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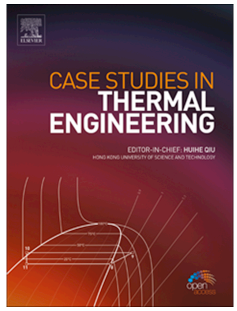

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Non-Singular Fractional Approach for Natural Convection Nanofluid with Damped Thermal Analysis and Radiation

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Abstract:

The studies of ordinary derivaives based nanofluids have limitations and some restrictions to solve and analyze the integer ordered leading partial differential equations and also have some memory effect complications. Fractional order in nanofluids can enhance and analyze more efficiently the memory effects on nanofluid behavior by different fractional derivatives techniques. In this study, the analytical solution of nanofluids containing water as a base fluid with copper oxide and silver as nanoparticles with heat and mass characteristics is investigated. The water-based nanofluid is flowing on an infinite sheet with constant temperature and thermal radiation. The dimensionless partial differential governing equations are solved in the sense of the most recent definition of fractional derivatives that is the Atangana-Baleanu fractional derivative. To dig out the mathematical solution of the developed fractional model of temperature and velocity field, the Laplace transformation technique and some of its inverse method i.e. Zakians method are utilized. To enhance the innovation of this article, the graphical and numerical representation of temperature and velocity fields are described and discussed by varying the values of different constraints such as fractional parameter and volume fraction. As a result, we concluded from the graphical illustration of the parameters, in comparison to copper oxide and silver nanofluid, CUo-water nanofluids has always slightly greater heat transfer rate as compared to Ag-water fractional nanofluid, which also depends on the enhancement of volume fraction. Furthermore, temperature and velocity profile shows decaying behavior with the enhancement in the fractional parameter β.

Keywords: Fractional nanofluids, Natural convection, Analytical solution, Thermal radiation

1. Introduction

Nanofluids are very diminutive elements in the base fluid which can exaggerate the process. To conquer the failure of heat transfer, nanofluids are unique in properties for altering the thermal features of different fluids [1, 2]. The process of heat transfer mostly is contingent on the thermophysical properties of nanoparticles and volume fraction it also is contingent on the thermophysical properties of the base fluid. The nanofluids flow under a magnetic field has a lot of real-world applications in many engineering fields i.e. hot rolling metal extrusion, energy extraction, and fiberglass. The best way to develop a heat transfer ratio is to use the use of nanoparticles in the base fluid. Firstly the idea of nanofluids was given by Choi [3]. For conventional liquids, nanofluids are considered the best substitute technique. Many examples of the contemporary use of forces and nanofluids have been discussed in the literature. Rostami *et al*. [4] considered the thermophysical possessions of nanofluids to get better thermal conductivity of nanofluids. The most studied model is Maxwell modal due to viscosity and elasticity properties i.e. Aman *et al.* [5] examined the Maxwell nanofluids solution in the sense of Laplace transformation and another Aman *et al*. [6] considered Maxwell graphene nanofluids for the exact solution of governed equations. Single-wall carbon nanotubes and multi-wall nanotubes have been discussed by Alzahrani *et al*. [7] with rotating plates. Recently, the influence of magnetic field and non-linear thermal radiation effect on hybrid bio-nanofluid flowing in a peristaltic channel by varying the values of Reynolds number with copper and gold as nanoparticles is studied in [8]. The intra-uterine nanofluid with gold as nanoparticles flowing through an asymmetric channel is studied in [9] under the magnetic field and thermal radiation effect. A comprehensive review of heat transfer in the cavities with their applications is analyzed by Hussien *et al.* [10]. In which they discussed the fluid thermal properties, the magnetic field effect, the thermal source discretion, and the entropy generation. Gul *et al*. [11] observed the effects of nanofluids on a needle base medium with carbon CNTs (SWCNTs and MWCNTs). In which governing equations are solved by Caputo fractional derivatives and the Laplace transformation technique. The numerical analysis of Non-Newtonian flow in the closed cavities with the buoyant convection effect is analyzed by the Rehman *et al.* [12]. The study of natural convection fluid mixed with nanoparticles flowing on an infinite vertical plate under the magnetic field and radiation effect is investigated by mohankrishana *et al.* [13]. Other comprehensive and interesting applications of nanofluids with their applications can be seen in [14-18].

In the mathematical study, the non-integer order modals of PDEs are solved by different mathematical techniques. Fractional calculus treats with non-local integration and differentiation [19]**.** For mathematical modeling and many physical phenomenons, fractional calculus has been widely used. An analytic solution of fractional type fluid for tangential stress and velocity was considered by Wang *et al*. [20]**.** Fractional derivative is the best approach to improve these mathematical models and numerical results. Fractional derivative models can clarify more efficiently numerical and analytical results of the real word problems such as diffusive transport, viscoelastic materials, electromagnetic theory, electrical networks, fluid flows, and rheology [21]. Makris *et al*. [22] accomplished the Maxwell model in the sense of fractional derivative rather than the classical derivatives and proved by his numerical results that the non-integer order model has more suitable results rather than the classical derivative model. With time, different definitions and algorithms were concluded by different researchers and mathematicians**.** To solve different physical and numerical phenomenon's different fractional-order definitions were used i.e. Riemann-Liouville [23], Caputo [24], Caputo-Fabrizio [25], and Atangana-Baleanu [26]. Recently, the solution for different fluids on mass and heat transfer with different mechanical and thermal conditions has been introduced by Caputo fractional derivatives by different authors [27-30]. The most recent approach and a modified form of Caputo-Fabrizio (CF) is the Atangana-Baleanu (AB) time-fractional derivative, which has the ideal properties in which non-singularity, non-locality of the kernel, good memory effects, and heredity effects are most common.

The Non-local and non-singular kernels of the AB fractional derivative utilized, namely as Mittag-Leffler function, and in some cases, this can accurately represent the dynamics of non-local phenomena. Prompted by this innovative methodology to fractional derivatives, namely AB-fractional derivative, the following points are remarkably noted for the comparison of AB and CF-fractional derivatives, that what are the advantages of AB- fractional derivative over CF-fractional derivatives;**(a):** The CF-fractional derivative is non-Markovian and some well Riemann-Liouville derivative is just Markovian. Whereas AB-fractional derivative has both Markovian and non-Markovian characteristics. **(b):** Caputo-Fabrizio is an exponential progression, and the Riemann-Liouville derivative is a power rule. While Atangana-Baleanu fractional derivative contains all properties power law, stretched exponential and Brownian motion. **(c):** RL-fractional derivative is still just power law and scale-invariant, whereas Atangana-Baleanu fractional derivative's mean square displacement is a transition from normal distribution to sub-diffusion. This demonstrates that the AB derivative is capable of explaining problems of various sizes in the real world [31]. This concept of the AB operator is implemented by Atangana and Koca [32] to a non-linear structure and demonstrates the presence and distinctiveness of the conceptual model solution of the fractional order. In [33], the authors discussed the hybrid nanofluid with alumina oxide and copper nanoparticles with water as base fluid flowing in a micro-channel under the magnetic field effect. They investigated the solution of the problem with the help of the AB-fractional derivative and the Laplace transformation scheme. And they concluded that temperature was enhanced by varying the volume fraction and heat generation parameter. In [34], the analysis of radiative heat transfer rate with different nanoparticles of hybrid nanofluid flowing on an inclined plate with ramped temperature is studied by using the AB-fractional derivative technique. The analysis of stoke's second problem by using AB-fractional derivative approach for nanofluids is studied by Abro *et al.* in [35]. The analytical solution of blood-based nanofluid with SWCNTs and MWCNTs is investigated by Saqib *et al.* [36] in the sense of AB-fractional derivative.

