Modified single variable shear deformation plate theory for free vibration analysis of rectangular FGM plates

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Abstract

In this paper, a modified single variable shear deformation plate theory is developed for free vibration analysis of rectangular functionally graded plates. By using the equilibrium equations of forces, the relationship between bending and shear parts is established, so that the displacement fields and the governing equation of the modified single variable plate theory contain only one unknown variable. As a consequence, the number of unknown variables is reduced, so it is effective for computation of solid mechanics. The advantage of proposed theory is that it consists of only one unknown variable but it is capable for analysis of heterogeneous plate such as functionally graded plates. The numerical results of the free vibration of functionally graded plates are carried out and compared with other published data to ensure the accurateness and effectiveness of this modification. The paper also explores the impact of some parameters on the free vibration of the functionally graded plates.

Keywords: modified single variable shear deformation; vibration; equilibrium equation; FGM plates.

1. Introduction

Nowadays, functionally graded materials (FGMs) have been produced and consumed in numerous fields of engineering, for example, civil engineering, transportation as well as nuclear power plants and the use of these materials is rising rapidly. The reason is that FGMs have a lot of advantages in comparison with traditional materials. As a consequence, many scientists attended to research the mechanical behavior of FGM structures by employing many different theories. These theories were applied to study dynamic responses, static behaviors, buckling as well as post-buckling behaviors of FGM frames, shells, plates, and beams.

The first one is classical plate theory (CPT) which has been applied to study FGM structures. Because the shear stresses and strains are neglected in CPT, therefore, it can be only applied to investigate thin plates. Some worthy works have been done by using CPT can be considered by numerous scientists, for instance, Javaheri and his coworkers [1], Mohammadi and his associates [2], Ghannadpour and his colleagues [3] and Damanpack et al. [4]. In these studies, the authors applied CPT to investigate thin FGM plates. The results of these pioneering works are significantly important at the beginning of the application of FGMs in many fields of engineering as well as industries.

To overcome the disadvantage of CPT, the Reissner-Mindlin plate theory was developed to deal with moderate and thick plates, this theory has been also well-known for the first-order shear deformation theory (FSDT). In which, the shear stresses and strains are taken into account and assumed to not vary in the thickness direction. Because the shear stresses and strains in FSDT are based on that proposal, this theory needs a correction coefficient, which depends on the material, boundary conditions as well as the geometry of structures, and also it is extremely difficult to get an exact value in any cases. Croce and his associates [5] applied the Reissner-Mindlin plate theory in the mixture with the finite element method (FEM) to investigate FGM plates. In their work, they developed some new plate elements to scrutinize the FGM plates in a thermal environment. Besides, the inspiration of high temperature on the mechanical behaviors of FGM plates has been studied by Nguyen and his associates [6] by using FSDT. In this work, the material properties varied belong to the change of the temperature, thus, it led to the change of mechanical response of FGM plates. A novel model based on FSDT had been developed by Nguyen and his colleagues [7] to analyze FGM plates. Furthermore, some simple FSDTs (S-FSDTs) and its variants have been developed by Shimpi and his collaborators [8], Thai and his associates [9] to research FGM plates. Nguyen and his co-workers [10] developed a refined simple FSDT to study advanced composite plates. In this work, by using an assumption of shear stresses with a distributed shape function, it did not need any correction coefficients and the plate was free of shear stresses at its two surfaces. Yu and his coworkers [11,12] used a combination of the FSDT and isogeometric analysis (IGA) to research linear and nonlinear behaviors of FGM plates. Tan and his associates [13] employed a combination of the FSDT and meshless methods to analyze FGM plates. It can conclude that, although the FSDT can be applied to study moderate and thick plates, and it still needs a correction factor, this is an enormous drawback of FSDT.

To address the inconvenience of FSDT, many higher-order shear deformation plate theories (HSDTs) have been developed by scientists all over the world. One worthy example is the HSDT which was developed by Reedy [14] to investigate moderate and thick FGM plates. The HSDT does not need any correction factors and shear stresses, as well as shear strains are zeros at the upper and lower surfaces. After the success of Reddy's theory, numerous noteworthy plate theories have been established by many scientists, and these theories have been employed to study FGM plates. Some published works can be counted as follows, Javaheri et al. [15], Ferreira and his collaborators [16] as well as Talha and his co-workers [17] proposed some novel HSDTs to scrutinize thick FGM plates. Tran et al. [18], Thai and his colleagues [19] used HSDT in combination with IGA to study FGM plates. Zenkour [20,21] developed some other shear deformation theories such as generalized shear deformation theory (GSDT), HSDT with trigonometric function as well as 3-D elasticity theory to research thick FGM plates. Bui and his coworkers [22] investigated the mechanical behaviors of FGM plates with the temperature-dependent material properties. Do and his colleagues [23] presented another extended work, in which the authors used HSDT in cooperating with FEM to study bi-directional FGM plates. Mantari and his colleagues [24-26] developed some innovative HSDTs to analyze FGM plates and shells. Besides, some worthy refined plate theories (RPT) which consist of four unknown variables have been proposed by Mechab and his collaborators [27], Benachour et al. [28] to examine FGM plates. To reduce the number of unknown variables, some RPTs were proposed by Shimpi [29-31], Thai and his colleagues [32] as well as Mechab and his coworkers [33]. In these theories, there are only two unknown variables in its displacement formulation. It is obvious that HSDTs are more efficient and precise than CPT, FSDT and they do not need any shear correction factors.

