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Proper efficiency in linear fractional vector optimization via Benson's characterization

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ABSTRACT

Linear fractional vector optimization problems are special non-convex vector optimization problems. They were introduced and first studied by E.U. Choo and D.R. Atkins in the period 1982–1984. This paper investigates the properness in the sense of Geoffrion of the efficient solutions of linear fractional vector optimization problems with unbounded constraint sets. Sufficient conditions for an efficient solution to be Geoffrion's properly efficient solution are obtained via Benson's characterization [An improved definition of proper efficiency for vector maximization with respect to cones. J Math Anal Appl. 1979;71:232–241] of Geoffrion's proper efficiency.

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1. Introduction

Introduced and firstly studied by Choo and Atkins [1–3], *linear fractional vector optimization problems* (LFVOPs) have many applications in management science and other fields. The problems have noteworthy properties and theoretical importance.

Topological properties of the solution sets of those problems and monotone affine vector variational inequalities have been studied by Choo and Atkins [2,3], Benoist [4,5], Yen and Phuong [6], Hoa et al. [7–9], Huong et al. [10,11], and other authors. Necessary and sufficient conditions for a feasible point to be an efficient solution, stability properties, solution methods, and applications of LFVOPs can be seen in [12–15]. Observe that linear fractional vector optimization problems and generalized linear fractional vector optimization problems on infinite-dimensional normed spaces were introduced and studied by Yen and Yang [16].

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Geoffrion's proper efficiency concept [17], which was proposed for vector optimization problems with the standard ordering cone (the non-negative orthant of a Euclidean space), has been extended to the case of problems with an arbitrary closed convex ordering cone by Borwein [18] and Benson [19]. Borwein's proper efficiency may differ from that of Geoffrion even if the ordering cone is the standard one. In this situation, Benson's concept of proper efficiency [19], which coincides with that of Geoffrion when the ordering cone is the standard one, deserves special attention.

It is well known that there is no difference between efficiency and Geoffrion's proper efficiency in linear vector optimization problem (see [[20, Corollary 3.1.1 and Theorem 3.1.4], [21, Remark 2.4]]). By using necessary and sufficient conditions for efficiency in linear fractional vector optimization, Choo [1] has proved that the efficient solution set of a solution of a LFVOP with a *bounded constraint set* coincides with Geoffrion's properly efficient solution set.

Recently, Huong et al. [21] have given sufficient conditions for an efficient solution of a LFVOP with *an unbounded constraint set* to be Geoffrion's properly efficient solution via a direct approach. The recession cone of the constraint set and the derivatives of the scalar objective functions at the point in question are used in these sufficient conditions. Two new theorems on Geoffrion's properly efficient solutions of LFVOPs with unbounded constraint sets and seven illustrative examples can be found in a subsequent paper [22] of these authors. Provided that all the components of the objective function are properly fractional, Theorem 3.2 from [22] gives sufficient conditions for the efficient solution set to coincide with the Geoffrion properly efficient solution set. Allowing the objective function to have some affine components, Theorem 3.4 of [22] states sufficient conditions for an efficient solution to be Geoffrion's properly efficient solution.

Verifiable sufficient conditions for an efficient point of a LFVOP to be a Borwein's properly efficient point have been obtained in [23].

In the present paper, sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to belong to Geoffrion's properly efficient solution set are obtained via Benson's characterization of Geoffrion's proper efficiency. The conditions rely on the recession cone of the constraint set, the derivatives of the scalar objective functions, and the tangent cone of the constraint set at the efficient solution. Our results complement Theorems 3.1 and 3.2 of [21], Theorems 3.2 and 3.4 of [22], and generalize the theorem of Choo [1, p.218] to the case of LFVOPs with arbitrary polyhedral convex constraint sets.

This paper is dedicated to Prof. Phan Quoc Khanh, who has made remarkable research works on proper solutions of vector optimization problems [24] and approximate proper solutions of vector equilibrium problems [25], on the occasion of his 75th birthday. It would be interesting to know what do the Kuhn-Tucker properness of Type I and of Type II from [24, p.108] mean for LFVOPs and for generalized linear fractional vector optimization problems on infinite-dimensional normed spaces [16].

The paper organization is as follows. Section 2 recalls some notations, definitions, and known results. Section 3 establishes the main result. Illustrative examples are given in Section 4.

2. Preliminaries

We denote by $\mathbb{N}$ the set of the positive integers. The scalar product and the norm in $\mathbb{R}^n$ are denoted, respectively, by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Vectors in $\mathbb{R}^n$ are represented by columns of real numbers in matrix calculations, but they are written as rows of real numbers in the text. If A is a matrix, then A^{T} stands for the transposed matrix of A. Thus, for any $x, y \in \mathbb{R}^n$, one has $\langle x, y\rangle = x^{\mathrm{T}}y$.

