \left(I + (-1)^n \sum_{k=1}^{\infty} A^k \binom{-k\alpha - 1}{n-2} \right) x_0, \qquad n \in \mathbb{N}_2, \tag{39}$$

$$\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, x_0) = \sum_{k=0}^{\infty} A^k \binom{-k\alpha - \alpha}{n-1} (I - A) x_0, \qquad n \in \mathbb{N}_2. \tag{40}$$

Proof (a) The solution formulas for $\varphi_{\mathrm{C}}^{\Delta}(\cdot, x_0)$ and $\varphi_{\mathrm{R-L}}^{\Delta}(\cdot, x_0)$ are proved in [7, Remark 1].

(b) The solution representations for $\varphi_{\mathrm{R-L}}^{\overline{\nabla}}(\cdot, x_0)$ and $\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(\cdot, x_0)$ follow from Remark 4(c) and [15, Theorem 18] or [8, Theorem 1], see also [10, Theorem 4.4] for an alternative proof using the Z-transform. Using Remark 4(c), the formula for $\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(\cdot, x_0)$ follows from the formula (37) for $\varphi_{\mathrm{C}}^{\overline{\nabla}}(\cdot, x_0)$. To show (37) we follow the line of reasoning from Example 48 in [1] and show that the solution $\varphi_{\mathrm{C}}^{\overline{\nabla}}(\cdot, x_0)$ of the initial value problem for

$$({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(n) = Ax(n), \qquad x \in \mathbb{N}_1, \tag{41}$$

under the assumption that all the eigenvalues of A lie inside the unit circle, is given by

$$\varphi_{\mathrm{C}}^{\overline{\nabla}}(0, x_0) = x_0$$

and

$$\varphi_{\mathrm{C}}^{\overline{\nabla}}(n, x_0) = (I - A)^{-1} \left(I + (-1)^{n-1} \sum_{k=1}^{\infty} A^k \binom{-k\alpha - 1}{n-1} \right) x_0, \; n \in \mathbb{N}_1. \tag{42}$$

For each $x_0 \in \mathbb{R}^d$ let us define a sequence $x = (x_m)_{m \in \mathbb{N}_0}$, of sequences $x_m : \mathbb{N}_1 \to \mathbb{R}^d$ recursively by

$$\begin{aligned} x_0(n) &= x_0, \qquad n \in \mathbb{N}_1, \\ x_m(n) &= x_0 + A \sum_{k=1}^{n} (-1)^{n-k} \binom{-\alpha}{n-k} x_{m-1}(k), \qquad m, n \in \mathbb{N}_1. \end{aligned} \tag{43}$$

In our further consideration we will use the following identities:

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{-\alpha}{n-k} = (-1)^{n-1} \binom{-\alpha - 1}{n-1}, \qquad n \in \mathbb{N}_0, \tag{44}$$

and the Chu–Vandermonde identity

$$\sum_{k=0}^{n}\binom{s}{n-k}\binom{\nu}{k}=\binom{s+\nu}{n}, \qquad s,\nu\in\mathbb{R}, n\in\mathbb{N}_0, \tag{45}$$

(see [27, p. 165, (5.16)] and [27, Table 174]). For $m=1$ we get

$$\begin{aligned}x_1(n)&=x_0+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}x_0(k)\\&=\left(I+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}\right)x_0\\&\overset{(44)}{=}\left(I+A(-1)^{n-1}\binom{-\alpha-1}{n-1}\right)x_0.\end{aligned}$$

Next, we show by the induction that

$$x_m(n)=\left(I+(-1)^{n-1}\sum_{k=1}^{m}A^k\binom{-k\alpha-1}{n-1}\right)x_0. \tag{46}$$

As we have checked this formula is true for $m\in\{0,1\}$. Assume that it is true for each $m\in\{0,1,\dots,l\}$ for an $l\in\mathbb{N}_1$. For $m=l+1$ we get

$$\begin{aligned}x_{l+1}(n)&=x_0+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}x_l(k)\\&=x_0+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}\left(\left(I+(-1)^{k-1}\sum_{j=1}^{l}A^j\binom{-j\alpha-1}{k-1}\right)x_0\right)\\&=\left(I+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}\right.\\&\quad\left.+A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}\left((-1)^{k-1}\sum_{j=1}^{l}A^j\binom{-j\alpha-1}{k-1}\right)\right)x_0\\&\overset{(44)}{=}\left(I+A(-1)^{n-1}\binom{-\alpha-1}{n-1}\right.\\&\quad\left.+(-1)^{n-1}\sum_{j=1}^{l}A^{j+1}\left(\sum_{t=0}^{n-1}\binom{-\alpha}{n-1-t}\binom{-j\alpha-1}{t}\right)\right)x_0\\&\overset{(45)}{=}\left(I+(-1)^{n-1}\sum_{k=1}^{l+1}A^k\binom{-k\alpha-1}{n-1}\right)x_0.\end{aligned}$$

The proof of (46) for $m \in \mathbb{N}_0$ is completed.

We now show that for each $n \in \mathbb{N}_1$ the limit $\lim_{m\to\infty} x_m(n)$ exists. According to Lemma 5.6.10 in [19], the assumption that all the eigenvalues of A lie inside the unit circle is equivalent to the fact that there exists a matrix norm $\|\cdot\|_*$ such that $\|A\|_* < 1$. Therefore, according to the Cauchy-Hadamard theorem, to prove the existence of the limit $\lim_{m\to\infty} x_m(n)$ or equivalently to prove the convergence of the series $\sum_{k=1}^{\infty} A^k \binom{-k\alpha-1}{n-1}$, it is enough to show that

$$\lim_{k\to\infty} \left| \binom{-k\alpha - 1}{n - 1} \right|^{\frac{1}{k}} = 1 \quad \text{for each } n \in \mathbb{N}_1. \tag{47}$$

Since

$$\binom{-k\alpha - 1}{n - 1} = (-1)^{n-1} \binom{n - 1 + k\alpha}{n - 1} = (-1)^{n-1} \frac{\Gamma(n + k\alpha)}{\Gamma(n)\,\Gamma(k\alpha + 1)},$$

we have to show that

$$\lim_{k\to\infty} \left(\frac{\Gamma(n + k\alpha)}{\Gamma(k\alpha + 1)} \right)^{\frac{1}{k}} = 1.$$

To compute the last limit we use the following well-known property of the Euler Gamma function [28, p. 415]

$$\Gamma(1 + z) = z\Gamma(z),$$

from which it follows that

$$\Gamma(n + z) = (n - 1 + z)(n - 2 + z) \cdots z\Gamma(z), \qquad n \in \mathbb{N}_1.$$

Let us take $z = k\alpha$ in the last identity, then we have

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\Gamma(n + k\alpha)}{\Gamma(k\alpha + 1)} \right|^{\frac{1}{k}} &= \lim_{k\to\infty} \left| \frac{\Gamma(n - 1 + 1 + k\alpha)}{\Gamma(k\alpha + 1)} \right|^{\frac{1}{k}} \\
&= \lim_{k\to\infty} \left| \frac{(n - 1 + k\alpha)(n - 2 + k\alpha) \cdots (k\alpha + 1)\,\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 1)} \right|^{\frac{1}{k}} \\
&= \lim_{k\to\infty} \left| (n - 1 + k\alpha)(n - 2 + k\alpha) \cdots (k\alpha + 1) \right|^{\frac{1}{k}}.
\end{aligned}$$

Each of the factors in the last limit has the form $|k\alpha + b|$, where $b \in \{1, \ldots, n - 1\}$, in particular, $b \geq 0$. Let us notice that

$$\lim_{k\to\infty} |k\alpha + b|^{\frac{1}{k}} = \lim_{k\to\infty} k^{\frac{1}{k}} \left| \alpha + \frac{b}{k} \right|^{\frac{1}{k}} = 1.$$

The proof of (47) is completed.

Observe that the sequence $x = (x(n))_{n\in\mathbb{N}_0}$ is given by

$$x(n) = \left(I + (-1)^{n-1}\sum_{k=1}^{\infty} A^k \binom{-k\alpha-1}{n-1}\right) x_0$$

and satisfies the equation

$$x(n) = x_0 + A\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}x(k), \qquad n \in \mathbb{N}_1. \tag{48}$$

In fact, we have

$$\begin{aligned}\lim_{m\to\infty}\sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}Ax_m(k) &= \sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}A\lim_{m\to\infty}x_m(k)\\ &= \sum_{k=1}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}Ax(k).\end{aligned}$$

Using the last equality and passing to the limit for $m \to \infty$ in (43), we get (48). From (48) we have

$$x(n) = (I-A)^{-1}\left(x(0) + \sum_{k=1}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}Ax(k)\right)$$

and this is the Volterra convolution equation (25) which is equivalent to (41). This completes the proof of (42). □

Remark 4 leads us to the following useful analog observations for linear time-invariant systems.