Not to mention, numerous works have been accomplished to study the vibration of FGM plates. Hosseini-Hashemi and his collaborators [34] researched the vibration of rectangular plates which are made of FGM, in which the authors used FSDT. In the other work achieved by Hosseini-Hashemi [35], he suggested a new exact analytical approach which was constructed with the support of the Reissner-Mindlin plate theory. This method has been employed to investigate the vibration of moderate FGM plates. Chakraverty and his colleagues [36] studied the effects of the Winkler foundation on the vibration of FGM plates. In his work, the Rayleigh-Ritz method has been applied and boundary conditions of the plates were arbitrary. An extended Kantorovich method has been developed by Fallah and his teammates [37] to research the free vibration of FGM plate with moderate thickness. Zhao and his colleagues [38] scrutinized the free vibration of FGM plates. The procedure in this work is the kp-Ritz method. Yang and his colleagues [39] studied vibration appearances and momentary behavior of advanced material plates. Besides, they also studied carefully the influence of thermal environments on the behavior of the panels. An investigation of nonlinear vibration and dynamic behavior of FGM plates embedded in a high-temperature environment has been presented by Huang and his colleagues [40]. Kim [41] studied FGM plates with the vibration problem, and he also studied the impact of heat on its behavior. A large number of worthy studies were also completed by many scientists such as Thai and his colleagues [42, 43], Meiche and his coworkers [44], Akavci [45], Matsunaga [46] as well as Mantari and his associates [47], in which, many precious results and parameter studies were accomplished. Neves and his coworkers [48, 49] used some different quasi-3D theories to study FGM plates. Kumar and his colleagues [50] utilized an accurate method to study FGM plates with two different kinds of varying material properties, which are sigmoid and exponential FGM plates. In this work, Kumar and his associates applied the dynamic stiffness method to research FGM plates. Since the free vibration of the FGM plate is an important problem in many fields of engineering, more studies are still required.

Although there are numerous works about FGM plates, there is no work using single variable plate theory to analyze the plate with a continuous varying of material properties such as FGM plates. This paper aims to modify single variable plate theory of Shimpi and apply this modification to analyze free vibration of FGM plates. This modified single variable plate theory is very simple, well-organized and accurate. The efficiency and accuracy of the proposed plate theory will be demonstrated via verification study, and then it will be applied to investigate the free vibration of the FGM plates. Besides, the authors also examine the effects of some parameters such as material properties and geometry on the free vibration of the FGM plates to show more valuable details.

2. Material properties of FGM plates

In this study, an FGM plate as shown in Fig. 1 is examined.

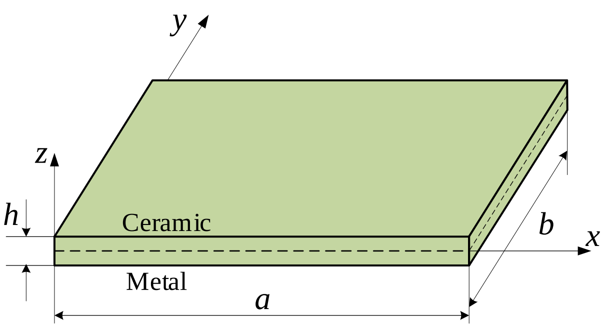


Figure 1. An FGM plate model

The following formulas are used to calculate the volume fraction of two ingredients

$$V_c = \left(\frac{1}{2}+\frac{z}{h}\right)^p, V_m = 1-\left(\frac{1}{2}+\frac{z}{h}\right)^p \tag{1}$$

in which p is a factor which causes the variation of volume fraction and h is the thickness.

To compute the material properties at any point of the plate, the following formula is used

$$P(z) = P_m V_m + P_c V_c \quad (2)$$

where P_c, P_m denote ceramic and metal properties such as Young's modulus, mass density and Poisson's ratio, respectively.

3. Formulation of modified single variable shear deformation plate theory

3.1. Assumption of single variable shear deformation plate theory

In this subsection, a short review of Shimpi's plate theory is given [29-31]. The basic notion of RPT is shown as follows

1) The displacement w is separated into two parts, the first part is bending part w_b and the second part is shear part w_s. As a consequence, the transverse displacement is

$$w = w_b + w_s \quad (3)$$

2) The normal stress σ_z is very small in contrast with σ_x, σ_y, so the normal stress σ_z is neglected. Therefore, by applying the Hook's law, the relationship between σ_x, σ_y and $\varepsilon_x, \varepsilon_y$ is written as

$$\begin{aligned} \sigma_x &= \frac{E(z)}{1-\nu(z)^2}\left[\varepsilon_x + \nu(z)\varepsilon_y\right] \\ \sigma_y &= \frac{E(z)}{1-\nu(z)^2}\left[\varepsilon_y + \nu(z)\varepsilon_x\right] \end{aligned} \quad (4)$$