Let $M \subset \mathbb{R}^n$ and $\bar{x} \in \bar{M}$, where $\bar{M}$ stands for the topological closure of M. The *Bouligand-Severi tangent cone* (see, e.g. [26, p.197]) of M at $\bar{x}$ is the set

$$T(\bar{x}; M) := \{v \in \mathbb{R}^n \;:\; \exists\{t_k\} \subset \mathbb{R}_+ \setminus \{0\},\ t_k \to 0,\ \exists\{v^k\} \subset \mathbb{R}^n,\ v^k \to v,$$
$$\bar{x} + t_k v^k \in M\ \forall k \in \mathbb{N}\}.$$

If $\{x^k\} \subset M$, $x^k \neq \bar{x}$ for all k, $\lim_{k\to\infty} x^k = \bar{x}$, and $\lim_{k\to\infty} \frac{x^k-\bar{x}}{\|x^k-\bar{x}\|} = v$, then one has $v \in T(\bar{x}; M)$. To verify this assertion, it suffices to set $t_k = \|x^k - \bar{x}\|$, $v^k = \frac{x^k-\bar{x}}{\|x^k-\bar{x}\|}$, and note that $x^k = \bar{x} + t_k v^k$ for all k.

It is well known that $T(\bar{x}; M)$ is a closed cone, which may be non-convex if M is a non-convex set. When M is convex, one has $T(\bar{x}; M) = \overline{\mathrm{cone}}(M - \bar{x})$ with

$$\mathrm{cone}\, Q = \{\lambda u \;:\; \lambda > 0,\ u \in Q\}$$

for any $Q \subset \mathbb{R}^n$ and $\overline{\mathrm{cone}}\, Q := \overline{\mathrm{cone}\, Q}$.

A non-zero vector $v \in \mathbb{R}^n$ (see [27, p.61]) is said to be a *direction of recession* of a non-empty convex set $M \subset \mathbb{R}^n$ if $x + tv \in M$ for every $t \geq 0$ and every $x \in M$. The set composed by $0 \in \mathbb{R}^n$ and all the directions $v \in \mathbb{R}^n \setminus \{0\}$ satisfying the last condition, is called the *recession cone* of M and denoted by 0^+M. If M is closed and convex, then $0^+M = \{v \in \mathbb{R}^n \;:\; \exists x \in M \text{ s.t. } x + tv \in M \text{ forall } t > 0\}$.

Lemma 2.1: *[See, e.g. [21, Lemma 2.10]] Let $C \subset \mathbb{R}^n$ be closed and convex, $\bar{x} \in C$, and let $\{x^k\}$ be a sequence in $C \setminus \{\bar{x}\}$ with $\lim_{k\to\infty} \|x^k\| = +\infty$. If $\lim_{k\to\infty} \frac{x^k-\bar{x}}{\|x^k-\bar{x}\|} = z$, then $z \in 0^+C$.*

For any $\bar{x} \in K$, where K is a convex set, one has $0^+K \subset T_K(\bar{x})$. Consider *linear fractional functions* $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, of the form

$$f_i(x) = \frac{a_i^{\mathrm{T}} x + \alpha_i}{b_i^{\mathrm{T}} x + \beta_i},$$

where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let K be a *polyhedral convex set*, i.e. there exist $p \in \mathbb{N}$, a matrix $C = (c_{ij}) \in \mathbb{R}^{p\times n}$, and a vector $d = (d_i) \in \mathbb{R}^p$ such that $K = \{x \in \mathbb{R}^n \;:\; Cx \leq d\}$.

We assume that $b_i^{\mathrm{T}}x + \beta_i > 0$ for all $i \in I$ and $x \in K$, where $I := \{1, \dots, m\}$. Put $f(x) = (f_1(x), \dots, f_m(x))$ and let

$$\Omega = \{x \in \mathbb{R}^n \,:\, b_i^{\mathrm{T}}x + \beta_i > 0,\ \forall i \in I\}.$$

Clearly, Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. The *linear fractional vector optimization problem* (LFVOP) given by f and K is formally written as

$$\text{(VP)} \qquad \text{Minimize } f(x) \quad \text{subject to } x \in K.$$

Definition 2.2: A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $(f(K) - f(x)) \cap (-\mathbb{R}^m_+ \setminus \{0\}) = \emptyset$, where $\mathbb{R}^m_+$ denotes the non-negative orthant in $\mathbb{R}^m$. One calls $x \in K$ a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $(f(K) - f(x)) \cap (-\operatorname{int}\mathbb{R}^m_+) = \emptyset$, where $\operatorname{int}\mathbb{R}^m_+$ abbreviates the topological interior of $\mathbb{R}^m_+$.

The efficient solution set (resp., the weakly efficient solution set) of (VP) are denoted, respectively, by E and E^w.

Lemma 2.3: *[See, e.g. [[12, Lemma 8.1], [13]]] Let $\varphi(x) = \frac{a^{\mathrm{T}}x+\alpha}{b^{\mathrm{T}}x+\beta}$ be a linear fractional function defined by $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Suppose that $b^{\mathrm{T}}x + \beta \neq 0$ for every $x \in K_0$, where $K_0 \subset \mathbb{R}^n$ is an arbitrary polyhedral convex set. Then, one has*

$$\varphi(y) - \varphi(x) = \frac{b^{\mathrm{T}}x + \beta}{b^{\mathrm{T}}y + \beta}\langle \nabla\varphi(x), y - x\rangle,$$

for any $x, y \in K_0$, where $\nabla\varphi(x)$ denotes the Fréchet derivative of φ at x.