Remark 5 *(Relation between solutions of backward time-invariant systems)* Assume that $I - A$ is invertible.

(a) If $x : \mathbb{N}_0 \to \mathbb{R}^d$ satisfies

$$x(n) = (I-A)^{-1}A\sum_{k=1}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}x(k) + (I-A)^{-1}x(0), \quad n \in \mathbb{N}_1, \tag{49}$$

or equivalently

$$
\begin{aligned}
x(n) = &-(I-A)^{-1}\sum_{k=1}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k) \\
&+(I-A)^{-1}(-1)^{n-1}\binom{\alpha-1}{n-1}x(0), \quad n\in\mathbb{N}_1,
\end{aligned}
\tag{50}
$$

then $\widetilde{x}\colon \mathbb{N}_1 \to \mathbb{R}^d$, $\widetilde{x}(n) = x(n-1)$, satisfies

$$
\widetilde{x}(n) = (I-A)^{-1}A\sum_{k=2}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}\widetilde{x}(k) + (I-A)^{-1}\widetilde{x}(1), \quad n\in\mathbb{N}_2, \tag{51}
$$

or equivalently

$$
\begin{aligned}
\widetilde{x}(n) = &-(I-A)^{-1}\sum_{k=2}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}\widetilde{x}(k) \\
&+(I-A)^{-1}(-1)^{n-2}\binom{\alpha-1}{n-2}\widetilde{x}(1), \quad n\in\mathbb{N}_2.
\end{aligned}
\tag{52}
$$

Similarly if $x\colon \mathbb{N}_0 \to \mathbb{R}^d$ satisfies

$$
\begin{aligned}
x(n) = &\,(I-A)^{-1}A\sum_{k=1}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}x(k) \\
&+(I-A)^{-1}(-1)^{n}\binom{-\alpha}{n}x(0), \quad n\in\mathbb{N}_1
\end{aligned}
\tag{53}
$$

or equivalently

$$
x(n) = -(I-A)^{-1}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k), \quad n\in\mathbb{N}_1, \tag{54}
$$

then $\widetilde{x}\colon \mathbb{N}_1 \to \mathbb{R}^d$, $\widetilde{x}(n) = x(n-1)$, satisfies

$$
\begin{aligned}
\widetilde{x}(n) = &\,(I-A)^{-1}A\sum_{k=2}^{n-1}(-1)^{n-k}\binom{-\alpha}{n-k}\widetilde{x}(k) \\
&+(I-A)^{-1}(-1)^{n-1}\binom{-\alpha}{n-1}\widetilde{x}(1), \quad n\in\mathbb{N}_2,
\end{aligned}
\tag{55}
$$

or equivalently

$$\widetilde{x}(n) = -(I - A)^{-1} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} \widetilde{x}(k), \quad n \in \mathbb{N}_2. \tag{56}$$

(b) Conversely, if $x\colon \mathbb{N}_1 \to \mathbb{R}^d$ satisfies (51) (or equivalently (52)), then $\overline{x}\colon \mathbb{N}_0 \to \mathbb{R}^d$, $\overline{x}(n) := x(n+1)$, satisfies (49) (or equivalently (50)). Similarly if $x\colon \mathbb{N}_1 \to \mathbb{R}^d$ satisfies (55) (or equivalently (56)) then $\overline{x}\colon \mathbb{N}_1 \to \mathbb{R}^d$, $\overline{x}(n) := x(n+1)$, satisfies (53) (or equivalently (54)).

(c) For $x_0 \in \mathbb{R}^d$

$$\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0) = \varphi_{\mathrm{C}}^{\overline{\nabla}}(n-1, x_0) \quad \text{and} \quad \varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, (I-A)x_0) = \varphi_{\mathrm{R-L}}^{\overline{\nabla}}(n-1, x_0), \qquad n \in \mathbb{N}_1.$$

In the next lemma we rewrite solutions of Volterra convolution equations as sums with recursively defined coefficients.

Lemma 3 (Volterra convolution representation)
Let $B, C \in \mathbb{R}^{d\times d}$, $a \in \mathbb{R}$, $c\colon \mathbb{N}_1 \to \mathbb{R}$, $g\colon \mathbb{N}_{a+1} \to \mathbb{R}$ and $x\colon \mathbb{N}_a \to \mathbb{R}^d$. If

$$x(n+a) = B \sum_{k=1}^{n-1} c(n-k)x(k+a) + Cg(n)x(a), \qquad n \in \mathbb{N}_1,$$

then

$$x(n+a) = \sum_{k=0}^{n-1} B^k C b(n,k) x(a), \qquad n \in \mathbb{N}_1, \tag{57}$$

where $b(n,k)$ for $n \in \mathbb{N}_1$, $k \in \{0, 1, \ldots, n-1\}$, are defined recursively by $b(n,0) := g(n+a)$ and

$$b(n+1,k) := \sum_{j=k-1}^{n-1} c(n-j) b(j+1, k-1), \quad n \in \mathbb{N}_1, k \in \{1, 2, \ldots, n\}.$$

Proof We show the statement by induction over $n \in \mathbb{N}_1$. For $n = 1$ the statement is true. Suppose that (57) holds for $n \in \{1, \ldots, m\}$ for an $m \in \mathbb{N}_1$. Then

$$\begin{aligned}
x(m+1+a) &= B \sum_{k=1}^{m} c(m+1-k)x(k+a) + Cg(m+1+a)x(a) \\
&= B \sum_{k=1}^{m} c(m+1-k) \sum_{j=0}^{k-1} B^j C b(k,j) x(a) + Cg(m+1+a)x(a) \\
&= \left(\sum_{k=1}^{m} \sum_{j=0}^{k-1} c(m+1-k) B^{j+1} C b\,(k,j) + Cg(m+1+a) \right) x(a) \\
&= \left(\sum_{i=0}^{m-1} B^{i+1} C \sum_{j=i}^{m-1} c(m-j) b\,(j+1,i) + Cg(m+1+a) \right) x(a)
\end{aligned}$$

$$= \left(\sum_{k=1}^{m} B^k C \sum_{j=k-1}^{m-1} c(m-j)b(j+1,k-1) + Cg(m+1+a) \right) x(a).$$

The last equality completes the proof. □

Applying the recursive representation of Volterra convolution equations in Lemma 3 to the time-invariant equations (49)–(56) of Remark 5, leads to the following result.

Theorem 4 (Recursive solution representation for backward time-invariant fractional systems)
Let $x_0 \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and $\alpha \in (0,1)$. Assume that $I - A$ is invertible.
(a) The solutions of (20) with time-invariant linear part A satisfy

$$\begin{aligned}\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x_0) &= \sum_{k=0}^{n-1} (I-A)^{-k-1} A^k b_1(n,k) x_0 \\ &= \sum_{k=0}^{n-1} (-1)^k (I-A)^{-k-1} b_2(n,k) x_0, \qquad n \in \mathbb{N}_1,\end{aligned}$$

where $b_1(n,k)$ for $n \in \mathbb{N}_1$, $k \in \{0,1,\dots,n-1\}$, are defined recursively by $b_1(n,0) := 1$ and

$$b_1(n+1,k) := \sum_{j=k-1}^{n-1} (-1)^{n-j} \binom{-\alpha}{n-j} b_1(j+1,k-1), \quad n \in \mathbb{N}_1, k \in \{1,2,\dots,n\},$$

and $b_2(n,k)$ for $n \in \mathbb{N}_1$, $k \in \{0,1,\dots,n-1\}$, are defined by $b_2(n,0) := (-1)^{n-1} \binom{\alpha-1}{n-1}$ and

$$b_2(n+1,k) := \sum_{j=k-1}^{n-1} (-1)^{n-j} \binom{\alpha}{n-j} b_2(j+1,k-1), \quad n \in \mathbb{N}_1, k \in \{1,2,\dots,n\}.$$