However, by inspiring the above literature study, we have investigated the analytical solution of fractional nanofluids flowing on an infinite sheet with constant temperature with heat flux and radiation effect. Initially, the infinite plate is at rest but after some time the plate starts to oscillate with constant velocity and fluid starts to flow. As fractional partial differential equations can explain and represent the physical behavior of any physical problem rather than simple partial differential equations. So to dig out the solution of different nanofluids modals many other definitions and techniques have been used but in this study, we have used the most recent definition of fractional derivatives i.e Atanga-Baleanu fractional derivatives and the Laplace transformation technique is utilized to find out the numerical results of temperature and velocity field. Furthermore, at the end of this study to check out the graphical behavior of different constraints on temperature and velocity field, the numerical discussion of obtained results is discussed and concluded.

2. Statement of the Problem

Suppose an unsteady viscous free convection nanofluid flow flowing on an infinite sheet lying in the XY-plane. Initially, both plate and fluid velocity are at rest with constant temperature T_∞ , then at $t = 0^+$ the plate starts to oscillate with some constant velocity. In addition, the radiation effect is also considered and radiative heat flux is applied on the plate in a perpendicular direction, as shown in Fig 1. As here we have considered an infinite plate so we assumed the components of velocity and temperature are ζ and $t.$ And Copper oxide CuO and Silver (Ag) are considered as nanoparticles with base fluid water H_2O whose physical properties are shown in Tab 1.

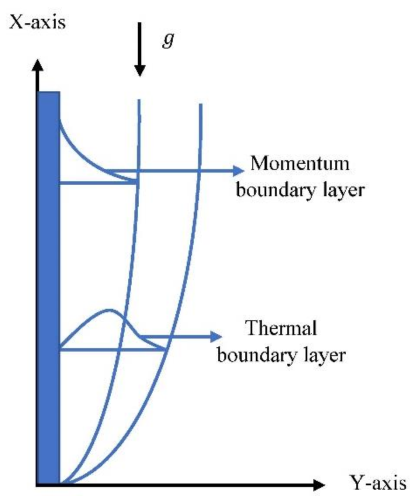


Fig 1: Physical representation of the problem

Now by using the Boussinesq's approximation[37, 38] and Rosseland approximation [39], the unsteady flow of nanofluid governed by the following governing partial differential equations [40]

$$\rho_{nf}\frac{\partial v(\zeta,t)}{\partial t} = \mu_{nf}\frac{\partial^2 v(\zeta,t)}{\partial \zeta^2} + g(\rho\beta_T)_{nf}(T(\zeta,t) - T_\infty)\ ; \quad \zeta, t > 0 \tag{1}$$

$$\left(\rho C_p\right)_{nf}\frac{\partial T(\zeta,t)}{\partial t} = k_{nf}\left(1 + \frac{q_r}{k_{nf}}\right)\frac{\partial^2 T(\zeta,t)}{\partial \zeta^2}\ ; \quad \zeta, t > 0 \tag{2}$$

Where q_r is the radiative heat flux which can be defined mathematically as $q_r = -\frac{16\sigma T_\infty^3}{3k}\frac{\partial T(\zeta,t)}{\partial \zeta}$, and the suitable initial and boundary conditions are

$$v(\zeta,t) = 0, \quad T(\zeta,t) = T_\infty\ ; \quad \zeta > 0, \quad t = 0 \tag{3}$$

$$v(\zeta,t) = 0, \quad T(\zeta,t) = T_w\ ; \quad \zeta = 0, \quad t > 0 \tag{4}$$

$$v(\zeta,t) \to 0, \quad T(\zeta,t) \to 0 \quad as \quad \zeta \to \infty, \quad t > 0 \tag{5}$$

$v(\zeta,t),\ \rho_{nf},\ g,\ T_w,\ (\beta_T)_{nf},\ \mu_{nf}$ are the fluid velocity in the XY-plane, Density of the fluid, the force of gravity, the temperature of the plate, thermal expansion, and dynamic viscosity respectively. Which can be summarized as

Table 1: Thermophysical characteristics of nanoparticles [41]

Base Fluid/Nanoparticles	$\rho(Kg/m^3)$	$C_p(J/Kg\ K)$	$k(w/m.K)$	$\beta_T \times 10^5(1/K)$
H_2O	997.1	4179	0.613	21
CuO	6320	531.8	76.5	1.80
Ag	10500	235	429	1.89

$$\rho_{nf} = (1-\varphi)\rho_f + \varphi\rho_s\ , \quad \mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}}$$

$$(\rho\beta)_{nf} = (1-\varphi)(\rho\beta_T)_f + \varphi(\rho\beta_T)_s \ , \qquad (\rho C_p)_{nf} = (1-\varphi)(\rho C_p)_f + \varphi(\rho C_p)_s$$

$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\varphi(k_f - k_s)}{k_s + 2k_f + \varphi(k_f - k_s)}$$

Where $\varphi, \rho_f, \rho_s, C_p, k_{nf}, k_f, k_s$ are volumetric fraction, the density of the base fluid, density of solid particles, specific heat at constant pressure, thermal conductivity, nanofluid, and solid particles respectively. Now to non-dimensionalize the governing equations introducing the dimensionless variables

$$t^* = \frac{\upsilon_f}{L^2}t, \quad \zeta^* = \frac{\zeta}{L}, \quad v^* = \frac{L}{\upsilon_f}v, \quad T^* = \frac{T - T_\infty}{T_w - T_\infty}, \quad L = \left[\frac{\upsilon_f{}^2}{g(\beta_T)_{nf}(T_w - T_\infty)}\right]^{1/3} \tag{6}$$

By utilizing these non-dimensional variables of Eq. (6) and ignoring the star notation, governing equations with conditions in the non-dimensional form will turn into as

$$\frac{\partial v(\zeta,t)}{\partial t} = \frac{1}{\lambda_1}\frac{\partial^2 v(\zeta,t)}{\partial \zeta^2} + \lambda_2 T(\zeta,t)\ ; \qquad \zeta, t > 0 \tag{7}$$

$$\frac{\partial T(\zeta,t)}{\partial t} = \frac{1}{\lambda_3}\frac{\partial^2 T(\zeta,t)}{\partial \zeta^2} \ ; \qquad \zeta, t > 0 \tag{8}$$

with initial and boundary conditions:

$$v(\zeta,t) = 0, \ \ T(\zeta,t) = T_\infty \ ; \qquad \zeta > 0, t = 0 \tag{9}$$

$$v(\zeta,t) = 0, \quad T(\zeta,t) = 1 \ ; \qquad \zeta = 0, t > 0 \tag{10}$$

$$v(\zeta,t) \to 0, \quad T(\zeta,t) \to 0 \quad as \ \ \zeta \to \infty, t > 0 \tag{11}$$