3) The translations in the $x-$axis and $y-$axis is also divided into two parts, which are bending part and shear part

$$\begin{aligned} u &= u_b + u_s \\ v &= v_b + v_s \end{aligned} \quad (5)$$

The first parts u_b as well as v_b are analogous to the displacements which are given by the CPT. They relate to the moments M_x, M_y and M_{xy}, but do not relate to shear forces Q_x, Q_y. So, they are calculated by

$$u_b = -z\frac{\partial w_b}{\partial x}, v_b = -z\frac{\partial w_b}{\partial y} \tag{6}$$

The second parts u_s and v_s give rise to shear strain γ_{xz}, γ_{yz} and therefore to the transverse shear stress τ_{xz}, τ_{yz} which have a nonlinear distribution through the thickness and they equal to zero at the upper and lower surfaces. The shear parts u_s, v_s are caused by shear forces Q_x, Q_y and they are calculated via the angle between the tangent to the cubic fiber at the reference surface $(z=0)$ and the normal sections $\partial w_s/\partial x$, $\partial w_s/\partial y$. As a result, the formulas for u_s and v_s are obtained by

$$u_s = f(z)\frac{\partial w_s}{\partial x}, v_s = f(z)\frac{\partial w_s}{\partial y} \tag{7}$$

In Eq. (7), $f(z)$ stands for the shear distributed shape function, its derivative describes the distribution of τ_{xz}, τ_{yz} through the thickness of the plates. It is noticed that τ_{xz}, τ_{yz} should be free at the upper surface as well as the lower surface. In this study, a new hybrid shear distributed shape function $f(z)$ is introduced as

$$f(z) = \left(\frac{127z^4}{200h^3} - \frac{77z^2}{125h} + \frac{39h}{500}\right)\sinh\left(\frac{\pi z}{h}\right) \tag{8}$$

It is noticed that u_s and v_s do not provide to the moments M_x, M_y as well as M_{xy}.

4) It is assumed that only bending deflection produces the rotation of the plate cross-section and inertia loading and distributed inertia moments as

$$\bar{q} = -\bar{m}\frac{\partial^2 w}{\partial t^2}, m_x = -\bar{J}\frac{\partial^3 w_b}{\partial x \partial t^2}, m_y = -\bar{J}\frac{\partial^3 w_b}{\partial y \partial t^2} \tag{9}$$

where $\bar{m}, \bar{J}$ are calculated by

$$\begin{aligned} \bar{m} &= \int_{-h/2}^{h/2} \rho(z)dz, \\ \bar{J} &= \int_{-h/2}^{h/2} \rho(z)z^2 dz \end{aligned} \tag{10}$$

3.2. Formulation of modified single variable plate theory

The formulation of two-variable plate theory of Shimpi [29-31] is specified as starting point for its modification. By applying the assumptions which are deliberated in the earlier subsection, the expressions of the displacement are

$$\begin{aligned} u &= -z\frac{\partial w_b}{\partial x} + f(z)\frac{\partial w_s}{\partial x} \\ v &= -z\frac{\partial w_b}{\partial y} + f(z)\frac{\partial w_s}{\partial y} \\ w &= w_b + w_s \end{aligned} \tag{11}$$

The formulas of the strain fields are obtained as

$$\begin{aligned} \varepsilon_x &= -z\frac{\partial^2 w_b}{\partial x^2} + f(z)\frac{\partial^2 w_s}{\partial x^2} \\ \varepsilon_y &= -z\frac{\partial^2 w_b}{\partial y^2} + f(z)\frac{\partial^2 w_s}{\partial y^2} \\ \gamma_{xy} &= -z\left(2\frac{\partial^2 w_b}{\partial x \partial y}\right) + f(z)\left(2\frac{\partial^2 w_s}{\partial x \partial y}\right) \\ \gamma_{xz} &= g(z)\frac{\partial w_s}{\partial x} \\ \gamma_{yz} &= g(z)\frac{\partial w_s}{\partial y} \end{aligned} \tag{12}$$

where $g(z) = 1 + f'(z)$.

The formulas of normal stresses σ_x and σ_y are obtained via Eq. (4) and Eq. (12), while the formulas of shear stresses τ_{xy}, τ_{xz} and τ_{yz} are calculated by using Eq. (12) and following constitutive equations

$$\begin{aligned} \tau_{xy} &= G(z)\gamma_{xy} \\ \tau_{xz} &= G(z)\gamma_{xz} \\ \tau_{yz} &= G(z)\gamma_{yz} \end{aligned} \tag{13}$$

where $G(z) = E(z)/\left[2(1+\nu(z))\right]$.