(b) The solutions of (21) with time-invariant linear part A satisfy

$$\begin{aligned}\varphi_{\mathrm{R\text{-}L}}^{\overline{\nabla}}(n,x_0) &= \sum_{k=0}^{n-1} (I-A)^{-k-1} A^k b_3(n,k) x_0 \\ &= \sum_{k=0}^{n-1} (I-A)^{-k-1} b_4(n,k) x_0, \qquad n \in \mathbb{N}_1,\end{aligned}$$

where $b_3(n,k)$ *for* $n \in \mathbb{N}_1$, $k \in \{0, 1, \dots, n-1\}$, *are defined by* $b_3(n,0) := (-1)^n \binom{-\alpha}{n}$ *and*

$$b_3(n+1,k) := \sum_{j=k-1}^{n-1} (-1)^{n-j} \binom{-\alpha}{n-j} b_3(j+1, k-1), \quad n \in \mathbb{N}_1, k \in \{1, 2, \dots, n\},$$

and $b_4(n,k)$ *for* $n \in \mathbb{N}_1$, $k \in \{0, 1, \dots, n-1\}$, *are defined by* $b_4(n,0) := (-1)^n \binom{\alpha}{n}$ *and*

$$b_4(n+1,k) := \sum_{j=k-1}^{n-1} (-1)^{n-j} \binom{\alpha}{n-j} b_4(j+1, k-1), \quad n \in \mathbb{N}_1, k \in \{1, 2, \dots, n\}.$$

(c) The solutions of (22) *with time-invariant linear part* A *satisfy*

$$\begin{aligned} \varphi_{\mathrm{C}}^{\tilde{\nabla}}(n, x_0) &= \sum_{k=0}^{n-2} (I-A)^{-k-1} A^k b_1(n-1,k) x_0 \\ &= \sum_{k=0}^{n-2} (-1)^k (I-A)^{-k-1} b_2(n-1,k) x_0, \qquad n \in \mathbb{N}_2. \end{aligned}$$

(d) The solutions of (23) *with time-invariant linear part* A *satisfy*

$$\begin{aligned} \varphi_{\mathrm{R-L}}^{\tilde{\nabla}}(n, x_0) &= \sum_{k=0}^{n-2} (I-A)^{-k-1} A^k b_3(n-1,k) x_0 \\ &= \sum_{k=0}^{n-2} (I-A)^{-k-1} b_4(n-1,k) x_0, \qquad n \in \mathbb{N}_2. \end{aligned}$$

Proof (a) Applying Lemma 3 with $B := (I-A)^{-1}A$, $C := (I-A)^{-1}$, $c(n) := (-1)^n \binom{-\alpha}{n}$ and $g(n) := 1$ for $n \in \mathbb{N}_1$ to (49) yields $\varphi_{\mathrm{C}}^{\nabla}(n, x_0) = \sum_{k=0}^{n-1} (I-A)^{-k-1} A^k b_1(n,k) x_0$, and the second formula for $\varphi_{\mathrm{C}}^{\nabla}(n, x_0)$ follows from Eq. (50) and again Lemma 3 with $B := -(I-A)^{-1}$, $C := (I-A)^{-1}$, $c(n) := (-1)^n \binom{\alpha}{n}$ and $g(n) := (-1)^{n-1} \binom{\alpha-1}{n-1}$ for $n \in \mathbb{N}_1$.

(b) As in (a), applying Lemma 3 to Eqs. (53) and (54) yields the result.

(c)–(d) This follows from Remark 5(c). □

We cite two results on asymptotic stability of (18) in the time-invariant case.

Theorem 5 (Characterization of asymptotic stability for time-invariant Caputo forward equations [3, Theorem 3.2])
Let $A \in \mathbb{R}^{d \times d}$, $\alpha \in (0, 1)$. *Then the following two statements are equivalent.*

(i) $({}_{\mathrm{C}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ *is asymptotically stable, i.e.* $\lim_{n\to\infty}\varphi_{\mathrm{C}}^{\Delta}(n, x_0) = 0$ *for all* $x_0 \in \mathbb{R}^d$.

(ii) The isolated zeros, off the non-negative real axis, of $z \mapsto det(I - z^{-1}(1 - z^{-1})^{-\alpha}A)$ *lie inside the unit circle.*

To present another sufficient condition for asymptotic stability of (18) in the time-invariant case, let us denote

$$S_1^{\alpha} = \left\{ z \in \mathbb{C}\colon |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\alpha}\right)^{\alpha} \text{ and } |\arg z| > \frac{\alpha\pi}{2} \right\}.$$

Theorem 6 (Asymptotic stability of time-invariant Caputo forward equations [13, Theorem 1.4])
Let $A \in \mathbb{R}^{d\times d}$.

(a) If $\lambda \in S_1^{\alpha}$ *for all eigenvalues* λ *of* A, *then* $({}_{\mathrm{C}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ *is asymptotically stable. In this case, the solutions decay towards zero algebraically (and not exponentially), more precisely, for* $x_0 \in \mathbb{R}^d$

$$\|\varphi_{\mathrm{C}}^{\Delta}(n, x_0)\| = O(n^{-\alpha}) \quad \text{as } n \to \infty.$$

(b) If $\lambda \in \mathbb{C} \setminus \mathrm{cl}\, S_1^{\alpha}$ *for an eigenvalue* λ *of* A, *then* $({}_{\mathrm{C}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ *is not stable and there exist* $x_0 \in \mathbb{R}^d$, $C > 0$ *and* $r > 1$ *with*

$$\|\varphi_{\mathrm{C}}^{\Delta}(n, x_0)\| \geq Cr^n, \qquad n \in \mathbb{N}_0.$$

The next result provides a sufficient condition for asymptotic stability of (19) in the time-invariant case.

Theorem 7 (Asymptotic stability of time-invariant Riemann-Liouville forward equations)
Let $A \in \mathbb{R}^{d\times d}$ *and* $\alpha \in (0, 1)$.

(a) If $\lambda \in S_1^{\alpha}$ *for all eigenvalues* λ *of* A, *then* $\varphi_{\mathrm{R-L}}^{\Delta}(\cdot, x_0) \in l^1$ *for each* $x_0 \in \mathbb{R}^d$, *hence* $({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ *is asymptotically stable.*

(b) If $\lambda \in \mathbb{C} \setminus \mathrm{cl}\, S_1^{\alpha}$ *for an eigenvalue* λ *of* A, *then* $({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ *is not stable and there exist* $x_0 \in \mathbb{R}^d$, $C > 0$ *and* $r > 1$ *with*

$$\|\varphi_{\mathrm{R-L}}^{\Delta}(n, x_0)\| \geq Cr^n, \qquad n \in \mathbb{N}_0. \tag{58}$$

Proof (a) The fact that the condition $\lambda \in S_1^{\alpha}$ for all eigenvalues λ of A implies stability of $({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha}x)(n) = Ax(n)$ is proved in the first step of the proof of Theorem 1.4 in [13].

(b) Suppose that there exists an eigenvalue $\lambda \in \mathbb{C} \setminus \mathrm{cl}\, S_1^{\alpha}$ of A. As it has been shown in steps 1 and 2 of the proof of Theorem 1.4 in [13], this implies that the equation

$$\det\left(A - z(1 - z^{-1})^{\alpha}\right) = 0$$

has a solution $z_0 \in \mathbb{C}$ with $|z_0| > 1$. The Z-transform $y(z)$ of $\varphi^{\Delta}_{\mathrm{R-L}}(n, x_0)$ is given by

$$y(z) = -\left(A - z(1 - z^{-1})^{\alpha}\right) z x_0,$$

and has a non-removable singularity at z_0. Hence the radius of convergence r of at least one of the coordinates $y_i(z)$ of $y(z)$ satisfies $r > 1$. Using the Cauchy-Hadamard theorem we get

$$r = \limsup_{n\to\infty} \sqrt[n]{|y_i(n)|} > 1$$

and consequently we get (58). □

To present stability results of the Riemann-Liouville backward equation (23) in the time-invariant case, let us denote

$$S_2^{\alpha} = \left\{ z \in \mathbb{C}\colon |z| > \left(2\cos\frac{|\arg z|}{\alpha}\right)^{\alpha} \text{ or } |\arg z| > \frac{\alpha\pi}{2} \right\}$$

and the interior of its complement in $\mathbb{C}$

$$U^{\alpha} = \left\{ z \in \mathbb{C}\colon |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\alpha}\right)^{\alpha} \text{ and } |\arg z| < \frac{\alpha\pi}{2} \right\}.$$

Theorem 8 (Asymptotic stability of time-invariant Riemann-Liouville backward equations [15, Theorem 6])
Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0, 1)$. Assume that $I - A$ is invertible.