Where

$$\lambda_1 = (1-\varphi)^{2.5}\left(1 - \varphi + \varphi\frac{\rho_s}{\rho_s}\right), \quad \lambda_2 = \frac{1 - \varphi + \varphi\dfrac{(\rho\beta_T)_s}{(\rho\beta_T)_f}}{1 - \varphi + \varphi\dfrac{\rho_s}{\rho_f}} \ , \quad \lambda_3 = Pr\frac{1 - \varphi + \varphi\dfrac{(\rho C_p)_s}{(\rho C_p)_f}}{\dfrac{k_{nf}}{k_f} + Nr}$$

$$Pr = \frac{\mu_f C_{pf}}{k_f} \quad , \quad Nr = \frac{16\sigma^* T_\infty^3}{3k^* k_f} \tag{12}$$

Now the fractional modal i.e. Atangana-Baleanu time-fractional derivative modal of the above governing equations (7) and (8) can be defined as

$$^{AB}\mathfrak{D}_t^\beta v(\zeta,t) = \frac{1}{\lambda_1}\frac{\partial^2 v(\zeta,t)}{\partial \zeta^2} + \lambda_2 T(\zeta,t) \; ; \quad \zeta, t > 0 \tag{13}$$

$$^{AB}\mathfrak{D}_t^\beta T(\zeta,t) = \frac{1}{\lambda_3}\frac{\partial^2 T(\zeta,t)}{\partial \zeta^2} \; ; \quad \zeta, t > 0 \tag{14}$$

Where $^{AB}\mathfrak{D}_t^\beta$ is the AB-time fractional derivative with fractional operator β which can be defined as mathematically

$$^{AB}\mathfrak{D}_t^\beta \; u(\xi,t) = \frac{1}{1-\beta}\int_0^t E_\beta\left[\frac{\beta(t-z)^\beta}{1-\beta}\right] u'_{(\xi,t)} dt \quad ; \quad 0 < \beta < 1 \tag{15}$$

Where $E_\beta(z)$ is a Mittage-Laffer function which can be expressed as

$$E_\beta(z) = \sum_{r=0}^{\infty} \frac{z^\beta}{\Gamma(r\beta+1)} \quad ; \quad 0 < \beta < 1, \; z \in \mathbb{C} \tag{16}$$

And the Laplace transformation of $^{AB}\mathfrak{D}_t^\beta$ is

$$\mathcal{L}\left\{^{AB}\mathfrak{D}_t^\beta u(\xi,t)\right\} = \frac{q^\beta \mathcal{L}[u(\xi,t)] - q^{\beta-1} u(\xi,0)}{(1-\beta)q^\beta + \beta} \tag{17}$$

and

$$\lim_{\beta \to 1} {^{AB}\mathfrak{D}_t^\beta} \; u(\xi,t) = \frac{\partial v(\zeta,t)}{\partial t} \tag{18}$$

3. Solution of the Problem

To dig out the solution of fractional modeled governing equations (13)-(14) and its corresponding initial conditions (9)-(11), the Laplace transformation method will be used.

3.1.Temperature field

By applying the Laplace transformation on the governed equation (14) and its corresponding initial and boundary conditions, and utilizing the result of equation (17)

$$\frac{q^{\beta}\mathcal{L}[T(\zeta,t)]-q^{\beta-1}T(\zeta,0)}{(1-\beta)q^{\beta}+\beta}=\frac{1}{\lambda_3}\frac{\partial^2\bar{T}(\zeta,q)}{\partial\zeta^2} \tag{19}$$

Where q is the transformation of t, the Laplace of $\mathrm{T}(\xi,\mathrm{t})$ to $\bar{T}(\zeta,q)$, satisfying the flowing conditions

$$\bar{T}(0,q)=\frac{1}{q}\ ;\quad \bar{T}(\zeta,q)\to 0 \ \text{ as } \ \zeta\to\infty \tag{20}$$

Utilizing the above conditions, the solution of temperature field (19) will become as

$$\bar{T}(\zeta,q)=\frac{1}{q}e^{-\zeta\sqrt{\frac{\lambda_3 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}} \tag{21}$$

By applying the inverse of Laplace transformation on equation (21), and utilizing the Appendix equation (A1), the inverse of equation (21) can be written as

$$T(\zeta,t)=\phi\left(1,\frac{-\beta}{2};\ -\zeta\sqrt{\lambda_3 q^{\beta}}\right)\ ;\quad \zeta>0 \tag{22}$$

Where:

$$\phi(a,-b;\ z)=\sum_{n=1}^{\infty}\frac{z^n}{n!\Gamma(a-nb)}\ ;\quad b\ \epsilon\ (0,1) \tag{23}$$

is the Wright function. When $\beta=1$ then equation (21) will become as in the conditional form as

$$T(\zeta,t)=erfc\left(\frac{\zeta\sqrt{\lambda_3}}{2\sqrt{\beta t}}\right)\quad ;\quad \zeta\sqrt{\frac{\lambda_3}{\beta}}>0 \tag{24}$$

3.2.Solution of Velocity Field

Now utilizing the Laplace transformation on fractional modeled equation (13) and utilizing the result (17), the Laplace of AB-fractional derivative, the equation (13) with its corresponding conditions will be yield as

$$\frac{q^{\beta}\mathcal{L}[v(\zeta,t)]-q^{\beta-1}v(\zeta,0)}{(1-\beta)q^{\beta}+\beta}=\frac{1}{\lambda_1}\frac{\partial^2\bar{v}(\zeta,q)}{\partial\zeta^2}+\lambda_2\bar{T}(\zeta,q)\ ;\quad 0<\beta<1 \tag{25}$$

Where q is the transformed parameter of $v(\zeta,t)$ to $\bar{v}(\zeta,q)$ satisfying the subsequent conditions:

$$\bar{v}(0,q)=0\ ;\quad \bar{v}(\zeta,q)\to 0 \text{ as } \zeta\to\infty \tag{26}$$

A particular solution of the velocity field (25) is

$$\bar{v}(\zeta,q)=-\frac{\lambda_1\lambda_2}{\lambda_3-\lambda_1}\frac{(1-\beta)q^{\beta}+\beta}{q^{\beta+1}}e^{-\zeta\sqrt{\frac{\lambda_3 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}} \tag{27}$$

With its general solution

$$\bar{v}(\zeta,q)=Ae^{\zeta\sqrt{\frac{\lambda_1 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}}+Be^{-\zeta\sqrt{\frac{\lambda_1 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}}-\frac{\lambda_1\lambda_2}{\lambda_3-\lambda_1}\frac{(1-\beta)q^{\beta}+\beta}{q^{\beta+1}}e^{-\zeta\sqrt{\frac{\lambda_3 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}} \tag{28}$$