The normal and shear stresses are calculated as the following formulas

$$\sigma_x = \frac{-zE(z)}{1-\nu(z)^2}\left(\frac{\partial^2 w_b}{\partial x^2} + \nu(z)\frac{\partial^2 w_b}{\partial y^2}\right) + \frac{f(z)E(z)}{1-\nu^2}\left(\frac{\partial^2 w_s}{\partial x^2} + \nu(z)\frac{\partial^2 w_s}{\partial y^2}\right)$$
$$\sigma_y = \frac{-zE(z)}{1-\nu(z)^2}\left(\frac{\partial^2 w_b}{\partial y^2} + \nu(z)\frac{\partial^2 w_b}{\partial x^2}\right) + \frac{f(z)E(z)}{1-\nu^2}\left(\frac{\partial^2 w_s}{\partial y^2} + \nu(z)\frac{\partial^2 w_s}{\partial x^2}\right) \quad (14)$$
$$\tau_{xy} = \frac{-zE(z)}{1-\nu(z)^2}(1-\nu(z))\frac{\partial^2 w_b}{\partial x \partial y} + \frac{f(z)E(z)}{1-\nu(z)^2}(1-\nu(z))\frac{\partial^2 w_s}{\partial x \partial y}$$

$$\tau_{yz} = \frac{g(z)E(z)}{2(1+\nu(z))}\frac{\partial w_s}{\partial y}$$
$$\tau_{xz} = \frac{g(z)E(z)}{2(1+\nu(z))}\frac{\partial w_s}{\partial x} \quad (15)$$

The moments and shear forces are obtained as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (16)$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \quad (17)$$

Based on third assumption of the proposed theory in section 3.1 and Eq. (4), the moments and shear forces are calculated as following formulas

$$M_x = \int_{-h/2}^{h/2} \frac{-z^2 E(z)}{1-\nu(z)^2}\left[\frac{\partial^2 w_b}{\partial x^2} + \nu(z)\frac{\partial^2 w_b}{\partial y^2}\right] dz$$
$$M_y = \int_{-h/2}^{h/2} \frac{-z^2 E(z)}{1-\nu(z)^2}\left[\frac{\partial^2 w_b}{\partial y^2} + \nu(z)\frac{\partial^2 w_b}{\partial x^2}\right] dz \quad (18)$$
$$M_{xy} = \int_{-h/2}^{h/2} \frac{-z^2 E(z)}{(1-\nu(z)^2)}\left[(1-\nu(z))\frac{\partial^2 w_b}{\partial x \partial y}\right] dz$$

$$Q_x = \int_{-h/2}^{h/2} \frac{g(z)E(z)}{2(1+\nu(z))}\frac{\partial w_s}{\partial x} dz$$
$$Q_y = \int_{-h/2}^{h/2} \frac{g(z)E(z)}{2(1+\nu(z))}\frac{\partial w_s}{\partial y} dz \quad (19)$$

After integrating Eq. (18) and Eq. (19) through the depth h, the moments and shear forces are expressed as following formulas

$$M_x = -\alpha \frac{\partial^2 w_b}{\partial x^2} - \alpha_1 \frac{\partial^2 w_b}{\partial y^2}$$
$$M_y = -\alpha \frac{\partial^2 w_b}{\partial y^2} - \alpha_1 \frac{\partial^2 w_b}{\partial x^2} \quad (20)$$
$$M_{xy} = -\alpha \frac{\partial^2 w_b}{\partial x \partial y} + \alpha_1 \frac{\partial^2 w_b}{\partial x \partial y}$$

$$Q_x = \beta \frac{\partial w_s}{\partial x}$$
$$Q_y = \beta \frac{\partial w_s}{\partial y} \quad (21)$$

In which, the coefficients α, α_1, β are calculated as the following integral expression

$$\alpha = \int_{-h/2}^{h/2} \frac{z^2 E(z)}{1-\nu(z)^2} dz \quad (22)$$

$$\alpha_1 = \int_{-h/2}^{h/2} \frac{z^2 \nu(z) E(z)}{1-\nu(z)^2} dz \quad (23)$$

$$\beta = \int_{-h/2}^{h/2} \frac{g(z) E(z)}{2(1+\nu(z))} dz \quad (24)$$

It is clear that if ν is constant through the z-direction, we then have $\alpha_1 = \nu\alpha$. The expressions for moments do not include w_s and the expressions for shear forces do not include w_b.

The equilibrium equations of the force can be written as

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = m_x$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = m_y \quad (25)$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\bar{q}$$

Substituting Eq. (20) and Eq. (21) into Eq. (25), one gets

$$-\frac{\partial}{\partial x}\left[\alpha\frac{\partial^2 w_b}{\partial x^2}+\alpha_1\frac{\partial^2 w_b}{\partial y^2}\right]-\frac{\partial}{\partial y}\left[(\alpha-\alpha_1)\frac{\partial^2 w_b}{\partial x\partial y}\right]-\beta\frac{\partial w_s}{\partial x}=-\bar{J}\frac{\partial^3 w_b}{\partial x\partial t^2}$$

$$-\frac{\partial}{\partial x}\left[(\alpha-\alpha_1)\frac{\partial^2 w_b}{\partial x\partial y}\right]-\frac{\partial}{\partial y}\left[\alpha\frac{\partial^2 w_b}{\partial y^2}+\alpha_1\frac{\partial^2 w_b}{\partial x^2}\right]-\beta\frac{\partial w_s}{\partial y}=-\bar{J}\frac{\partial^3 w_b}{\partial y\partial t^2} \quad (26)$$

$$\beta\frac{\partial^2 w_s}{\partial x^2}+\beta\frac{\partial^2 w_s}{\partial y^2}=\bar{m}\frac{\partial^2 w}{\partial t^2}$$