(a) If all eigenvalues of A lie in S_2^{α}, then $\varphi^{\widetilde{\nabla}}_{\mathrm{R-L}}(\cdot, x_0) \in l^1$, hence $({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ is asymptotically stable. Moreover, if all eigenvalues of $(I - A)^{-1}$ lie inside the open unit disc, then $\|\varphi^{\widetilde{\nabla}}_{\mathrm{R-L}}(n, x_0)\| = O(n^{-\alpha-1})$ as $n \to \infty$ for all $x_0 \in \mathbb{R}^d$.

(b) If there exists an eigenvalue λ of A such that $\lambda \in U^{\alpha}$, then $({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ is not stable and there exist $x_0 \in \mathbb{R}^d$, $C > 0$ and $r > 1$ with

$$\|\varphi^{\widetilde{\nabla}}_{\mathrm{R-L}}(n, x_0)\| \geq C r^n, \qquad n \in \mathbb{N}_0.$$

Theorem 8 does not answer the stability problem if some of the eigenvalues of A lie on the boundary of S_2^{α}. The following assertion from [15] demonstrates that all stability variants are possible in such a case.

Theorem 9 (Asymptotic behavior on the stability boundary of time-invariant Riemann-Liouville backward equations [15, Theorem 9])
Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0, 1)$. Assume that $I - A$ is invertible, that zero is an eigenvalue of A and that all nonzero eigenvalues of A belong to S_2^{α}. Denote by $r \in \mathbb{N}_1$ the maximal size of the Jordan blocks corresponding to the zero eigenvalue.

(a) If $r < \alpha^{-1}$, then $({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ is asymptotically stable and $\|\varphi^{\widetilde{\nabla}}_{\mathrm{R-L}}(n, x_0)\| = O(n^{r\alpha-1})$ as $n \to \infty$.

(b) If $r = \alpha^{-1}$, then $({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ is stable but not asymptotically stable.

(c) If $r > \alpha^{-1}$*, then* $({}_{\text{R-L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ *is not stable.*

Remark 6 Remark 5(c) implies that Theorems 8 and 9 are also valid for $({}_{\text{R-L}}\overline{\nabla}^{\alpha} x)(n) = A(n)x(n)$.

To discuss stability of the Caputo backward equation (52) in Remark 4 we rewrite it, using the fact [27] that $\binom{\alpha}{n-1} = \binom{\alpha-1}{n-1} + \binom{\alpha-1}{n-2}$, to get

$$\begin{aligned} x(n) = &-(I-A)^{-1}\sum_{k=1}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k) \\ &+ (I-A)^{-1}(-1)^{n-1}\binom{\alpha-1}{n-1}x(1), \quad n \in \mathbb{N}_2. \end{aligned} \tag{59}$$

We will also use the following solution representation of inhomogeneous Volterra equations.

Theorem 10 (Inhomogeneous Volterra equation)
Let $C \in \mathbb{R}^{d\times d}$*,* $a\colon \mathbb{N}_1 \to \mathbb{R}$*,* $g\colon \mathbb{N}_2 \to \mathbb{R}^d$ *and* $R\colon \mathbb{N}_1 \to \mathbb{R}^{d\times d}$*. If*

$$R(n) = C\sum_{k=1}^{n-1} a(n-k)R(k), \qquad n \in \mathbb{N}_2$$

with initial condition $R(1) = I$*, then the unique solution* $x\colon \mathbb{N} \to \mathbb{R}^d$ *of the equation*

$$x(n) = C\sum_{k=1}^{n-1} a(n-k)x(k) + g(n), \qquad n \in \mathbb{N}_2$$

with the initial condition $x(1) = x_1 \in \mathbb{R}^d$*, is given by*

$$x(n) = R(n)x_1 + \sum_{k=1}^{n-1} R(k)g(n-k+1), \qquad n \in \mathbb{N}_1. \tag{60}$$

Proof For $n = 1$ the statement is true. Let $m \in \mathbb{N}_2$. Suppose that (60) holds for all $n \in \{1, \ldots, m\}$. Then it also holds for $m+1$, since

$$\begin{aligned} x(m+1) &= C\sum_{k=1}^{m} a(m+1-k)x(k) + g(m+1) \\ &= C\sum_{k=1}^{m} a(m+1-k)\Big(R(k)x_1 + \sum_{j=1}^{k-1} R(j)g(k-j+1)\Big) + g(m+1) \end{aligned}$$

$$= C \sum_{k=1}^{m} a(m+1-k) R(k) x_1$$
$$+ C \sum_{k=1}^{m} a(m+1-k) \sum_{j=1}^{k-1} R(j) g(k-j+1) + g(m+1)$$
$$= R(m+1) x_1 + \sum_{k=1}^{m} R(k) g(m+1-k+1).$$

When we compare the Caputo backward equation (52) in its equivalent form (59) to the Riemann-Liouville backward equation (56) and use Theorem 10, we obtain the following relation between Caputo and Riemann-Liouville equations.

Lemma 4 (Relation between backward Caputo and Riemann Liouville equations) *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0,1)$. If $X\colon \mathbb{N}_1 \to \mathbb{R}^{d\times d}$ is the solution of the Riemann-Liouville matrix equation*

$$X(n) = -(I-A)^{-1} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} X(k), \qquad n \in \mathbb{N}_2, \tag{61}$$

with initial condition $X(1) = I$, then the solution $x\colon \mathbb{N} \to \mathbb{R}$ of the Caputo equation (59) *with initial condition $x(1) = x_1 \in \mathbb{R}^d$, is given by*

$$x(n) = X(n)x_1 + \sum_{k=1}^{n-1} X(k) g(n-k+1), \qquad n \in \mathbb{N}_2, \tag{62}$$

where

$$g(n) = (I-A)^{-1}(-1)^{n-1}\binom{\alpha-1}{n-1} x(1), \qquad n \in \mathbb{N}_2.$$

Using the representation (62) we show the following stability result for (22) in the time-invariant case.

Theorem 11 (Asymptotic stability of time-invariant Caputo backward equations) *Let $A \in \mathbb{R}^{d\times d}$. Assume that $I - A$ is invertible. If all eigenvalues of A lie in S_2^α, then*

$$({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$$

is asymptotically stable and $\|\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0)\| = O(n^{-\alpha})$ as $n \to \infty$ for all $x_0 \in \mathbb{R}^d$.

Proof Suppose that all the eigenvalues of A lie in S_2^α and consider the sequence $X(n)$, given by (61) and $X(1) = I$. Since $\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, x_0) = X(n)x_0$, by Theorem 8 we know that $X(n) \in l^1$ and therefore there exists a constant $C > 0$ such that

$$\|X(n)\| \le \frac{C}{n}, \qquad n \in \mathbb{N}_1. \tag{63}$$

It is also known [27] that for the binomial coefficients we have

$$\left|\binom{\alpha-1}{n}\right| \leq \frac{C}{(n+1)^{\alpha}}, \qquad n \in \mathbb{N}_1, \tag{64}$$

for certain $C > 0$ (without loss of generality we may assume that the constants C are the same in the last two inequalities (63) and (64)). From (62) we have

$$\begin{aligned}\|\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0)\| &= \left\| X(n)x_0 + (I-A)^{-1}\sum_{k=1}^{n-1}(-1)^{n-k}X(k)\binom{\alpha-1}{n-k}x_0\right\| \\ &\leq C_1\sum_{k=1}^{n}\|X(k)\|\left|\binom{\alpha-1}{n-k}\right| = C_1\sum_{j=0}^{n-1}\|X(n-j)\|\left|\binom{\alpha-1}{j}\right|,\end{aligned}$$

where $C_1 = \|x_0\| \max\left\{1, \|(I-A)^{-1}\|\right\}$. Using (64) and dividing the sum into two parts, we get

$$\begin{aligned}\|\varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0)\| &\leq C_1 C\sum_{j=0}^{n-1}\frac{\|X(n-j)\|}{(j+1)^{\alpha}} \\ &= C_1 C\sum_{j=0}^{\lfloor n-1\rfloor}\frac{\|X(n-j)\|}{(j+1)^{\alpha}} + C_1 C\sum_{j=\lfloor n-1\rfloor+1}^{n-1}\frac{\|X(n-j)\|}{(j+1)^{\alpha}}.\end{aligned} \tag{65}$$