Definition 2.4: [See [17, p.618]] One says that $\bar{x} \in E$ is *Geoffrion's properly efficient solution* of (VP) if there exists a scalar $M > 0$ such that, for each $i \in I$, whenever $x \in K$ and $f_i(x) < f_i(\bar{x})$ one can find an index $j \in I$ such that $f_j(x) > f_j(\bar{x})$ and $A_{i,j}(\bar{x}, x) \le M$ with $A_{i,j}(\bar{x}, x) := \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})}$.

For LFVOPs, the ordering cone is the standard one. So, the notion of properly efficient solution in the sense of Benson [19] is as follows.

Definition 2.5: [[19, Def. 2.4]] An element $\bar{x} \in K$ is called a *Benson properly efficient solution* of (VP) if

$$\overline{\operatorname{cone}}(f(K) + \mathbb{R}^m_+ - f(\bar{x})) \cap (-\mathbb{R}^m_+) = \{0\}. \tag{1}$$

The Benson properly efficient solution set of (VP) is denoted by E^{Be}. Since (1) surely yields $(f(K) - f(\bar{x})) \cap (-\mathbb{R}^m_+) = \{0\}$, property (1) implies that $\bar{x} \in E$. Applying [19, Theorem 3.2] to (VP), we get the following result.

Proposition 2.6: *One has $E^{Ge} = E^{Be}$, i.e. the Benson properly efficient solution set of (VP) coincides with the Geoffrion properly efficient solution set of that problem.*

The equality $E^{Ge} = E^{Be}$ allows us to use the criterion (1) to verify whether $\bar{x}$ is a properly efficient solution of (VP) in the sense of Geoffrion, or not. Sometimes, checking (1) is easier than checking the condition in Definition 2.4. The next theorem belongs to Choo [1].

Proposition 2.7: *[See [1, p.218]] If K is bounded, then $E = E^{Ge}$.*

The following lemma was suggested with full proof by one of the two anonymous referees of this paper. Thanks to this lemma, the original proof of our main result can be significantly shortened.

Lemma 2.8: *For any non-empty subset A of $\mathbb{R}^m$, one has*

$$\overline{\mathrm{cone}}(A + \mathbb{R}^m_+) \cap (-\mathbb{R}^m_+) = \{0\} \tag{2}$$

if and only if

$$\overline{\mathrm{cone}}(A) \cap (-\mathbb{R}^m_+) = \{0\}. \tag{3}$$

Proof: The implication (2) $\Rightarrow$ (3) is clear because $A \subset A + \mathbb{R}^m_+$. To prove the reverse implication, suppose to the contrary that (3) holds, but there are a vector $v \in -\mathbb{R}^m_+$, $v \neq 0$, a sequence $\{t_k\}$ of positive real numbers, and sequences $\{r^k\} \subset \mathbb{R}^m_+$, $\{a^k\} \subset A$, such that $v = \lim_{k\to\infty}[t_k(a^k + r^k)]$. Setting $u^k = t_k r^k$ for all $k \in \mathbb{N}$, we have $\{u^k\} \subset \mathbb{R}^m_+$ and

$$v = \lim_{k\to\infty} (t_k a^k + u^k). \tag{4}$$

If the sequence $\{u^k\}$ is bounded, we may assume that it converges to some $u \in \mathbb{R}^m_+$. From (4), it follows that

$$\lim_{k\to\infty} (t_k a^k) = v - u \in -\mathbb{R}^m_+ \setminus \{0\} - \mathbb{R}^m_+ = -\mathbb{R}^m_+ \setminus \{0\}.$$

Since $\lim_{k\to\infty}(t_k a^k) \in \overline{\mathrm{cone}}(A)$, this contradicts (3).

If $\{u^k\}$ is unbounded, we may assume that $\lim_{k\to\infty} \|u^k\| = +\infty$, $u^k \neq 0$ for all k, and $\lim_{k\to\infty} \frac{u^k}{\|u^k\|} = z$, where $\|z\| = 1$. Then, by (4),

$$0 = \lim_{k\to\infty} \frac{v}{\|u^k\|} = \lim_{k\to\infty} \left(\frac{t_k}{\|u^k\|} a^k + \frac{u^k}{\|u^k\|} \right).$$

Therefore, $-z = \lim_{k\to\infty}(\frac{t_k}{\|u^k\|} a^k) \in \overline{\mathrm{cone}}(A)$. Since $-z \in -\mathbb{R}^m_+$, this comes in conflict with (3) and completes the proof. ■

3. Sufficient conditions for the Geoffrion proper efficiency

In this section, we will establish a new theorem on the Geoffrion proper efficiency LFVOPs. It is proved by using the criterion of Benson for the Geoffrion proper efficiency, which has been recalled in Proposition 2.6.

Note that some objective functions of (VP) may be linear (affine, to be more precise), i.e. one may have $f_i(x) = a_i^{\mathrm{T}} x + \alpha_i$ for some $i \in I$. Let $I_1 := \{i \in I \,:\, b_i \neq 0\}$. Then, $b_i = 0$ and $\beta_i = 1$ for all $i \in I_0$, where $I_0 := I \setminus I_1$.