Now utilizing the above-transformed conditions (26) the solution of the velocity field can be concluded as

$$\bar{v}(\zeta,q)=\frac{\lambda_1\lambda_2}{\lambda_3-\lambda_1}\frac{(1-\beta)q^{\beta}+\beta}{q^{\beta+1}}\left(e^{-\zeta\sqrt{\frac{\lambda_1 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}}-e^{-\zeta\sqrt{\frac{\lambda_3 q^{\beta}}{(1-\beta)q^{\beta}+\beta}}}\right) \tag{29}$$

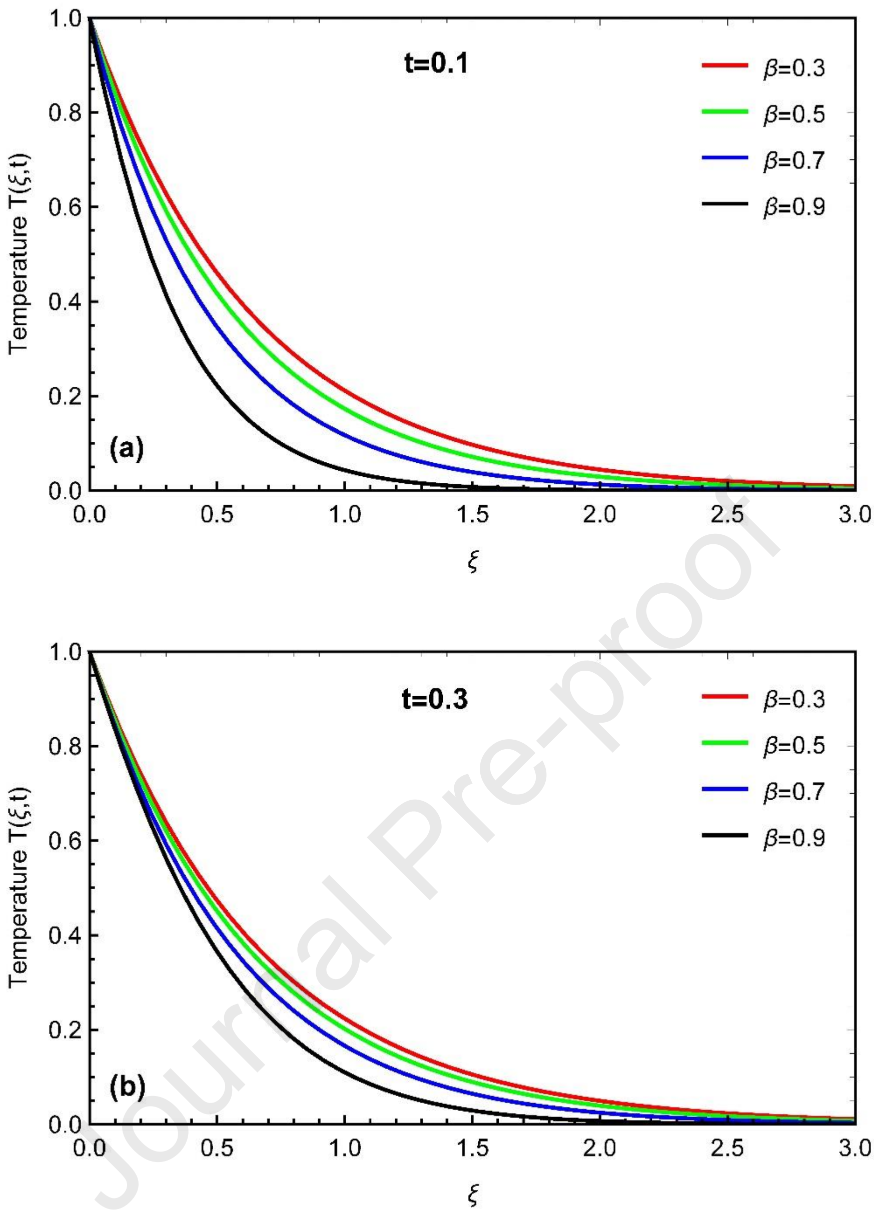


Fig 2: Variation in fractional parameter β for temperature distribution and CuO nano-particles at $t = 0.1$ and $t = 0.3$

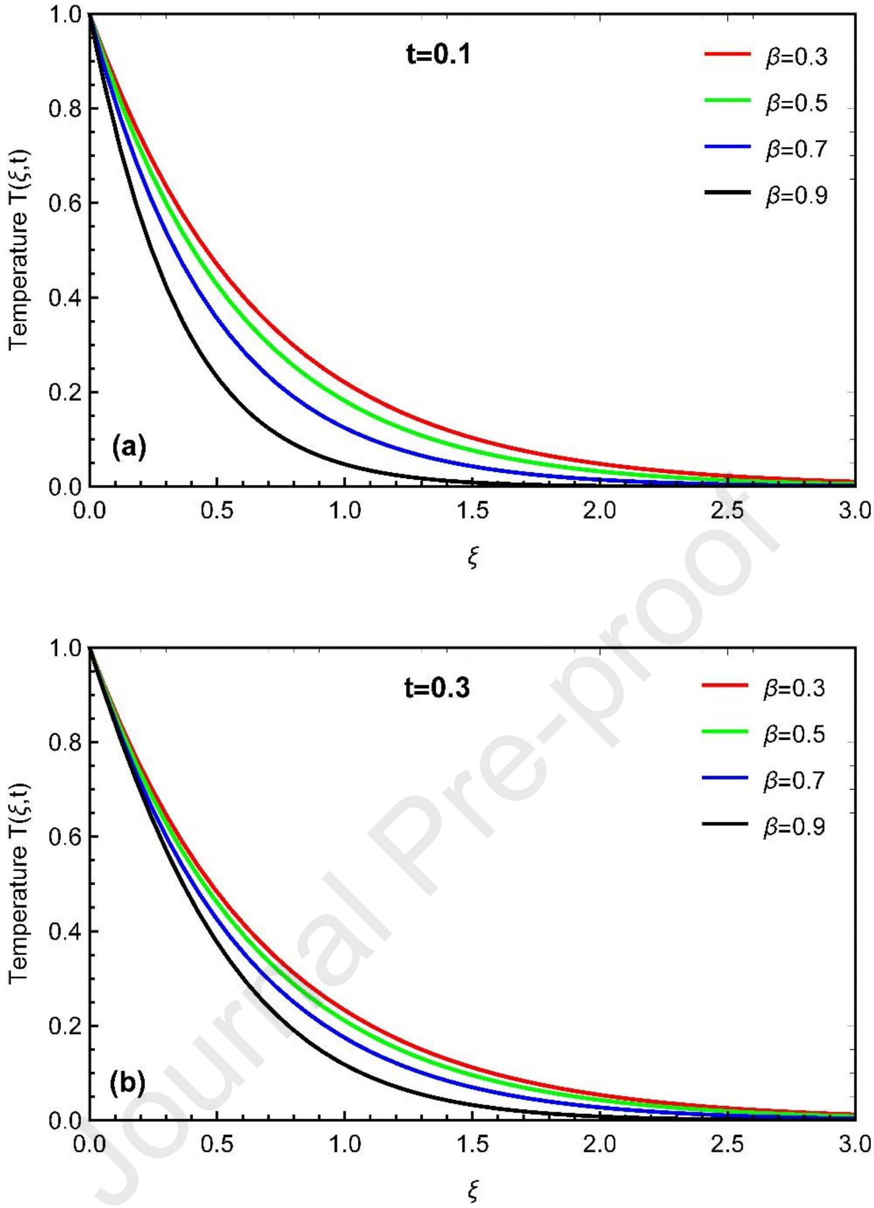


Fig 3: Variation in fractional parameter β for temperature distribution and Ag nano-particles at $t = 0.1$ and $t = 0.3$