To estimate the first term we use (63) and the inequality

$$\sum_{i=1}^{l}\frac{1}{i^{\alpha}} \leq \int_1^{l+1} x^{-\alpha}dx = \frac{(l+1)^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}, \qquad l \in \mathbb{N}_1,$$

as follows

$$\begin{aligned}\sum_{j=0}^{\lfloor n-1\rfloor}\frac{\|X(n-j)\|}{(j+1)^{\alpha}} &\leq \frac{C}{n-\lfloor n-1\rfloor}\sum_{j=0}^{\lfloor n-1\rfloor}\frac{1}{(j+1)^{\alpha}} \\ &= \frac{C}{n-\lfloor n-1\rfloor}\sum_{j=1}^{\lfloor n-1\rfloor+1}\frac{1}{j^{\alpha}} \\ &\leq \frac{C}{n-\lfloor n-1\rfloor}\left(\frac{(\lfloor n-1\rfloor+2)^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right).\end{aligned}$$

From the last inequality it is clear that

$$\sum_{j=0}^{\lfloor n-1 \rfloor} \frac{\|X(n-j)\|}{(j+1)^{\alpha}} \leq \frac{C_2}{n^{\alpha}}, \tag{66}$$

for certain $C_2 > 0$ and all $n \in \mathbb{N}_1$. To estimate the second term in (65) we proceed as follows

$$\sum_{j=\lfloor n-1 \rfloor+1}^{n-1} \frac{\|X(n-j)\|}{(j+1)^{\alpha}} \leq \frac{C_3}{n^{\alpha}} \sum_{j=\lfloor n-1 \rfloor+1}^{n-1} \|X(n-j)\| \leq \frac{C_4}{n^{\alpha}}, \tag{67}$$

for certain $C_3, C_4 > 0$ and all $n \in \mathbb{N}_1$. Applying (66) and (67) to (65), we obtain the statement of the theorem. □

6 Asymptotic Properties of Scalar Linear Fractional Equations

In this section we investigate one-dimensional linear fractional equations and discuss their asymptotic behavior in the time-invariant case in the first subsection. In the second subsection we study asymptotic behavior of time-varying backward equations.

6.1 Scalar Time-Invariant Equations

We consider one-dimensional systems (18)–(23) in the time-invariant case, i.e. we assume that $A(n) = \lambda \in \mathbb{R}$, $n \in \mathbb{N}_0$. For multi-dimensional time-invariant systems the picture of stability is not complete and the theorems from the previous section leave some cases unsolved. For one-dimensional equations the problem of stability and asymptotic stability is much more exhaustively described but even in this relatively simple situation it is not completely solved.

Theorem 12 (Asymptotic behavior of scalar time-invariant fractional equations) *Let $\lambda \in \mathbb{R}$ and $\alpha \in (0, 1)$.*

(a) $({}_{\mathrm{C}}\overline{\Delta}^{\alpha} x)(n) = \lambda x(n)$ is asymptotically stable if and only if $\lambda \in (-2^{\alpha}, 0)$, and $({}_{\mathrm{C}}\overline{\Delta}^{\alpha} x)(n) = \lambda x(n)$ is stable, but not asymptotically stable if $\lambda = 0$ or $\lambda = -2^{\alpha}$.
(b) $({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha} x)(n) = \lambda x(n)$ is asymptotically stable if and only if $\lambda \in (-2^{\alpha}, 0]$, and $({}_{\mathrm{R\text{-}L}}\overline{\Delta}^{\alpha} x)(n) = \lambda x(n)$ is stable, but not asymptotically stable if $\lambda = -2^{\alpha}$.

(c) $({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(n) = \lambda x(n)$ with $\lambda \neq 1$ is asymptotically stable if $\lambda \in (-\infty, 0] \cup (2^{\alpha}, \infty)$.

(d) $({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(n) = \lambda x(n)$ with $\lambda \neq 1$ is asymptotically stable if and only if $\lambda \in (-\infty, 0] \cup (2^{\alpha}, \infty)$, $({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(n) = \lambda x(n)$ with $\lambda \neq 1$ is not stable if $\lambda \in (0, 2^{\alpha})$.

(e) $({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}x)(n) = \lambda x(n)$ *with* $\lambda \neq 1$ *is asymptotically stable if* $\lambda \in (-\infty, 0] \cup (2^{\alpha}, \infty)$.

(f) $({}_{\text{R-L}}\widetilde{\nabla}^{\alpha}x)(n) = \lambda x(n)$ *with* $\lambda \neq 1$ *is asymptotically stable if and only if* $\lambda \in (-\infty, 0] \cup (2^{\alpha}, \infty)$,
$({}_{\text{R-L}}\overline{\nabla}^{\alpha}x)(n) = \lambda x(n)$ *with* $\lambda \neq 1$ *is not stable if* $\lambda \in (0, 2^{\alpha})$.

Proof (a) and (b) have been proved in [13] (see also [6, Example 29]). (c) follows from Theorem 11. (f) has been proved in [14]. (d) and (e) follow from Remark 5(c) together with (c) and (f), respectively. □

Theorem 12 leaves the following questions for (20)–(23) with $\lambda \neq 1$ open:

Scalar linear fractional difference equation	Open question
$({}_{\mathrm{C}}\overline{\nabla}^{\alpha}x)(n) = \lambda x(n)$ and $({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}x)(n) = \lambda x(n)$	asymptotic behavior for $\lambda \in (0, 2^{\alpha}]$
$({}_{\text{R-L}}\overline{\nabla}^{\alpha}x)(n) = \lambda x(n)$ and $({}_{\text{R-L}}\widetilde{\nabla}^{\alpha}x)(n) = \lambda x(n)$	stability for $\lambda = 2^{\alpha}$

For scalar time-invariant fractional equations, also some results about the convergence and divergence rates of solutions are known. These results are collected in the next two theorems.

Theorem 13 (Growth and decay rates for scalar linear time-invariant fractional equations)
Consider (20)–(23) *with linear part* $\lambda \in \mathbb{R}$. *Let* $\alpha \in (0, 1)$ *and* $x_0 \in \mathbb{R}$. *Then*

(a) $\lim_{n\to\infty} \varphi_{\mathrm{C}}^{\Delta}(n, x_0)n^{\alpha} = \frac{-x_0}{\lambda\Gamma(1-\alpha)}$ *if* $\lambda \in (-2\alpha, 0)$,

(b) $\lim_{n\to\infty} \varphi_{\mathrm{R-L}}^{\Delta}(n, x_0)n^{\alpha+1} = \frac{-x_0}{\lambda^2\Gamma(-\alpha)}$ *if* $\lambda \in (-2\alpha, 0)$,

(c) $\lim_{n\to\infty} \varphi_{\mathrm{C}}^{\widetilde{\nabla}}(n, x_0)n^{\alpha} = \frac{-x_0}{\lambda\Gamma(1-\alpha)}$ *if* $\lambda \in (-\infty, 0) \cup (2, \infty)$,

(d) $\lim_{n\to\infty} \varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, x_0)n^{\alpha+1} = \frac{\alpha(1-\lambda)x_0}{\lambda^2\Gamma(1-\alpha)}$ *if* $\lambda \in (-\infty, 0) \cup (2, \infty)$, *and*
$\lim_{n\to\infty} \varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, x_0)n^{\alpha+1} = \frac{x_0}{\Gamma(\alpha)}$ *if* $\lambda = 0$,

(e) $\lim_{n\to\infty} \varphi_{\mathrm{C}}^{\overline{\nabla}}(n, x_0)n^{\alpha} = \frac{-x_0}{\lambda\Gamma(1-\alpha)}$ *if* $\lambda \in (-\infty, 0) \cup (2, \infty)$,

(f) $\lim_{n\to\infty} \varphi_{\mathrm{R-L}}^{\overline{\nabla}}(n, x_0)n^{\alpha+1} = \frac{\alpha(1-\lambda)^2x_0}{\lambda^2\Gamma(1-\alpha)}$ *if* $\lambda \in (-\infty, 0) \cup (2, \infty)$, *and*
$\lim_{n\to\infty} \varphi_{\mathrm{R-L}}^{\overline{\nabla}}(n, x_0)n^{\alpha+1} = \frac{x_0}{\Gamma(\alpha)}$ *if* $\lambda = 0$,

(g) If $0 < \lambda < 1$ *and* $x_0 > 0$, *then* $\varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(\cdot, x_0)$ *grows geometrically. More precisely,*

$$\frac{\lambda^{1/\alpha}x_0}{(1-\lambda^{1/\alpha})^n} < \varphi_{\mathrm{R-L}}^{\widetilde{\nabla}}(n, x_0) < \frac{x_0}{(1-\lambda^{1/\alpha})^n}, \qquad n \in \mathbb{N}_2.$$

In the proof of Theorem 13 we use the following Lemma from [5].