Theorem 3.1: *Assume that $\bar{x} \in E$. If K is bounded, then $\bar{x} \in E^{Ge}$. In the case where K is unbounded, if the regularity assumptions*

$$\begin{cases} \text{There is no } z \in T(\bar{x}; K) \setminus \{0\} \text{ such that,} \\ \langle \nabla f_i(\bar{x}), z \rangle = 0 \quad \text{for all } i \in I \end{cases} \tag{5}$$

and

$$\begin{cases} \text{For any } z \in (0^+K) \setminus \{0\},\ a_i^{\mathrm{T}} z > 0 \quad \text{for all } i \in I_0 \\ \text{and } b_i^{\mathrm{T}} z > 0 \quad \text{for all } i \in I_1, \end{cases} \tag{6}$$

are satisfied, then $\bar{x} \in E^{Ge}$.

One referee of this paper has given a nice proof for Theorem 3.1, which is shorter than our original proof. Upon the advice of the handling Associate Editor, we will present the shorter proof. As noted by the referee, the next statement was used in the original proof of Theorem 3.1.

Lemma 3.2: *If for some $u \in T(\bar{x}; K) \setminus \{0\}$, where $\bar{x} \in K$, one has $\langle \nabla f_i(\bar{x}), u \rangle \leq 0$ for all $i \in I$ and at least one inequality is strict, then $\bar{x}$ is not efficient.*

Proof: Assume that $\bar{x} \in K$ and there is $u \in T(\bar{x}; K) \setminus \{0\}$ such that $\langle \nabla f_i(\bar{x}), u \rangle \leq 0$ for all $i \in I$ and at least one inequality is strict. As K is a polyhedral convex set, there exists a number $\tau > 0$ such that $[\bar{x}, \bar{x} + \tau u] \subset K$. Hence, for any fixed $t \in (0, \tau]$, by Lemma 2.3 one has

$$f_i(\bar{x} + tu) - f_i(\bar{x}) = \frac{b_i^{\mathrm{T}} \bar{x} + \beta_i}{b_i^{\mathrm{T}} (\bar{x} + tu) + \beta_i} \langle \nabla f_i(\bar{x}), tu \rangle \quad (\forall i \in I). \tag{7}$$

Since $b_i^{\mathrm{T}} x + \beta_i > 0$ for all $x \in K$ and $i \in I$, our assumption and (7) imply that

$$f_i(\bar{x} + tu) \leq f_i(\bar{x}) \quad (\forall i \in I),$$

where at least one inequality is strict. Then, we have $\bar{x} \notin E$. ■

In the proof below, the symbol '→' signifies 'converges to' as k tends to ∞.

Proof: If K is bounded, then by Proposition 2.7 one has $\bar{x} \in E^{Ge}$. Now, consider the situation where K is unbounded. Suppose to the contrary that $\bar{x} \notin E^{Ge}$. Then, according to Proposition 2.6, Lemmas 2.8 and 2.3, we can find a sequence $\{t_k\}$ of positive real numbers and a sequence $\{x^k\} \subset K$ such that

$$\begin{aligned} v_i &:= \lim_{k\to\infty} [t_k(f_i(x^k) - f_i(\bar{x}))] \\ &= \lim_{k\to\infty} \left[\frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}}x^k + \beta_i} \langle \nabla f_i(\bar{x}), t_k(x^k - \bar{x}) \rangle \right] \leq 0 \quad (\forall i \in I), \end{aligned} \tag{8}$$

in which at least one inequality is strict. Due to the last property, there is no loss of generality to assume that $x^k \neq \bar{x}$ for all $k \in \mathbb{N}$. Putting $v_i^k = t_k(f_i(x^k) - f_i(\bar{x}))$ for all $i \in I$ and $k \in \mathbb{N}$, we observe by (8) that $v_i = \lim_{k\to\infty} v_i^k$ for all $i \in I$.

Thanks to Lemma 2.1, by choosing a subsequence of $\{x^k\}$ if necessary, it suffices to consider three cases only:

(C1) $x^k \to \bar{x}$ and $\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to u \in T(\bar{x}; K) \setminus \{0\}$.
(C2) $x^k \to \hat{x} \in K, \hat{x} \neq \bar{x}$.
(C3) $\|x^k\| \to +\infty$ and $\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to z \in 0^+K \setminus \{0\}$.

As for $\{t_k\}$, we may assume either

(S1) $t_k(x^k - \bar{x}) \to 0$; or
(S2) $t_k(x^k - \bar{x}) \to w \neq 0$; or
(S3) $t_k\|x^k - \bar{x}\| \to +\infty$.

Consider the case (C1) first. If the situation (S1) occurs, it follows from (8) that $v_i = 0$ for all $i \in I$, which is impossible. In the situation (S2), we deduce from (8) that

$$\begin{aligned} 0 \geq \frac{v_i}{\|w\|} &= \lim_{k\to\infty} \frac{v_i^k}{t_k\|x^k - \bar{x}\|} = \lim_{k\to\infty} \left[\frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}}x^k + \beta_i} \left\langle \nabla f_i(\bar{x}), \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] \\ &= \langle \nabla f_i(\bar{x}), u \rangle \end{aligned}$$

for all $i \in I$, in which at least one inequality is strict. Then, by Lemma 3.2, $\bar{x}$ is not efficient. If the situation (S3) happens, one has

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{v_i^k}{t_k\|x^k - \bar{x}\|} = \lim_{k\to\infty} \left[\frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}}x^k + \beta_i} \left\langle \nabla f_i(\bar{x}), \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] \\ &= \langle \nabla f_i(\bar{x}), z \rangle \end{aligned}$$

for all $i \in I$. This is in contradiction with the assumption (5).