First two equations of Eq. (26) become

$$\frac{\partial w_s}{\partial x}=-\frac{\alpha}{\beta}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)+\frac{\bar{J}}{\beta}\frac{\partial^3 w_b}{\partial x\partial t^2}$$

$$\frac{\partial w_s}{\partial y}=-\frac{\alpha}{\beta}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)+\frac{\bar{J}}{\beta}\frac{\partial^3 w_b}{\partial y\partial t^2} \quad (27)$$

Or

$$w_s=-\frac{\alpha}{\beta}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)+\frac{\bar{J}}{\beta}\frac{\partial^2 w_b}{\partial t^2}=\chi\left(\nabla w_b\right)+\frac{\bar{J}}{\beta}\frac{\partial^2 w_b}{\partial t^2} \quad (28)$$

In which $\chi=-\frac{\alpha}{\beta}$ and ∇ is Laplace operator $\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

By means of introducing Eq. (28) into Eq.(11), new displacement formulas of modified single variable plate theory are

$$u=-z\frac{\partial w_b}{\partial x}+f(z)\frac{\partial}{\partial x}\left[\chi\left(\nabla w_b\right)+\frac{\bar{J}}{\beta}\frac{\partial^2 w_b}{\partial t^2}\right]$$

$$v=-z\frac{\partial w_b}{\partial y}+f(z)\frac{\partial}{\partial y}\left[\chi\left(\nabla w_b\right)+\frac{\bar{J}}{\beta}\frac{\partial^2 w_b}{\partial t^2}\right] \quad (29)$$

$$w=w_b+\chi\left(\nabla w_b\right)+\frac{\bar{J}}{\beta}\frac{\partial^2 w_b}{\partial t^2}$$

It can see clearly that the formulas of displacement of the modified single variable plate theory consist of only one unknown variable, the bending component w_b. The coefficients α,β and χ depend on the thickness, material properties and the shape functions. This is a significantly different point of the proposed plate theory in comparison with the single variable plate theory of Shimpi. In Shimpi's plate theory, the substantial properties are constants, so α,β and χ are easily integrated through the plate thickness, so they are explicit expressions of the thickness of the plate. As a consequence, Shimpi's plate theory is only applied to analyze homogeneous plates, while this modified single variable plate theory can be employed to scrutinize heterogeneous plates.

3.3. Governing equations

By introducing Eq. (29) into Eq. (20) and Eq. (21), the expressions for moments as well as shear forces of the proposed theory are obtained as the following formulas

$$\begin{aligned} M_x &= -\alpha \frac{\partial^2 w_b}{\partial x^2} - \alpha_1 \frac{\partial^2 w_b}{\partial y^2} \\ M_y &= -\alpha \frac{\partial^2 w_b}{\partial y^2} - \alpha_1 \frac{\partial^2 w_b}{\partial x^2} \\ M_{xy} &= -\alpha \frac{\partial^2 w_b}{\partial x \partial y} + \alpha_1 \frac{\partial^2 w_b}{\partial x \partial y} \end{aligned} \tag{30}$$

$$\begin{aligned} Q_x &= \beta \frac{\partial}{\partial x}\left[\chi\left(\nabla w_b\right)\right] + \bar{J} \frac{\partial^3 w_b}{\partial x \partial t^2} \\ Q_y &= \beta \frac{\partial}{\partial y}\left[\chi\left(\nabla w_b\right)\right] + \bar{J} \frac{\partial^3 w_b}{\partial y \partial t^2} \end{aligned} \tag{31}$$

Substituting Eq. (30) and Eq. (31) into Eq. (25), the governing differential equation of the FGM plate is presented as

$$\beta \frac{\partial^2}{\partial x^2}\left[\chi\left(\nabla w_b\right)\right] + \bar{J} \frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \beta \frac{\partial^2}{\partial y^2}\left[\chi\left(\nabla w_b\right)\right] + \bar{J} \frac{\partial^4 w_b}{\partial y^2 \partial t^2} = -\bar{q} \tag{32}$$

After some efforts, the governing differential equation is

$$\alpha\left(\nabla\nabla w_b\right) - \left(\bar{J} - \chi \bar{m}\right) \frac{\partial^2}{\partial t^2}\left(\nabla w_b\right) + \bar{m} \frac{\partial^2 w_b}{\partial t^2} + \frac{\bar{J}\bar{m}}{\beta} \frac{\partial^4 w_b}{\partial t^4} = 0 \tag{33}$$

It is true that the governing differential equation of the FGM plate consists of only one unknown variable which is the bending component w_b (3 unknown functions if including in-plane displacements of the neutral plane). On the other hand, the coefficients $\alpha, \beta, \chi, \bar{m}$ as well as $\bar{J}$ depend on the plate thickness, the substantial properties as well as the shape function $f(z)$. It is a drastically different point of this modified single variable plate theory in comparison with Shimpi's theory. As a consequence, only this modified single variable plate theory can be applied to analyze heterogeneous plates such as FGM plates while Shimpi's theory can only be employed to research the isotropic homogeneous plates. A comparison between some different plate theories has been disclosed in Table 1.