Lemma 5 ([5, Lemma 6])
Let $\alpha \in (0, 1)$ *and* $r, f : \mathbb{N}_1 \to \mathbb{R}$. *If*

$$\sup_{n\in\mathbb{N}_1}\left|\frac{r(n)}{n^{-\alpha-1}}\right| < \infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{f(n)}{n^{-\alpha}} =: d \textit{ exists}, \tag{68}$$

then the convolution $r * f : \mathbb{N}_1 \to \mathbb{R}$, $(r * f)(n) = \sum_{i=0}^{n} r(n-i) f(i)$, *satisfies*

$$\lim_{n\to\infty} \frac{(r * f)(n)}{n^{-\alpha}} = d \sum_{i=1}^{\infty} r(i).$$

***Proof (of Theorem** 13)* (a) is proved in [5, Theorem 4].

(b) is shown in [13, Corollary 4.2].

(d) and (g) are proved in [14, Theorem 4.7].

(f) follows from (d) by Remark 5(c).

(e) is a consequence of (c) and Remark 5(c). It remains to prove (c).

(c) Let $x : \mathbb{N} \to \mathbb{R}$ be a solution of $({}_{\text{R-L}}\widetilde{\nabla}^{\alpha} x)(n) = \lambda x(n)$. Then by (36) in Lemma 2,

$$x(n) = -\frac{1}{1-\lambda} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k), \qquad n \in \mathbb{N}_2, \tag{69}$$

and by Theorem 8, if $\lambda \in (-\infty, 0] \cup (2^{\alpha}, \infty)$ then $x \in l^1(\mathbb{N}_1)$. We will calculate

$$S := \sum_{n=1}^{\infty} x(n).$$

Summing up the Eqs. (69) for n from 2 to ∞ we get

$$S - x(1) = -\frac{1}{1-\lambda} \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k). \tag{70}$$

Since the series $\sum_{i=1}^{\infty} (-1)^i \binom{\alpha}{i}$ and $\sum_{i=1}^{\infty} x(i)$ are absolutely convergent, their Cauchy product

$$\sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{\alpha}{n-k} x(k)$$

is also absolutely convergent and its sum is

$$\sum_{i=1}^{\infty} (-1)^i \binom{\alpha}{i} \sum_{i=1}^{\infty} x(i) = -S. \tag{71}$$

In the last step we use the well known formula

$$\sum_{i=0}^{\infty} \binom{\alpha}{i} w^i = (1+w)^{\alpha}, \qquad w \in [-1, 1],$$

with $w = -1$. Combining (70) with (71) we get

$$S = \frac{\lambda - 1}{\lambda} x(1). \tag{72}$$

From Lemma 4 we know that the solution $y(n), n \in \mathbb{N}_1$, of the Caputo time-invariant one-dimensional equation (22) is given by

$$y(n) = (x * g)(n) = \sum_{j=1}^{n} x(n+1-j)g(j), \tag{73}$$

where $x(n), n \in \mathbb{N}_1$, is the solution of (69) with $x(1) = 1$ and $g(n), n \in \mathbb{N}_1$, is given by

$$g(n) = \begin{cases} y(1) & \text{for } n = 1, \\ (1-\lambda)^{-1}(-1)^{n-1}\binom{\alpha-1}{n-1} y(1) & \text{for } n \in \mathbb{N}_2. \end{cases}$$

Using formula (6) in [5]

$$\lim_{n\to\infty} (-1)^{n-1} \binom{\alpha-1}{n-1} n^{\alpha} = \frac{1}{\Gamma(1-\alpha)},$$

we get

$$\lim_{n\to\infty} g(n)n^{\alpha} = \frac{y(1)}{(1-\lambda)\Gamma(1-\alpha)}.$$

Moreover, from Theorem 8 we know that the sequence $r(n) = x(n), n \in \mathbb{N}_1$ satisfies condition (68) and therefore we may apply Lemma 5 to the sequences $r(n) = x(n)$ and $f(n) = g(n), n \in \mathbb{N}_1$. This leads, in light of (72) and (73) to

$$\lim_{n\to\infty} \frac{y(n)}{n^{-\alpha}} = -\frac{x(1)}{\lambda\Gamma(1-\alpha)}.$$

The last equality completes the proof of (c). □

6.2 *Scalar Time-Varying Backward Equations*

In this subsection we consider one dimensional time-varying fractional backward equations (20)–(23) with linear part $\lambda\colon \mathbb{N}_0 \to \mathbb{R}$, i.e.

$$({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(n) = \lambda(n)x(n) \quad \text{and} \quad ({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha} x)(n) = \lambda(n)x(n), \quad n \in \mathbb{N}_1, \tag{74}$$

$$({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(n) = \lambda(n)x(n) \quad \text{and} \quad ({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha} x)(n) = \lambda(n)x(n), \quad n \in \mathbb{N}_2. \tag{75}$$

We first provide conditions under which an order relation between two linear parts and initial conditions implies an order of the two corresponding solutions.

Theorem 14 (Comparison theorem for scalar backward equations)
Let $x_1, x_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 \colon \mathbb{N}_0 \to \mathbb{R}$, $\alpha \in (0,1)$, and assume the order relations

$$x_1 \geq x_2 > 0 \quad \text{and} \quad \lambda_1(n) \geq \lambda_2(n) \quad \text{for each } n \in \mathbb{N}_0.$$

For $\widehat{\nabla}^{\alpha} \in \{{}_{\mathrm{C}}\overline{\nabla}^{\alpha}, {}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}, {}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}, {}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}\}$ let $\varphi_1^{\widehat{\nabla}}(\cdot, x_1)$ and $\varphi_2^{\widehat{\nabla}}(\cdot, x_2)$ denote the solutions of

$$(\widehat{\nabla}^{\alpha} x)(n) = \lambda_1(n)x(n) \quad \text{and} \quad (\widehat{\nabla}^{\alpha} x)(n) = \lambda_2(n)x(n)$$

with initial condition x_1 and x_2, respectively.

If either $\lambda_1(n) < 1$ for each $n \in \mathbb{N}_0$, or $\lambda_2(n) > 1$ for each $n \in \mathbb{N}_0$, then

$$\varphi_1^{\widehat{\nabla}}(n, x_1) \geq \varphi_2^{\widehat{\nabla}}(n, x_2), \qquad n \in \mathbb{N}_0. \tag{76}$$

Proof Under the assumption that $\lambda_1(n) < 1$ for each $n \in \mathbb{N}_0$, the statement has been proved for $\widehat{\nabla}^{\alpha} = {}_{\mathrm{C}}\overline{\nabla}^{\alpha}$ in [21, Theorem 2.4] and for $\widehat{\nabla}^{\alpha} = {}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}$ in [20, Theorem 2.5]. Remark 4(c) implies the statement also for $\widehat{\nabla}^{\alpha} = {}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}$ and $\widehat{\nabla}^{\alpha} = {}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}$.

Assume now that $\lambda_2(n) > 1$ for each $n \in \mathbb{N}_0$. We consider first the case $\widehat{\nabla}^{\alpha} = {}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}$. For $n = 1$ the statement is true, since a direct calculation shows that for $i = 1, 2$

$$\varphi_i^{\widehat{\nabla}}(n, x_i) = \frac{\alpha x_i}{-\lambda_i(1) + 1}.$$

Suppose that (76) holds for $n \in \{0, 1, \ldots, m\}$ for an $m \in \mathbb{N}_1$, then according to (32) we have

$$\begin{aligned}
(1 - \lambda_1(m+1))\varphi_1^{\widehat{\nabla}}(m+1, x_1) &= -\sum_{k=0}^{m} (-1)^{m+1-k} \binom{\alpha}{m+1-k} \varphi_1^{\widehat{\nabla}}(k, x_1) \\
&\geq -\sum_{k=0}^{m} (-1)^{m+1-k} \binom{\alpha}{m+1-k} \varphi_2^{\widehat{\nabla}}(k, x_1) \\
&= (1 - \lambda_2(m+1))\varphi_2^{\widehat{\nabla}}(m+1, x_1),
\end{aligned}$$

i.e. (76) holds for $n = m + 1$ and the proof is completed. In the same way, using the representation (30), the statement follows for $\widehat{\nabla}^{\alpha} = {}_{\mathrm{C}}\overline{\nabla}^{\alpha}$. Finally using Remark 4(c) again, the statement follows also for $\widehat{\nabla}^{\alpha} = {}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}$ and $\widehat{\nabla}^{\alpha} = {}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}$. □

The next theorem presents sufficient conditions for asymptotic stability of the Eqs. (74) and (75).