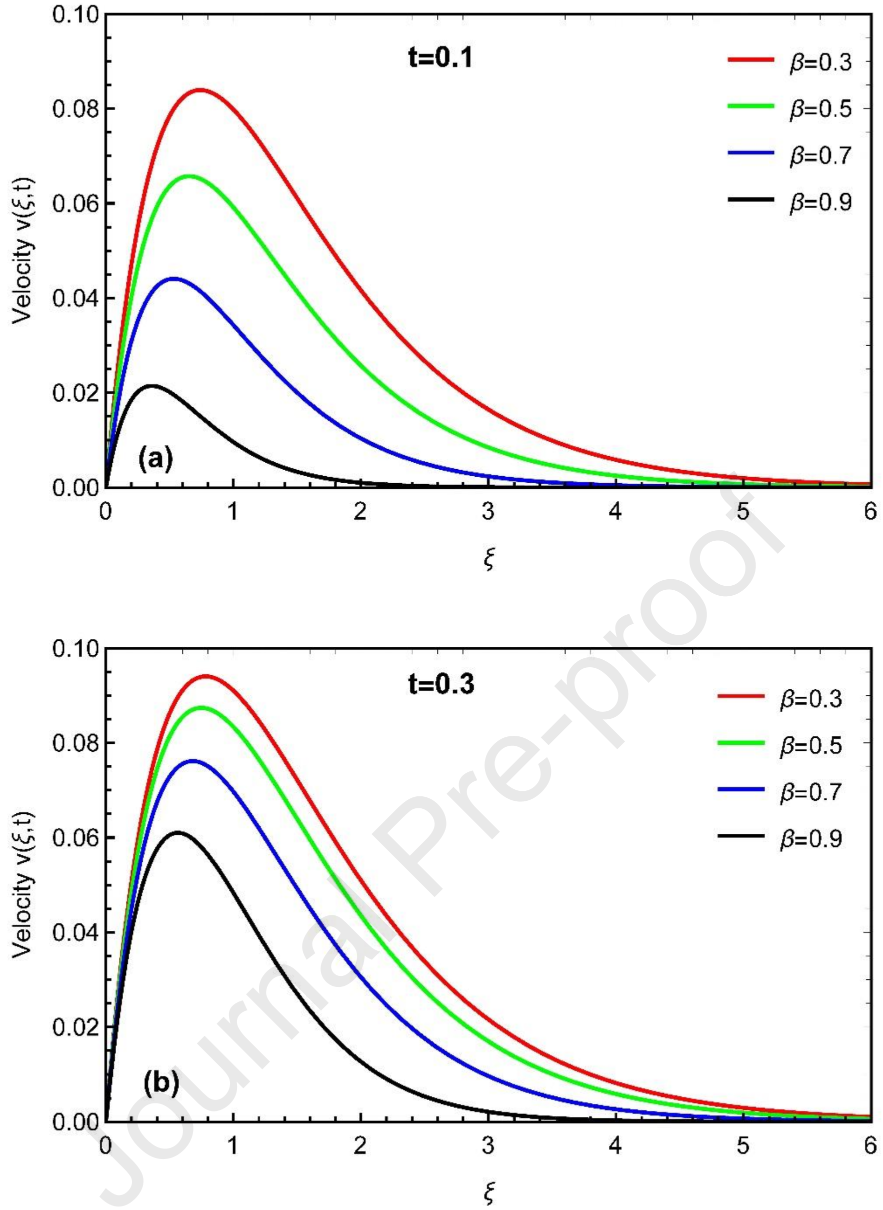


Fig 4: Variation in fractional parameter β for the velocity field and CuO nano-particles at $t = 0.1$ and $t = 0.3$

The Eq. (29) corresponds to the velocity field of fractional nanofluids is so complicated, so their Laplace inverse also difficult to solve analytically. For numerical inverse of Laplace, we have used here Zakians method w.r.t transformed variable t. Zakians method mathematically can be expressed as

$$u(\xi, t) = \frac{2}{t} \sum_{j=1}^{N} \mathrm{Re}\left(k_j \,.\, \bar{u}\left(\xi, \frac{\alpha_j}{t}\right)\right) \quad (30)$$

4. Discussion of Results

The analytical solution of free convection unsteady nanofluid flow of water-based fractional fluid on an infinite sheet with the constant temperature at boundaries is studied. The most recent definition of fractional derivatives, AB-time fractional derivative is utilized to generate the fractional modal of the under conversation problem. To dig up the analytical solution of dimensionless governing equations i.e. temperature and velocity field the Laplace transformation technique is utilized. The thermophysical properties of nanoparticles such as Silver (Ag) and Copper oxide (CuO) with water as a base fluid are shown in Tab. 1. To enhance the innovation of this work graphical and numerical results are represented in Figs 2-8 for altered values of different constraints such as fractional parameter β, and volumetric friction φ. The temperature field against the fractional parameter β is plotted in Fig 2 for the altered values of the fractional parameter and volume fraction φ at different times. By enhancing the value of the fractional parameter Fig 2 shows the decay in the temperature field of the fluid. Similarly, the effect of fractional parameter β at different values of time t for temperature field and Ag nanoparticles are plotted in Fig 3. As a result, this can be seen in Figs 2 and 3 that the temperature field decreases as the values of the fractional parameter increase due to the physical characteristics of nanoparticles. In Tab 2 the comparison of different nanoparticles that's are copper oxide and silver for changed values of fractional parameter β and ζ are represented at time $t = 0.5$ and volume fraction $\varphi = 0.2$. Furthermore, the copper oxide CuO nanoparticles show large temperature values as compared to silver Ag nanoparticles in Tab 2.