Now, let us consider the case (C2). Select an index $i_0 \in I$ such that $v_{i_0} < 0$. According to (8), we have

$$0 > v_{i_0} = \lim_{k\to\infty} [t_k(f_{i_0}(x^k) - f_{i_0}(\bar{x}))]. \tag{9}$$

Since

$$\lim_{k\to\infty} (f_{i_0}(x^k) - f_{i_0}(\bar{x})) = f_{i_0}(\hat{x}) - f_{i_0}(\bar{x}),$$

from (9) it follows that $\{t_k\}$ cannot have any subsequence converging to 0. Since $\hat{x} - \bar{x} \in T(\bar{x}; K) \setminus \{0\}$, the assumption (5) implies that $\langle \nabla f_{i_1}(\bar{x}), \hat{x} - \bar{x}\rangle \neq 0$ for some $i_1 \in I$. By (8),

$$0 \geq v_{i_1} = \lim_{k\to\infty} \left[\frac{b_{i_1}^{\mathrm{T}}\bar{x} + \beta_{i_1}}{b_{i_1}^{\mathrm{T}}x^k + \beta_{i_1}} \langle \nabla f_{i_1}(\bar{x}), t_k(x^k - \bar{x})\rangle \right]. \tag{10}$$

As

$$\lim_{k\to\infty} \left[\frac{b_{i_1}^{\mathrm{T}}\bar{x} + \beta_{i_1}}{b_{i_1}^{\mathrm{T}}x^k + \beta_{i_1}} \langle \nabla f_{i_1}(\bar{x}), x^k - \bar{x}\rangle \right] = \frac{b_{i_1}^{\mathrm{T}}\bar{x} + \beta_{i_1}}{b_{i_1}^{\mathrm{T}}\hat{x} + \beta_{i_1}} \langle \nabla f_{i_1}(\bar{x}), \hat{x} - \bar{x}\rangle \neq 0,$$

by (10) we can assert that $\{t_k\}$ does not have any subsequence converging to $+\infty$. Therefore, by considering a subsequence if necessary, we may assume that $\{t_k\}$ converges to some $\bar{t} > 0$. Since $f(x^k) \to f(\hat{x})$, from (8) it follows that $f(\hat{x}) - f(\bar{x}) = \bar{t}^{-1}v$ Since $v \leq 0$ and $v_{i_0} < 0$, this shows that $\bar{x}$ is not efficient.

Finally, consider the case (C3). Suppose that the situation (S1) occurs. If $i \in I_0$, then

$$v_i = \lim_{k\to\infty} [t_k(f_i(x^k) - f_i(\bar{x}))] = \lim_{k\to\infty} [a_i^{\mathrm{T}}(t_k(x^k - \bar{x}))] = 0.$$

Since $t_k\|x^k - \bar{x}\| \to 0$ under (S1) and $\|x^k\| \to +\infty$, we must have $t_k \to 0$. For every $i \in I_1$, as $b_i^{\mathrm{T}}z > 0$ by the assumption (6), it holds that

$$\begin{aligned} v_i &= \lim_{k\to\infty} \left[\frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\frac{b_i^{\mathrm{T}}(x^k - \bar{x})}{\|x^k - \bar{x}\|} + \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\|x^k - \bar{x}\|}} \left\langle \nabla f_i(\bar{x}), t_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] \\ &= \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}}z} \lim_{k\to\infty} \left[\left\langle \nabla f_i(\bar{x}), t_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] = 0. \end{aligned} \tag{11}$$

We have thus arrived at a contradiction that $v = 0$. In the situation (S2), setting $w^k = t_k(x^k - \bar{x})$, we get $w^k \to w \neq 0$. Note that

$$w^k = t_k(x^k - \bar{x}) = [t_k\|x^k - \bar{x}\|] \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|}. \tag{12}$$

So, $t_k\|x^k - \bar{x}\| = \|w^k\| \to \|w\|$. Therefore, passing (12) to the limit as $k \to \infty$ yields $w = \|w\|z$. Since $z \in 0^+K \setminus \{0\}$, this implies that $w \in 0^+K \setminus \{0\}$. If $i \in I_0$, then $v_i = \lim_{k\to\infty}[a_i^{\mathrm{T}}(t_k(x^k - \bar{x}))] = a_i^{\mathrm{T}} w$. Since $v_i \le 0$ and $w \in 0^+K \setminus \{0\}$, in view of (6), $I_0 = \emptyset$. Thus, $I_1 = I$. Observe that under (C3) and (S2), $t_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to 0$. Therefore, for each $i \in I = I_1$, the expression (11) holds true. Again we arrive at a contradiction that $v = 0$. In the situation (S3), for $i \in I_0$ we have

$$0 = \lim_{k\to\infty} \frac{v_i^k}{t_k\|x^k - \bar{x}\|} = \lim_{k\to\infty} \frac{a_i^{\mathrm{T}}(x^k - \bar{x})}{\|x^k - \bar{x}\|} = a_i^{\mathrm{T}} z.$$

By (6), $I_0 = \emptyset$. Hence, $I_1 = I$. Again, by choosing a subsequence of if necessary, we have to distinguish only three situations for $\{t_k\}$:

(a) $t_k \to 0$;
(b) $t_k \to \bar{t} > 0$;
(c) $t_k \to +\infty$.