Theorem 15 (Asymptotic stability of scalar backward equations)
Let $\lambda \colon \mathbb{N}_0 \to \mathbb{R}$, $\alpha \in (0,1)$.

(a) If $\sup_{n \in \mathbb{N}_0} \lambda(n) < 0$ then

$$({}_{\mathrm{C}}\overline{\nabla}^{\alpha} x)(n) = \lambda(n)x(n) \quad \text{and} \quad ({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha} x)(n) = \lambda(n)x(n)$$

are asymptotically stable.

(b) If $\inf_{n\in\mathbb{N}_0}|1-\lambda(n)|\geq 1$ *then*

$$({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}x)(n)=\lambda(n)x(n) \quad \textit{and} \quad ({}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}x)(n)=\lambda(n)x(n)$$

are asymptotically stable.

Proof (a) The asymptotic stability of $({}_{\mathrm{C}}\overline{\nabla}^{\alpha}x)(n)=\lambda(n)x(n)$ was proved in [21, Theorem B and D]. Remark 4(c) implies the asymptotic stability of $({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}x)(n)=\lambda(n)x(n)$.

(b) [20, Theorem $\widehat{\mathrm{B}}$] implies the asymptotic stability of $({}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}x)(n)=\lambda(n)x(n)$. Applying again Remark 4(c) completes the proof. □

Finally we present a result about divergence of solutions of (74) and (75).

Theorem 16 (Divergence of solutions of scalar backward equations)
Let $\lambda\colon\mathbb{N}_0\to\mathbb{R}$, $\alpha\in(0,1)$. *For* $\widehat{\nabla}^{\alpha}\in\{{}_{\mathrm{C}}\overline{\nabla}^{\alpha},{}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha},{}_{\mathrm{C}}\widetilde{\nabla}^{\alpha},{}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}\}$ *and* $x_0\in\mathbb{R}$, *let* $\varphi^{\widehat{\nabla}}(\cdot,x_0)$ *denote the solution of*

$$(\widehat{\nabla}^{\alpha}x)(n)=\lambda(n)x(n)$$

with initial condition x_0. *If there exists a* $\lambda_0>0$ *such that*

$$1>\lambda(n)\geq\lambda_0>0, \qquad n\in\mathbb{N}_0,$$

then for each $x_0\in\mathbb{R}$

$$\lim_{n\to\infty}|\varphi^{\widehat{\nabla}}(n,x_0)|=\infty.$$

Proof For $\widehat{\nabla}^{\alpha}\in\{{}_{\mathrm{C}}\overline{\nabla}^{\alpha},{}_{\mathrm{R\text{-}L}}\overline{\nabla}^{\alpha}\}$ the result is proved in [21, Theorems A and C] and [20, Theorems A and $\widehat{\mathrm{A}}$], respectively. Using Remark 4(c), we get the conclusion also for $\widehat{\nabla}^{\alpha}\in\{{}_{\mathrm{C}}\widetilde{\nabla}^{\alpha},{}_{\mathrm{R\text{-}L}}\widetilde{\nabla}^{\alpha}\}$. □

7 Separation of Solutions

The next theorem contains the main result of this paragraph.

Theorem 17 (Separation of solutions of Caputo equations)
Let $\alpha\in(0,1)$, $A\colon\mathbb{N}_0\to\mathbb{R}^{d\times d}$ *with* $\sup_{n\in\mathbb{N}_0}\|A(n)\|<\infty$, $\lambda\in\mathbb{R}$ *with* $\lambda>\frac{\alpha}{1-\alpha}$, $x,y\in\mathbb{R}^d$ *with* $x\neq y$, *and* $x_0\in\mathbb{R}^d\setminus\{0\}$.

(a) Forward equation $({}_{\mathrm{C}}\overline{\Delta}^{\alpha}x)(n)=A(n)x(n)$*:*

$$\limsup_{n\to\infty}n^{\lambda}\|\varphi_{\mathrm{C}}^{\Delta}(n,x)-\varphi_{\mathrm{C}}^{\Delta}(n,y)\|=\infty \quad \textit{and} \quad \limsup_{n\to\infty}\frac{1}{n}\ln\|\varphi_{\mathrm{C}}^{\Delta}(n,x_0)\|=\infty.$$

(b) Backward equations $({}_{\mathrm{C}}\overline{\nabla}^{\alpha}x)(n)=A(n)x(n)$ and $({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}x)(n)=A(n)x(n)$*:*

$$\limsup_{n\to\infty} n^{\lambda}\|\varphi_{\mathrm{C}}^{\overline{\nabla}}\Delta(n,x)-\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,y)\| = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{1}{n}\ln\|\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x_0)\| = \infty,$$

if $(I-A(n))^{-1}$ *exists for each* $n\in\mathbb{N}_0$. *The same holds for* $\varphi_{\mathrm{C}}^{\widetilde{\nabla}}$.

In the proof of this theorem we use the following fact.

Lemma 6 ([5, Lemma 1])
Let $\alpha>0$ *and the sequence* $(u_{-\alpha}(k))_{k\in\mathbb{N}_0}$ *be defined by*

$$u_{-\alpha}(k) = (-1)^k\binom{-\alpha}{k}, \qquad k\in\mathbb{N}_0. \tag{77}$$

Then the following statements hold:

(a) $u_{-\alpha}(k)>0$ *for* $k\in\mathbb{N}_0$.
(b) *If* $0<\alpha<1$, *then* $(u_{-\alpha}(k))_{k\in\mathbb{N}_0}$ *is a decreasing sequence.*
(c) $\sum_{k=0}^{n} u_{-\alpha}(k) = u_{-\alpha-1}(n)$ *for* $n\in\mathbb{N}_0$.
(d) *There exist* $\overline{m},\overline{M}>0$ *such that*

$$\frac{\overline{m}}{n^{1-\alpha}} < u_{-\alpha}(n) < \frac{\overline{M}}{n^{1-\alpha}}, \qquad n\in\mathbb{N}_1.$$

Proof ***(of Theorem 17)*** (a) This is proved in [5, Theorem 5].
(b) Assume that $I-A(n)$ is invertible for each $n\in\mathbb{N}_0$. We show the claim for $({}_{\mathrm{C}}\overline{\nabla}^{\alpha}x)(n) = A(n)x(n)$. The result for the solution $\varphi_{\mathrm{C}}^{\widetilde{\nabla}}$ of $({}_{\mathrm{C}}\widetilde{\nabla}^{\alpha}x)(n) = A(n)x(n)$ follows then by Remark 4(c). To this end let $x,y\in\mathbb{R}^d$ with $x\neq y$ and $\lambda>\frac{\alpha}{1-\alpha}$. Suppose the contrary, i.e. there exists $K\in\mathbb{R}$ such that

$$\limsup_{n\to\infty} n^{\lambda}\|\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x)-\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,y)\| < K,$$

which implies that

$$\lim_{n\to\infty}\|\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x)-\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,y)\| = 0 \tag{78}$$

and therefore by the boundedness of A that

$$\lim_{n\to\infty}\|(I-A(n))\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x)-\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,y)\| = 0.$$

Let us denote

$$L := \sup_{n\in\mathbb{N}_0}\|\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,x)-\varphi_{\mathrm{C}}^{\overline{\nabla}}(n,y)\| < \infty. \tag{79}$$

Furthermore, there exists $N\in\mathbb{N}_0$ such that

$$\|\varphi_{\mathrm{C}}^{\nabla}(n,x) - \varphi_{\mathrm{C}}^{\nabla}(n,y)\| \le K n^{-\lambda}, \qquad n \ge N. \tag{80}$$

Considering the Caputo equation in the form given by (29), we have

$$\begin{aligned}
&(I - A(n))(\varphi_{\mathrm{C}}^{\nabla}(n,x) - \varphi_{\mathrm{C}}^{\nabla}(n,y))\\
&= x - y + \sum_{k=1}^{n} u_{-\alpha}(n-k)A(k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y))\\
&= x - y + \sum_{k=1}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)),
\end{aligned}$$

where

$$B(n,k) := u_{-\alpha}(n-k)A(k),$$

with $u_{-\alpha}(\cdot)$ given by (77). Thus,

$$\|x - y\| \le \|(I - A(n))(\varphi_{\mathrm{C}}^{\nabla}(n,x) - \varphi_{\mathrm{C}}^{\nabla}(n,y))\| + \left\| \sum_{k=1}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\|.$$