Table 2: Comparison of temperature field for both CuO and Ag nanoparticles at $t = 0.5$ and volumetric fraction $\varphi = 0.2$

ζ	CuO	Ag

	$\beta=0.3$	$\beta=0.5$	$\beta=0.7$	$\beta=0.9$	$\beta=0.3$	$\beta=0.5$	$\beta=0.7$	$\beta=0.9$
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	0.8742	0.8904	0.9136	0.9387	0.8773	0.8931	0.9158	0.9403
0.2	0.7641	0.7922	0.8330	0.8786	0.7695	0.7971	0.8371	0.8818
0.3	0.6678	0.7044	0.7581	0.8201	0.6750	0.7110	0.7639	0.8247
0.4	0.5836	0.6260	0.6887	0.7634	0.5920	0.6339	0.6958	0.7693
0.5	0.5100	0.5559	0.6245	0.7086	0.5192	0.5648	0.6328	0.7158
0.6	0.4456	0.4934	0.5655	0.6561	0.4553	0.5029	0.5745	0.6643
0.7	0.3894	0.4377	0.5111	0.6058	0.3992	0.4476	0.5209	0.6149
0.8	0.3401	0.3881	0.4613	0.5579	0.3500	0.3981	0.4715	0.5678
0.9	0.2971	0.3439	0.4158	0.5125	0.3068	0.3540	0.4262	0.5230
1.0	0.2595	0.3045	0.3741	0.4694	0.3690	0.3146	0.3848	0.4806

Table 3: Comparison of the velocity field for both CuO and Ag nanoparticles at $t=0.7$ and volumetric fraction $\varphi=0.2$ by using Zakians Method

ζ	CuO				Ag			
	$\beta=0.3$	$\beta=0.5$	$\beta=0.7$	$\beta=0.9$	$\beta=0.3$	$\beta=0.5$	$\beta=0.7$	$\beta=0.9$
0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.3	0.0749	0.0874	0.1063	0.1313	0.0463	0.0541	0.0659	0.0816
0.6	0.1052	0.1269	0.1600	0.2045	0.0634	0.0768	0.0972	0.1248
0.9	0.1107	0.1379	0.1793	0.2356	0.0650	0.0815	0.1066	0.1409
1.2	0.1037	0.1329	0.1773	0.2379	0.0593	0.0767	0.1030	0.1390
1.5	0.0910	0.1198	0.1633	0.2222	0.0508	0.0674	0.0926	0.1265
1.8	0.0768	0.1035	0.1436	0.1967	0.0416	0.0567	0.0793	0.1088
2.1	0.0629	0.0868	0.1220	0.1671	0.0332	0.0463	0.0655	0.0896

2.4	0.0505	0.0712	0.1011	0.1374	0.0259	0.0369	0.0527	0.0712
2.7	0.0400	0.0574	0.0820	0.1100	0.0199	0.0289	0.0415	0.0549
3.0	0.0313	0.0456	0.0655	0.0860	0.0150	0.0223	0.0320	0.0412

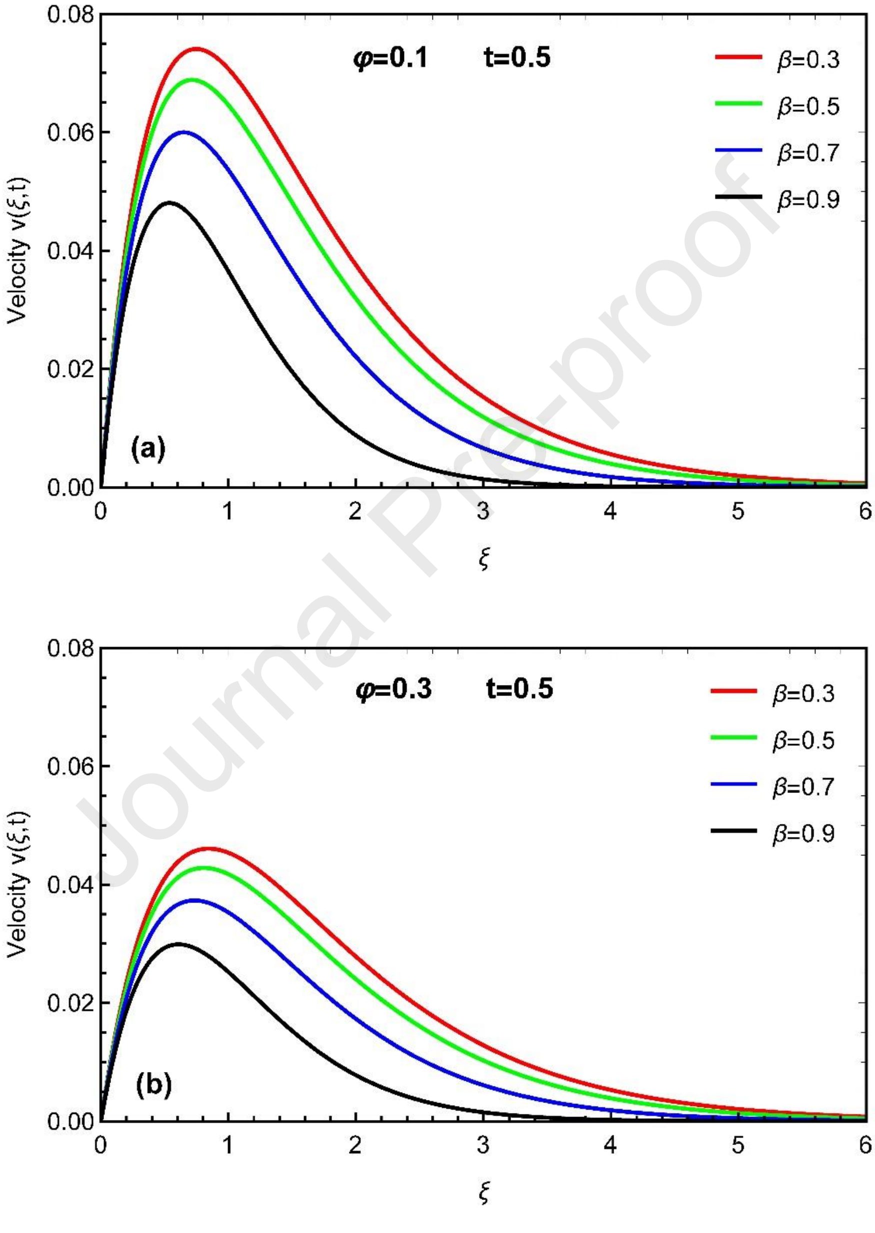


Fig 5: Variation in volume fraction φ for the velocity field and CuO nano-particles at $\varphi = 0.1$ and $\varphi = 0.3$