Under (a), expression (11) implies $v_i = 0$ for all $i \in I$ which is a contradiction because $v \neq 0$. Under (b),

$$\begin{aligned} 0 \ge \frac{v_i}{\bar{t}} = \lim_{k\to\infty} \frac{v_i^k}{\bar{t}} &= \lim_{k\to\infty} \left[\frac{t_k}{\bar{t}} \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\frac{b_i^{\mathrm{T}}(x^k - \bar{x})}{\|x^k - \bar{x}\|} + \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\|x^k - \bar{x}\|}} \left\langle \nabla f_i(\bar{x}), \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] \\ &= \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}} z} \langle \nabla f_i(\bar{x}), z \rangle \end{aligned}$$

for all $i \in I$. Therefore, $\langle \nabla f_i(\bar{x}), z \rangle \le 0$ for all $i \in I$ and at least one inequality is strict because $b_i^{\mathrm{T}} z > 0$ by (6) and $b_i^{\mathrm{T}}\bar{x} + \beta_i > 0$. By Lemma (3.2), $\bar{x}$ is not efficient. Under (c), one has

$$\begin{aligned} 0 = \lim_{k\to\infty} \frac{v_i^k}{t_k} &= \lim_{k\to\infty} \left[\frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\frac{b_i^{\mathrm{T}}(x^k - \bar{x})}{\|x^k - \bar{x}\|} + \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{\|x^k - \bar{x}\|}} \left\langle \nabla f_i(\bar{x}), \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right\rangle \right] \\ &= \frac{b_i^{\mathrm{T}}\bar{x} + \beta_i}{b_i^{\mathrm{T}} z} \langle \nabla f_i(\bar{x}), z \rangle \end{aligned}$$

for all $i \in I$, which contradicts (5) because $z \in (0^+K) \setminus \{0\} \subset T(\bar{x}; K) \setminus \{0\}$.

The proof is complete. ∎

4. Illustrative examples

To show the usefulness of Theorem 3.1, we will apply it to some examples, which were analysed in [21] by other results and methods.

Example 4.1: [See [3, Example 2]] Consider problem (VP) with

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 2,\ 0 \leq x_2 \leq 4\},$$
$$f_1(x) = \frac{-x_1}{x_1 + x_2 - 1}, \quad f_2(x) = \frac{-x_1}{x_1 - x_2 + 3}.$$

It is well known that $E = E^w = \{(x_1, 0) : x_1 \geq 2\} \cup \{(x_1, 4) : x_1 \geq 2\}$. Since $I_1 = I$ and $0^+K = \{v = (v_1, 0) : v_1 \geq 0\}$, condition (6) is fulfilled. For any $x = (x_1, x_2) \in K$, one has

$$\nabla f_1(x) = \begin{pmatrix} \dfrac{-x_2 + 1}{(x_1 + x_2 - 1)^2} \\ \dfrac{x_1}{(x_1 + x_2 - 1)^2} \end{pmatrix}, \quad \nabla f_2(x) = \begin{pmatrix} \dfrac{x_2 - 3}{(x_1 - x_2 + 3)^2} \\ \dfrac{-x_1}{(x_1 - x_2 + 3)^2} \end{pmatrix}.$$

So, for any $\bar{x} \in \{(\bar{x}_1, 0) : \bar{x}_1 \geq 2\} \cup \{(x_1, 4) : x_1 \geq 2\}$ and $v = (v_1, v_2) \in \mathbb{R}^2$, one sees that

$$\begin{cases} \langle \nabla f_1(\bar{x}), v \rangle = 0 \\ \langle \nabla f_2(\bar{x}), v \rangle = 0 \end{cases} \Longleftrightarrow \begin{cases} v_1 = 0, \\ v_2 = 0. \end{cases}$$

Hence, condition (5) is satisfied for any $\bar{x} \in E$. Thus, by Theorem 3.1 we can assert that $E^{Ge} = E$.

Example 4.2: [See [8, p.483]] Consider problem (VP) where $n = m = 3$,

$$K = \{x \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 \leq 1,\ x_1 - 2x_2 + x_3 \leq 1,$$
$$-2x_1 + x_2 + x_3 \leq 1,\ x_1 + x_2 + x_3 \geq 1\},$$

and

$$f_i(x) = \frac{-x_i + \frac{1}{2}}{x_1 + x_2 + x_3 - \frac{3}{4}} \quad (i = 1, 2, 3).$$

According to [8], one has

$$\begin{aligned} E = E^w = {} & \{(x_1, x_2, x_3) : x_1 \geq 1,\ x_3 = x_2 = x_1 - 1\} \\ & \cup \{(x_1, x_2, x_3) : x_2 \geq 1,\ x_3 = x_1 = x_2 - 1\} \\ & \cup \{(x_1, x_2, x_3) : x_3 \geq 1,\ x_2 = x_1 = x_3 - 1\}. \end{aligned} \tag{13}$$