Table 1. Comparison of some plate theories

Model	Theory	Number of unknown	Shear stress	Analysis FGM plates
CPT	Classical plate theory	3	No	Yes
FSDT	First shear deformation theory (Nguyen [6])	5	Constant	Yes
S-FSDT	Simple First shear deformation theory (Thai [9])	4	Constant	Yes
HSDT	Higher-order shear deformation theory (Reddy [14])	5	Parabolic	Yes
RPT-Variant I	Variant of Refined plate theory - I (Shimpi [29])	4	Parabolic	Yes
RPT-Variant II	Variant of Refined plate theory - II (Shimpi [29])	3	Parabolic	No
Present	Modified single variable shear deformation theory	3	Parabolic	Yes

4. Analytical solutions

In the current work, the Navier's solution is applied to analyze the free vibration of a rectangular FGM plate. Boundary conditions of the plate are simply supported at four edges. In the Navier's procedure, the solution of the displacement is implicit as the following formula

$$w_b(x, y) = \sum_{k=1}^{\infty}\sum_{r=1}^{\infty} W_{bkr} \sin \varphi x \sin \psi y \sin \omega_{kr} t \tag{34}$$

where $\varphi = k\pi/a$, $\psi = r\pi/b$; W_{bkr} are the quantities to be determined and ω_{kr} are the frequencies of the free vibration. By introducing Eq. (29) into Eq. (28), the following biquadratic frequency equation is achieved

$$\bar{J}\bar{m}\omega_{kr}^4 + \left[\left(-\bar{J}\varphi^2 - J\psi^2 - \bar{m}\right)\beta - \alpha\bar{m}\left(\varphi^2 + \psi^2\right)\right]\omega_{kr}^2 + \alpha\beta\left(\varphi^2 + \psi^2\right)^2 = 0 \tag{35}$$

This is a biquadratic algebraic equation which has four roots, the lower positive root is related to the bending mode and the higher positive root is related to the shear mode.

5. Illustrative examples and discussion

In this section, some examples will be performed to exhibit the accurateness and effectiveness of the modified single variable plate theory for free vibration analysis of FGM plates. Furthermore, some investigations about the effects of some parameters on free vibration of the FGM plate are also carried out cautiously.

5.1. Verification study

Example 1. In this first example, the free vibration of an Al/Al_2O_3 square plate is investigated in some cases of thin and thick plates. The material properties of Al are $E_m = 70$ GPa and $\rho_m = 2702\ \text{kg/m}^3$, the material properties of Al_2O_3 are $E_c = 380$ GPa and $\rho_c = 3800\ \text{kg/m}^3$. It can be assumed that the Poisson's ratio is constant and equal to 0.3. The non-dimensional frequencies in this example are calculated by $\hat{\omega} = \omega h\sqrt{\rho_c/E_c}$. The comparison for the first two non-dimensional frequencies is indicated in Table 2. In this table, the numerical solutions which are calculated by using the modified single variable plate theory are compared with those of Thai and his colleagues [9] and Matsunaga and his colleagues [46]. Wherein, the results of Matsunaga and his colleagues were calculated by applying a Quasi-3D theory while the numerical solutions of Thai and his colleagues were computed using S-FSDT. According to Table 2, it can conclude that the numerical solutions of modified single variable plate theory are very analogous with other available data.

Table 2. The first two non-dimensional frequencies $\hat{\omega}$ of the Al/Al_2O_3 square plate

Mode	a/h	Method	p				
			0	0.5	1	4	10
1	2	Quasi-3D [46]	0.9400	0.8233	0.7477	0.5997	0.5460
		S-FSDT [9]	0.9265	0.8062	0.7333	0.6116	0.5644
		Present	0.9264	0.8166	0.7469	0.6178	0.5549
	5	Quasi-3D [46]	0.2121	0.1819	0.1640	0.1383	0.1306
		S-FSDT [9]	0.2112	0.1805	0.1631	0.1397	0.1324
		Present	0.2112	0.1812	0.1641	0.1401	0.1316
	10	Quasi-3D [46]	0.0578	0.0492	0.0443	0.0381	0.0364
		S-FSDT [9]	0.0577	0.0490	0.0442	0.0382	0.0366
		Present	0.0577	0.0491	0.0443	0.0383	0.0365
2	2	Quasi-3D [46]	1.7406	1.5425	1.4078	1.1040	0.9847
		S-FSDT [9]	1.7045	1.4991	1.3706	1.1285	1.0254
		Present	1.7041	1.5264	1.4067	1.1444	1.0022
	5	Quasi-3D [46]	0.4658	0.4040	0.3644	0.3000	0.2790

	S-FSDT [9]	0.4618	0.3978	0.3604	0.3049	0.2856
	Present	0.4618	0.4009	0.3644	0.3068	0.2825
10	Quasi-3D [46]	0.1381	0.1180	0.1063	0.0905	0.0859
	S-FSDT [9]	0.1376	0.1173	0.1059	0.0911	0.0867
	Present	0.1376	0.1176	0.1063	0.0913	0.0863