The effects of fractional parameters on velocity field via AB-fractional derivative at different time values for copper oxide nanoparticles are plotted in Fig 4. By increasing the value of fractional parameter β Fig 4 represents the decay in the velocity field. Physically, the enhancement in the velocity fractional parameter increases the thickness of the boundary layers which causes the reduction in the velocity of nanofluids. The effects of volume fraction φ at altered values of fractional parameter β and ζ are plotted in Fig 5 for copper oxide CuO nanoparticles in the existence of thermal radiation. As shown in the figure, the velocity field again decreases by increasing the value of volume fraction and fractional parameter and the boundary layer viscosity is smaller for ordinary nanofluids as compared to fractional nanofluid. Physically enhancement in the volume fraction increases the viscousness and thermal conduction of the fluid that's clues to slow down the fluid velocity with the enhancement in volume fraction parameter and fractional parameter β. The rate of the fractional parameter corresponds to the velocity of nanofluid with Ag nanoparticles is brought to light in Fig 6, in the existence of thermal radiation at different values of the time. This Fig also represents the decay in the velocity field by amassed the value of fractional parameter β. The variation of volume fraction φ on velocity field is represented in Fig 7 which also decays fluid velocity, similar to water-based CUo nanoparticles. The comparison for the temperature field of copper oxide and silver-based fractional nanofluid when $\beta \to 1$ and ordinary nanofluid temperature is plotted in Fig 8(a). It can be seen that, in natural convection flow, the enhancement of heat transfer is lesser in copper oxide and silver-based fractional nanofluid as compared to ordinary nanofluid temperature. Because, for fractional nanofluid, the thermal boundary layer is denser. And the temperature field shows decay by enhancing the value of the fractional parameter for both cases i.e. silver nanofluid and copper oxide nanofluid. And the comparison of the velocity field obtained by the AB-fractional derivative and the velocity field of Fetecau *et al.* [42] is highlighted in Fig 8 (b). The curves of our study result velocity overlap to the velocity field obtained by Fetecau *et al.* [42]. The overlapping of the curves represents the validation of our obtained results. Furthermore, in comparison to copper oxide and silver nanofluid, copper oxide shows greater values as compared to silver-fractional nanofluid for velocity field which can be seen in Tab 3.

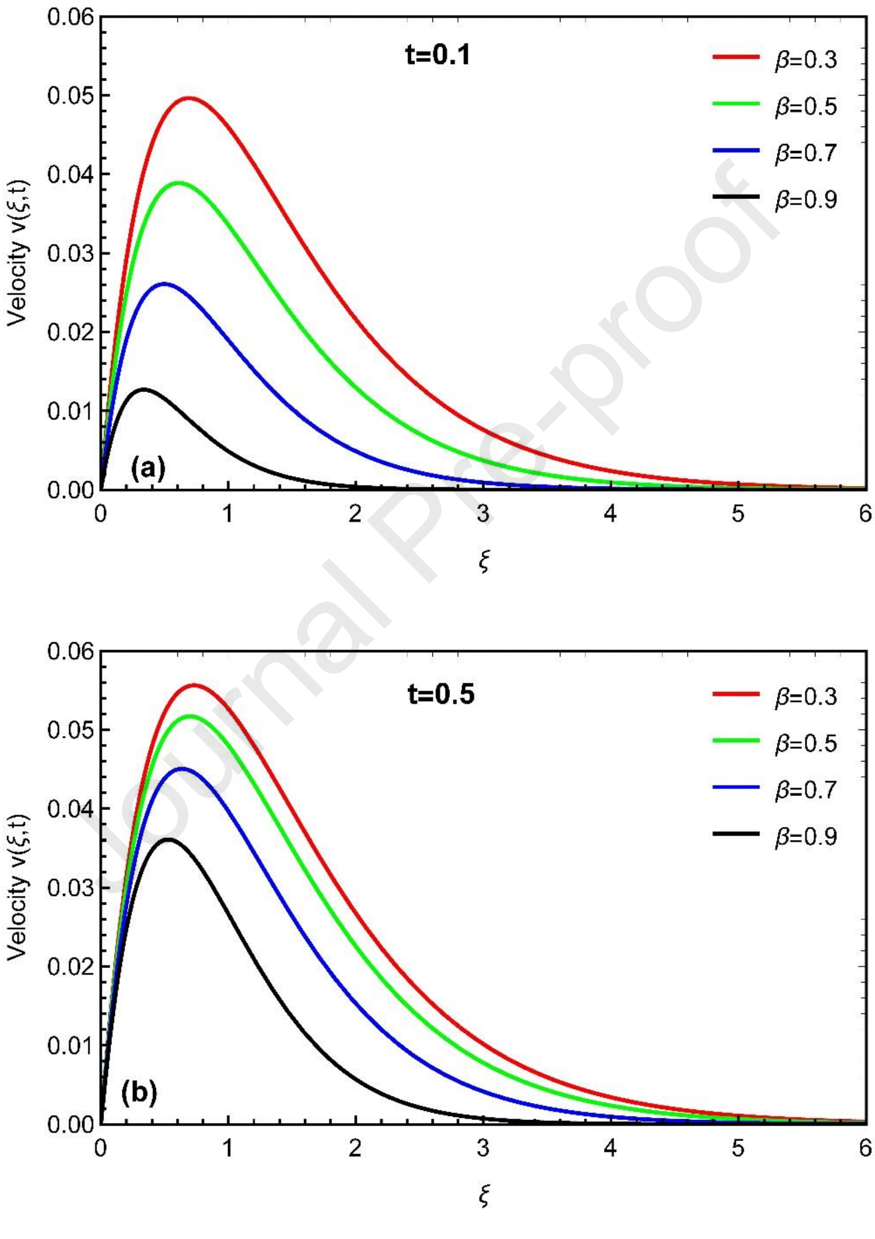
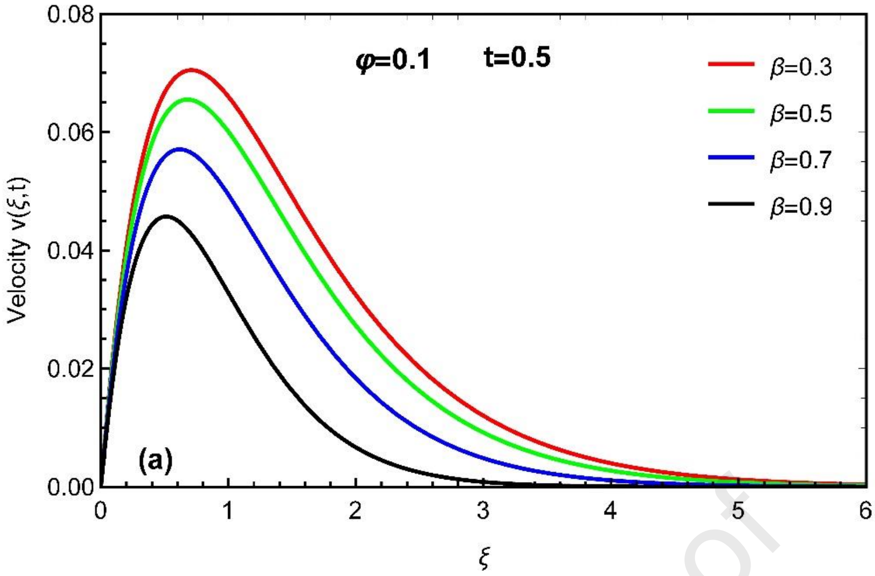


Fig 6: Variation in fractional parameter β for the velocity field of Water-Ag nanofluid at $t = 0.1$ and $t = 0.5$