Since $0^+K = \{v = (\tau, \tau, \tau) : \tau \geq 0\}$ and $I_1 = I$, it is easy to verify that condition (6) is satisfied. Now, setting $p(x) = (x_1 + x_2 + x_3 - \frac{3}{4})^2$, one has

$$\nabla f_1(x) = \frac{1}{p(x)} \left(-x_2 - x_3 + \frac{1}{4}, x_1 - \frac{1}{2}, x_1 - \frac{1}{2} \right),$$
$$\nabla f_2(x) = \frac{1}{p(x)} \left(x_2 - \frac{1}{2}, -x_1 - x_3 + \frac{1}{4}, x_2 - \frac{1}{2} \right),$$

$$\nabla f_3(x) = \frac{1}{p(x)}\left(x_3 - \frac{1}{2}, x_3 - \frac{1}{2}, -x_1 - x_2 + \frac{1}{4}\right).$$

Given any $\bar{x} \in E$ and $v = (\tau, \tau, \tau) \in 0^+K$, by (13) we see that one of the following situations must occur: (i) $x_1 \geq 1,\ x_3 = x_2 = x_1 - 1$; (ii) $x_2 \geq 1,\ x_3 = x_1 = x_2 - 1$; (iii) $x_3 \geq 1,\ x_2 = x_1 = x_3 - 1$. If (i) occurs (resp., (ii), or (iii) occurs), then the equality $\langle \nabla f_1(\bar{x}), v\rangle = 0$ (resp., $\langle \nabla f_2(\bar{x}), v\rangle = 0$, or $\langle \nabla f_3(\bar{x}), v\rangle = 0$) means that $\frac{1}{4}\tau = 0$. Thus, condition (5) is fulfilled for any $\bar{x} \in E$, and we have $E^{Ge} = E$ by Theorem 3.1.

Example 4.3: [See [8, pp.479–480]] Consider problem (VP) where $n = m$, $m \geq 2$,

$$K = \left\{x \in \mathbb{R}^m \ :\ x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_m \geq 0,\ \sum_{k=1}^{m} x_k \geq 1\right\},$$

and

$$f_i(x) = \frac{-x_i + \frac{1}{2}}{\sum_{k=1}^{m} x_k - \frac{3}{4}} \quad (i = 1, \ldots, m).$$

Note that $0^+K = \mathbb{R}^m_+$. Setting $q(x) = (\sum_{k=1}^{m} x_k - \frac{3}{4})^2$, we have

$$\nabla f_i(x) = \frac{1}{q(x)}\left(x_i - \frac{1}{2}, \ldots, -\sum_{k\neq i} x_k + \frac{1}{4}, \ldots, x_i - \frac{1}{2}\right)$$

where the expression $-\sum_{k\neq i} x_k + \frac{1}{4}$ is the i th component of $\nabla f_i(x)$. Hence, the equality $E^{Ge} = E$ can be proved by using Theorem 3.1 similarly as it has been done in the preceding example.

Example 4.4: [See [21, Example 2.6]] Consider the problem (VP) where

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \ :\ x_1 \geq 0,\ x_2 \geq 0\},$$
$$f_1(x) = -x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}.$$

As it has been shown in [21], $E = \{(x_1, 0) \ :\ x_1 \geq 0\}$ and $E^{Ge} = \emptyset$. To check the conditions in Theorem 3.1, note that $I_0 = \{1\}$, $I_1 = \{2\}$, $a_1 = (0, -1)^{\mathrm{T}}$, $b_2 = (1, 1)^{\mathrm{T}}$, and $0^+K = K$. For every efficient solution $\bar{x} = (\bar{x}_1, 0)$, $\bar{x}_1 > 0$, one has

$$\nabla f_1(\bar{x}) = (0, -1)^{\mathrm{T}}, \quad \nabla f_2(\bar{x}) = \begin{pmatrix} 0 \\ \dfrac{1}{\bar{x}_1 + 1} \end{pmatrix},$$

and $T_K(\bar{x}) = \{v = (v_1, v_2) \ :\ v_1 \in \mathbb{R},\ v_2 \geq 0\}$. Hence (5) and (6) are violated if one chooses $v = (1, 0) \in (0^+K) \setminus \{0\} \subset T_K(\bar{x}) \setminus \{0\}$. For $\bar{x} = (0, 0)$ we have

$T_K(\bar{x}) = \mathbb{R}^2_+$. Conditions (5) and (6) are violated if one chooses $v = (1, 0)$. The violation of the regularity conditions in Theorem 3.1 is a reason for $\bar{x} \notin E^{Ge}$.