Example 2. Continuously, another comparison for the first four non-dimensional frequencies of a rectangular FGM plate is explored. Once again, the FGM plate is made of Al/Al_2O_3, these substances have similar substantial properties as in the previous example. The first four non-dimensional frequencies of the plate are computed by $\bar{\omega} = \omega\sqrt{a^4\rho_c/h^2E_c}$. These numerical results of dimensionless frequencies using the modified single variable plate theory are compared with those of other authors. The results of Hosseini-Hashenmi [35] were calculated using FSDT, the numerical results of Reddy [14] were computed using third-order shear deformation theory (TSDT) while the numerical results of Thai et al. [43] were calculated using sinusoidal shear deformation theory (SSDT). The comparison is indicated in Table 3, the numerical calculations of the current modified single variable plate theory are very similar to other advertised solutions.

Table 3.The first four non-dimensional frequencies $\bar{\omega}$ of rectangular plate $(b/a = 2)$

a/h	Mode (m, n)	Method	*p*						
			0	0.5	1	2	5	8	10
5	1 (1,1)	FSDT [35]	3.4409	2.9322	2.6473	2.4017	2.2528	2.1985	2.1677
		TSDT [14]	3.4412	2.9347	2.6475	2.3949	2.2272	2.1697	2.1407
		SSDT [43]	3.4416	2.9350	2.6478	2.3948	2.2260	2.1688	2.1403
		Present	3.4408	2.9405	2.6578	2.4118	2.2544	2.1922	2.1581
	2 (1,2)	FSDT [35]	5.2802	4.5122	4.0773	3.6953	3.4492	3.3587	3.3094
		TSDT [14]	5.2813	4.518	4.0781	3.6805	3.3938	3.2964	3.2514
		SSDT [43]	5.2822	4.5187	4.0787	3.6804	3.3914	3.2947	3.2506
		Present	5.2799	4.5309	4.1012	3.7182	3.4532	3.3451	3.2886
	3 (1,3)	FSDT [35]	8.0710	6.9231	6.2636	5.6695	5.2579	5.1045	5.0253
		TSDT [14]	8.0749	6.9366	6.2663	5.6390	5.1425	4.9758	4.9055
		SSDT [43]	8.0772	6.9384	6.2678	5.6391	5.1378	4.9727	4.9044
		Present	8.0704	6.9645	6.3164	5.7203	5.2670	5.0762	4.9818
	4 (2,1)	FSDT [35]	9.7416	8.6926	7.8711	7.1189	6.5749	5.9062	5.7518
		TSDT [14]	10.1164	8.7138	7.8762	7.0751	6.4074	6.1846	6.0954

		SSDT [43]	10.1201	8.7167	7.8787	7.0756	6.4010	6.1806	6.0942
		Present	10.1080	8.7549	7.9509	7.1957	6.5889	6.3295	6.2047
10	1 (1,1)	FSDT [35]	3.6518	3.0983	2.7937	2.5386	2.3998	2.3504	2.3197
		TSDT [14]	3.6518	3.0990	2.7937	2.5364	2.3916	2.3411	2.3110
		SSDT [43]	3.6519	3.0991	2.7937	2.5364	2.3912	2.3408	2.3108
		Present	3.6517	3.1007	2.7967	2.5416	2.4002	2.3483	2.3166
	2 (1,2)	FSDT [35]	5.7693	4.8997	4.4192	4.0142	3.7881	3.7072	3.6580
		TSDT [14]	5.7694	4.9014	4.4192	4.0090	3.7682	3.6846	3.6368
		SSDT [43]	5.7697	4.9016	4.4194	4.0089	3.7673	3.6839	3.6365
		Present	5.7692	4.9057	4.4268	4.0216	3.7892	3.7022	3.6505
	3 (1,3)	FSDT [35]	9.1876	7.8145	7.0512	6.4015	6.0247	5.8887	5.8086
		TSDT [14]	9.1880	7.8189	7.0515	6.3886	5.9765	5.8341	5.7575
		SSDT [43]	9.1887	7.8194	7.0519	6.3885	5.9742	5.8324	5.7566
		Present	9.1874	7.8296	7.0704	6.4199	6.0276	5.8768	5.7905
	4 (2,1)	FSDT [35]	11.8310	10.0740	9.0928	8.2515	7.7505	7.5688	7.4639
		TSDT [14]	11.8315	10.0810	9.0933	8.2309	7.6731	7.4813	7.3821
		SSDT [43]	11.8326	10.0818	9.0940	8.2306	7.6696	7.4787	7.3808
		Present	11.8300	10.0980	9.1241	8.2816	7.7554	7.5497	7.4350
20	1 (1,1)	FSDT [35]	3.7123	3.1456	2.8352	2.5777	2.4425	2.3948	2.3642
		TSDT [14]	3.7123	3.1458	2.8352	2.5771	2.4403	2.3923	2.3619
		SSDT [43]	3.7123	3.1458	2.8353	2.5771	2.4401	2.3922	2.3618
		Present	3.7123	3.1462	2.8360	2.5784	2.4425	2.3942	2.3634
	2 (1,2)	FSDT [35]	5.9198	5.0175	4.5228	4.1115	3.8939	3.8170	3.7681
		TSDT [14]	5.9199	5.0180	4.5228	4.1100	3.8884	3.8107	3.7622
		SSDT [43]	5.9199	5.0180	4.5228	4.1100	3.8881	3.8105	3.7621
		Present	5.9198	5.0190	4.5248	4.1135	3.8942	3.8156	3.7660
	3 (1,3)	FSDT [35]	9.5668	8.1121	7.3132	6.6471	6.2903	6.1639	6.0843
		TSDT [14]	9.5669	8.1133	7.3132	6.6433	6.2760	6.1476	6.0690
		SSDT [43]	9.5671	8.1135	7.3133	6.6432	6.2753	6.1471	6.0688
		Present	9.5668	8.1163	7.3185	6.6522	6.2910	6.1602	6.0788
	4 (2,1)	FSDT [35]	12.4560	10.5660	9.5261	8.6572	8.1875	8.0207	7.9166
		TSDT [14]	12.4562	10.5677	9.5261	8.6509	8.1636	7.9934	7.8909
		SSDT [43]	12.4565	10.5680	9.5263	8.6508	8.1624	7.9925	7.8905
		Present	12.4560	10.5730	9.5351	8.6659	8.1888	8.0146	7.9074