Letting $n \to \infty$ and using (78), we obtain that

$$\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\| > 0. \tag{81}$$

Since $\lambda > \frac{\alpha}{1-\alpha}$, there exists $\delta \in (\frac{\alpha}{\lambda}, 1-\alpha)$. Thus, to get a contradiction to inequality (81), it is sufficient to show that

$$\limsup_{n\to\infty} \sum_{k=1}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) = 0 \tag{82}$$

and

$$\limsup_{n\to\infty} \sum_{k=\lceil n^{\delta}\rceil}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) = 0. \tag{83}$$

By definition of $B(n,k)$ and non-negativity of the sequence $(u_{-\alpha}(n))$ by Lemma 6(a), we have

$$\left\| \sum_{k=1}^{\lceil n^{\delta}\rceil - 1} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\|$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\lceil n^{\delta}\rceil -1} \|B(n,k)\| \|(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y))\| \\
&\leq \sum_{k=1}^{\lceil n^{\delta}\rceil -1} M u_{-\alpha}(n-k) \|(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y))\| \\
&\leq ML \sum_{k=1}^{\lceil n^{\delta}\rceil -1} u_{-\alpha}(n-k),
\end{aligned}
$$

where we used (79) to obtain the last inequality. By Lemma 6(b), the sequence $(u_{-\alpha}(n))$ is decreasing. Thus,

$$
\left\| \sum_{k=1}^{\lceil n^{\delta}\rceil -1} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\| \leq ML \lceil n^{\delta}\rceil u_{-\alpha}(n - \lceil n^{\delta}\rceil).
$$

Using Lemma 6(d), we obtain that

$$
\left\| \sum_{k=1}^{\lceil n^{\delta}\rceil -1} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\| \leq ML(n^{\delta}+1) \frac{\overline{M}}{(n-n^{\delta})^{1-\alpha}},
$$

which, together with the fact that $\delta < 1-\alpha$, proves (82). To conclude the proof we show (83). For this purpose, we use the estimate

$$
\begin{aligned}
&\left\| \sum_{k=\lceil n^{\delta}\rceil}^{n} B(n,k)(\varphi_{c}^{\nabla}(k,x) - \varphi_{c}^{\nabla}(k,y)) \right\| \\
&\leq \sum_{k=\lceil n^{\delta}\rceil}^{n} \|B(n,k)\| \|(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y))\| \\
&\leq M \sum_{k=\lceil n^{\delta}\rceil}^{n} u_{-\alpha}(n-k) \|(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y))\|.
\end{aligned}
$$

Let $n \in \mathbb{N}$ such that $n^{\delta} \geq N$. Using (80), we obtain that

$$
\left\| \sum_{k=\lceil n^{\delta}\rceil}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\nabla}(k,x) - \varphi_{\mathrm{C}}^{\nabla}(k,y)) \right\| \leq MK \lceil n^{\delta}\rceil^{-\lambda} \sum_{k=\lceil n^{\delta}\rceil}^{n} u_{-\alpha}(n-k).
$$

By Lemma 6(a) and the fact that

$$
\sum_{k=1}^{n} (-1)^k \binom{\alpha}{k} = (-1)^n \binom{\alpha-1}{n}, \qquad n \in \mathbb{N}_0,
$$

we have

$$\sum_{k=\lceil n^\delta \rceil}^{n} u_{-\alpha}(n-k) \le \sum_{k=1}^{n} u_{-\alpha}(n-k) = u_{-(\alpha+1)}(n).$$

Thus,

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\overline{\nabla}}(k,x) - \varphi_{\mathrm{C}}^{\overline{\nabla}}(k,y)) \right\| \le MK \lceil n^\delta \rceil^{-\lambda} u_{-(\alpha+1)}(n).$$

In light of Lemma 6(d) for $\alpha + 1$, we have

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi_{\mathrm{C}}^{\overline{\nabla}}(k,x) - \varphi_{\mathrm{C}}^{\overline{\nabla}}(k,y)) \right\| \le MKn^{-\delta\lambda} \frac{\overline{M}}{n^{-\alpha}}.$$

Note that $\delta\lambda > \alpha$, (83) is proved and the proof is complete. □

Our hypothesis is that a similar result holds true for the Riemann-Liouville forward equation $({}_{\text{R-L}}\overline{\Delta}^{\alpha} x)(n) = A(n)x(n)$ and backward equations $({}_{\text{R-L}}\overline{\nabla}^{\alpha} x)(n) = A(n)x(n)$ and $({}_{\text{R-L}}\widetilde{\nabla}^{\alpha} x)(n) = A(n)x(n)$ but we cannot provide a proof.

8 Conclusions

In this work we considered the asymptotic properties of six types of linear fractional equations in discrete time described by the equations (18)–(23). The first two are the Caputo and Riemann-Liouville forward equations in which the difference operator is defined as the composition of the classical forward difference with the fractional order sum. In the case of the Caputo equation, the order of these operators is such that the difference operator acts first and the fractional sum operator acts second and in the case of the Riemann-Liouville equation, the order of these operators is reversed. The next four equations, i.e. (20)–(23) are the Caputo and Riemann-Liouville backwards equation, in which the difference operator is defined as composing the classical backward difference with the sum of the fractional order. Additionally, in the case of these equations, we distinguish between two sum definitions, which include and do not include the initial condition.

For each of the equations under consideration we have given a precise formulation of the initial value problem (see Definition 4 and Remark 3) and discussed the existence and uniqueness of solutions to initial value problems (see Theorem 1). In Theorem 2 we show that each of the equations considered can be represented in two different ways as a convolution-type Volterra equation. These preparations play a key role in obtaining the further results of our work. One of them is included in Remark 5, which shows that the solutions of the Caputo backwards equations defined with a sum that takes into account the initial conditions and a sum that does not take it into account are closely related and in particular, have the same asymptotic properties. The same is true for the backwards Riemann-Liouville equations.

In Theorem 3, we present explicit formulas for solutions to stationary equations. It should be noted that the first two points of this theorem provide the formula for the solution in the form of a polynomial of the variable A, where A is the coefficient of the equation under consideration without additional assumptions about the matrix A. The remaining points present the formula for the solution in the form of a series and require an additional assumption about the matrix A: that it has all eigenvalues inside the unit circle, although a solution also exists when A has eigenvalues outside the unit circle. The formulas for solutions of the backward equations in the general case are an open problem. A step towards its solution may be Theorem 4, where such formulas are given in the form of polynomials of the variable $(I - A)^{-1}$, unfortunately the coefficients of these polynomials are given in recursive form and therefore these formulas cannot be considered as satisfactory, explicit formulas.

Theorems 5, 6, 7, 8, 9 and 11 provide sufficient conditions for the stability and instability of the equations (18)–(23). They have the following form: if the eigenvalues of the matrix A belong to an open set S, then the equation is asymptotically stable (the form of the set S depends on the equation) and if at least one eigenvalue of A belongs to the set $\mathbb{C} \setminus \operatorname{cl} S$, the equation is unstable. The problem of the asymptotic behavior of these equations when certain eigenvalues of the matrix A lie on the boundary of the stability region S remains an open problem. Moreover, these theorems say that in the case of stable Caputo equations, the rate of decay to zero as $n \to \infty$ is not greater than $n^{-\alpha}$ and in the case of the Riemann-Liouville equations, not greater than $n^{-\alpha-1}$, and that the rate of growth to infinity in the case of unstable equations is not less than r^n, $r > 1$. The problem of giving the exact growth and decay rates is also an open problem. In a special case when the system is one-dimensional, the stability problem is completely solved and its solution is given by Theorem 12. However even in the one-dimensional case, the problem of the exact rate of convergence to zero is a problem that is not completely solved. Its solution for some subsets of the stability set is given by Theorem 13. Finally, Theorems 14, 15 and 16 provide some conditions that are sufficient for the asymptotic stability and instability of one-dimensional equations with variable coefficients.

Theorem 17 is a complement of the picture of rate of convergence of solutions. It says that for the considered Caputo time-varying equations, this rate is not faster than $n^{-\lambda}$ with a certain $\lambda > 0$. Our hypothesis is that a similar result holds true for the Riemann-Liouville equations but we cannot provide a proof.

Acknowledgements The research of A. Czornik was supported by the Polish National Agency for Academic Exchange (NAWA), during the implementation of the project PPN/BEK/2020/1/00188 within the Bekker NAWA Programme. The research of the A. Babiarz was funded by the National Science Centre in Poland granted according to decision DEC-2017/01/X/ST7/00313. The research of the second author was done when he visited the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for its hospitality and financial support. The last author was partially supported by an Alexander von Humboldt Polish Honorary Research Fellowship.

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