Example 4.5: [See [21, Example 4.7]] Consider problem (VP) with $m = 3$, $n = 2$,

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \geq 0, \ x_2 \geq 0\},$$

$$f_1(x) = -x_1 - x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}, \quad f_3(x) = x_1 - x_2.$$

According to [21], $E = \{x = (x_1, x_2) \ : \ x_1 \geq 0, \ x_2 \geq 0, \ x_2 < x_1 + 1\}$, while

$$E^w = \{x = (x_1, x_2) \ : \ x_1 \geq 0, \ x_2 \geq 0, \ x_2 \leq x_1 + 1\}.$$

Let us prove that $E^{Ge} = \emptyset$. Taking any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$, one has $\bar{x}_1 \geq 0$, $\bar{x}_2 \geq 0$ and $\bar{x}_2 < \bar{x}_1 + 1$. Since $(1, 1) \in 0^+K$, we see that $x^p := \bar{x} + p(1, 1)$ belongs to K for any $p \in \mathbb{N}$. One has $f_1(x^p) < f_1(\bar{x})$ and $f_2(x^p) > f_2(\bar{x})$, while $f_3(x^p) = f_3(\bar{x})$. As observed in Section 2, we will have $\bar{x} \notin E^{Ge}$ if for every scalar $M > 0$ there exist $x \in K$ and $i \in I$ with $f_i(x) < f_i(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x) > f_j(\bar{x})$, one has $A_{i,j}(\bar{x}, x) > M$. For each $p \in N$, we choose $i = 1$. Then, $f_i(x^p) < f_i(\bar{x})$ and $j = 2$ is the unique index in I satisfying $f_j(x^p) > f_j(\bar{x})$. Moreover, for $(i, j) = (1, 2)$, we have

$$\begin{aligned} A_{i,j}(\bar{x}, x^p) = A_{1,2}(\bar{x}, x^p) &= \frac{f_1(\bar{x}) - f_1(x^p)}{f_2(x^p) - f_2(\bar{x})} \\ &= \frac{-\bar{x}_1 - \bar{x}_2 - (-\bar{x}_1 - \bar{x}_2 - 2p)}{\frac{\bar{x}_2 + p}{\bar{x}_1 + \bar{x}_2 + 1 + 2p} - \frac{\bar{x}_2}{\bar{x}_1 + \bar{x}_2 + 1}} \\ &= \frac{2(\bar{x}_1 + \bar{x}_2 + 1 + 2p)(\bar{x}_1 + \bar{x}_2 + 1)}{\bar{x}_1 + 1 - \bar{x}_2}. \end{aligned}$$

Since $\bar{x}_1 \geq 0$, $\bar{x}_2 \geq 0$ and $\bar{x}_2 < \bar{x}_1 + 1$, one has $\lim_{p \to \infty} A_{1,2}(\bar{x}, x^p) = +\infty$. So, for every $M > 0$, there exist $p \in N$ and $i \in I$ with $f_i(x^p) < f_i(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x^p) > f_j(\bar{x})$, one has $A_{i,j}(\bar{x}, x^p) > M$. This proves that $\bar{x} \notin E^{Ge}$.

The fact that $E^{Ge} = \emptyset$ can also be proved by using Proposition 2.6. Indeed, take an element $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$ and construct the sequence $\{x^p\} \subset K$ as above. We need to show that (1) is not satisfied. For every $p \in \mathbb{N}$, choosing $u^p = (0, 0, 0) \in \mathbb{R}^3_+$ and $t_p = \frac{1}{p}$, one has

$$\lim_{p \to \infty} t_p(f(x^p) + u^p - f(\bar{x})) = \lim_{p \to \infty} \frac{1}{p} \begin{pmatrix} f_1(\bar{x} + p(1,1)) - f_1(\bar{x}) \\ f_2(\bar{x} + p(1,1)) - f_2(\bar{x}) \\ f_3(\bar{x} + p(1,1)) - f_3(\bar{x}) \end{pmatrix}$$

$$= \lim_{p \to \infty} \frac{1}{p} \begin{pmatrix} -2p \\ \dfrac{p(\bar{x}_1 + 1 - \bar{x}_2)}{(\bar{x}_1 + \bar{x}_2 + 1 + 2p)(\bar{x}_1 + \bar{x}_2 + 1)} \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \in -\mathbb{R}^3_+.$$

This means that $\overline{\text{cone}}(f(K) + \mathbb{R}^m_+ - f(\bar{x})) \cap (-\mathbb{R}^m_+) \neq \{0\}$. Thus $\bar{x}$ is not Benson's properly efficient solution of (VP). So, by Proposition 2.6, $\bar{x} \notin E^{Ge}$. Since $\bar{x} \in E$ can be chosen arbitrarily, we can assert that $E^{Ge} = \emptyset$.

Now, let us check the regularity conditions (5) and (6) in Theorem 3.1. One has $I_0 = \{1, 3\}$, $I_1 = \{2\}$, and $0^+K = K = \mathbb{R}^2_+$. Since $\nabla f_1 \bar{x}) = (-1, -1)$ and $\nabla f_3(\bar{x}) = (1, -1)$ for every $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$, one simultaneously has $\langle \nabla f_1(\bar{x}), z\rangle = 0$ and $\langle \nabla f_3(\bar{x}), z\rangle = 0$ for $z = (z_1, z_2) \in T(\bar{x}; K) \setminus \{0\}$ only if $z = (0, 0)$. So, (5) is fulfilled for all $\bar{x} \in E$. However, choosing $i = 3$ and $z = (1, 1) \in (0^+K) \setminus \{0\}$, one has $i \in I_0$ and $a_i^T z = 0$. Hence, (6) is violated.

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