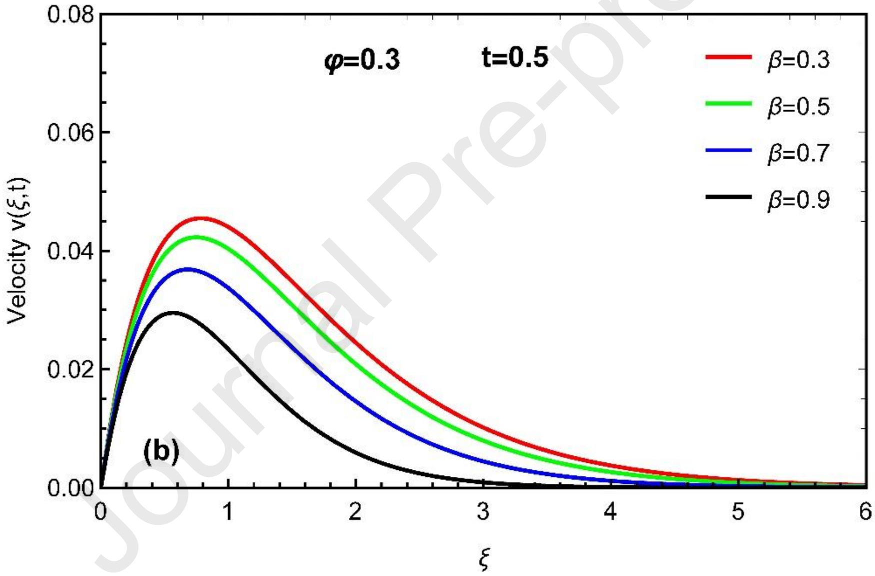


Fig 7: Variation in volume fraction φ for the velocity field and Ag nano-particles at $\varphi =$ 0.1 and $\varphi = 0.3$

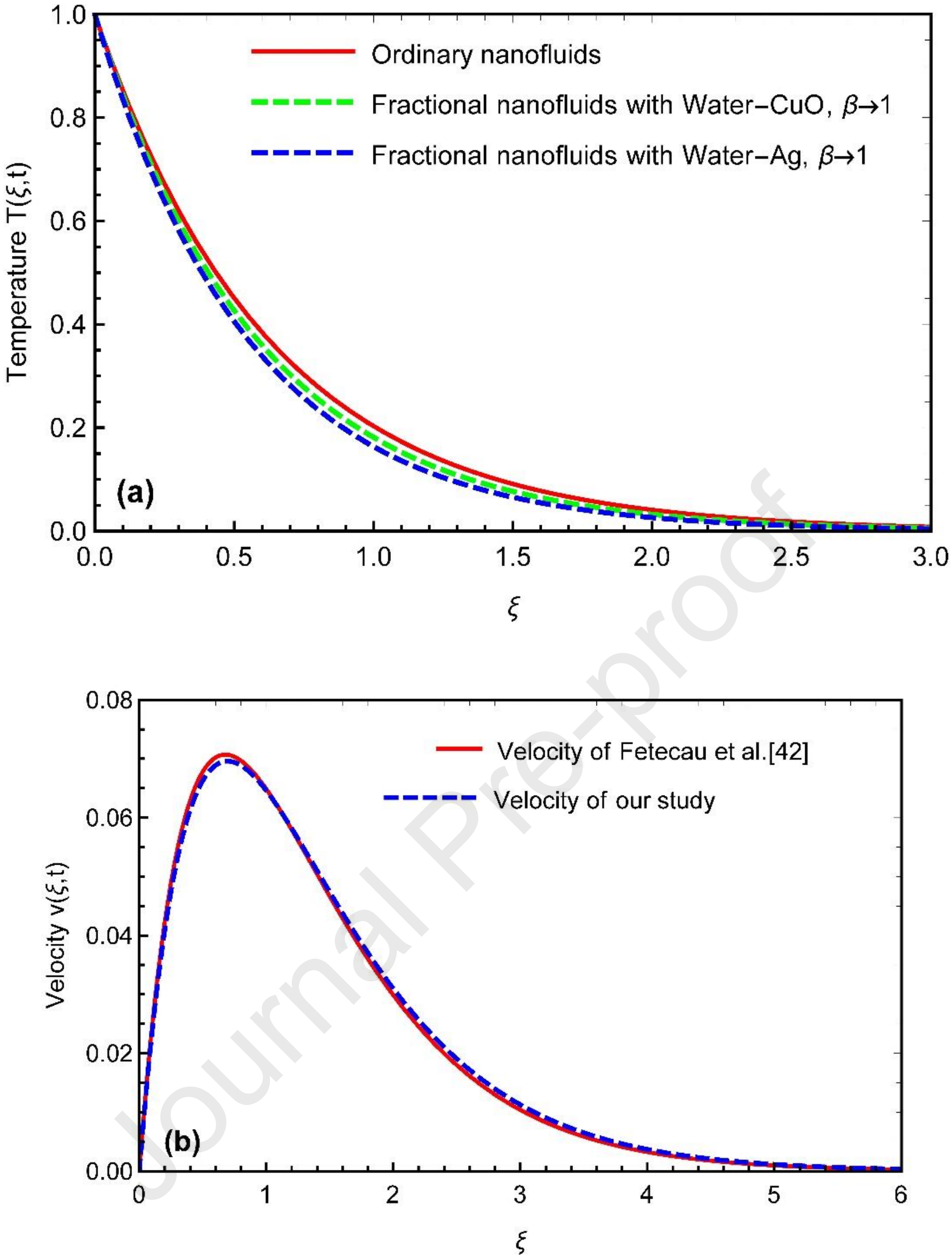


Fig 8: Comparison of **(a)**: temperature field of fractional and ordinary nanofluid **(b):** our study velocity with Fetecau *et al.* [42] solutions velocity

5. Conclusions

An analytical study of free convection nanofluid flow is studied with the most recent definition of fractional derivative i.e. Atangana-Baleanu time-fractional derivative with the uniform temperature at the boundaries of an infinite sheet. To find out the physical behavior of temperature and velocity field for altered values of different parameter fractional parameter, and volume fraction graphical and numerical representation is plotted and presented in figures and tables. The leading results concluded from the whole study of this work can be encapsulated as

- In natural convection flow, the enhancement of heat transfer is lesser in fractional nanofluid as compared to ordinary nanofluid. For fractional nanofluid, the thermal boundary layer is denser. And the temperature field shows decay by enhancing the value of the fractional parameter for both cases i.e. silver nanofluid and copper oxide nanofluid.
- The enhancement in volume fraction φ decreases the dimensionless velocity of water-based nanofluid and ordinary nanofluid.
- Copper oxide nanofluid increases the heat transfer in natural convection flow as compared to silver nanofluid flow.
- In this study, this can be concluded that the fractional derivatives have a substantially different behavior as compared to ordinary nanofluid. And on the heat transfer, the fractional parameter has a more solid impact as compared to other models.

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Conflict of Interest:

The authors declare that they don’t have any conflict of interest.