Example 3. Accordingly, we focus on the comparison of fundamental dimensionless frequencies of an Al/Al_2O_3 plate. The physical properties of Al are $E_m = 70$ GPa and $\rho_m = 2707$ kg/m^3, the physical properties of Al_2O_3 are $E_c = 380$ GPa and $\rho_c = 3800$ kg/m^3. In this example, the Poisson's ratio equals to 0.3 (constant). The fundamental non-dimensional

frequencies are computed by $\hat{\omega} = \omega h\sqrt{\rho_c / E_c}$. The numerical comparison is shown in Table 4, where the present numerical solutions are computed using proposed modified single variable plate theory, the results of Yin [12] were calculated using an analytical method with different plate theory and the results of Tan et al. [13] were obtained using S-FSDT and meshless method. It is true that the present numerical solutions are very closed to other published solutions.

Table 4.The fundamental non-dimensional frequencies $\hat{\omega}$ of an Al/Al_2O_3 square plate

a/h	Method	*p* = 0	*p* = 0.5	*p* = 1	*p* = 4	*p* = 10
2	2D-HOT [12]	0.9400	0.8232	0.7476	0.5997	0.5460
	S-HSDT [12]	0.9297	0.8110	0.7356	0.5924	0.5412
	FSDT-IGA [12]	0.9265	0.8060	0.7330	0.6111	0.5640
	S-FSDT [13]	0.9270	0.8070	0.7350	0.6136	0.5652
	Present	0.9264	0.8164	0.7466	0.6174	0.5545
10	2D-HOT [12]	0.0578	0.0492	0.0443	0.0381	0.0364
	S-HSDT [12]	0.0577	0.0490	0.0442	0.0381	0.0364
	FSDT-IGA [12]	0.0577	0.0490	0.0442	0.0382	0.0366
	S-FSDT [13]	0.0575	0.0489	0.0442	0.0383	0.0366
	Present	0.0577	0.0490	0.0443	0.0382	0.0365
20	2D-HOT [12]	0.0148	0.0125	0.0113	0.0098	0.0094
	S-HSDT [12]	0.0146	0.0124	0.0112	0.0097	0.0093
	FSDT-IGA [12]	0.0148	0.0125	0.0113	0.0098	0.0094
	S-FSDT [13]	0.0148	0.0125	0.0111	0.0098	0.0094
	Present	0.0148	0.0125	0.0113	0.0098	0.0094

According to the three above examples, we can conclude that the modified single variable plate theory estimates the free vibration of FGM plates with excellent accurateness and effectiveness.

5.2. Parameter study

In this subsection, the effects of some parameters on free vibration of FGM plates with four edges simple supported are examined. The plate is made of Aluminum (Al) and Alumina (Al_2O_3), the substantial properties are $E_m = 70\ \text{GPa}$, $\rho_m = 2707\ \text{kg/m}^3$ for Al and $E_c = 380\ \text{GPa}$, $\rho_c = 3800\ \text{kg/m}^3$ for Al_2O_3.

The non-dimensional frequencies of FGM plates are calculated using the following formula

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho_c}{E_c}} \tag{36}$$

Firstly, the influences of slender ratio, as well as material parameters on the dimensionless frequencies of a square FGM plate, are considered. Fig. 2 shows that the side-to-thickness ratio and material parameter have a significant effect on the behavior of the FGM plate. The first two frequencies of FGM plate increase rapidly as the increase of *a/h* ratio, especially when it grows from 2 to 5, then the rate of increase is smaller. On the contrary, when the material parameter grows, the first two frequencies decrease. The frequencies decrease at a higher speed when the material parameter increases in the range from 0 to 2, then the speed of the decrease is slower. The reason is that when the material parameter increases, the FGM plate becomes a metal-rick plate and leads to a softer plate in comparison with a ceramic-rick plate (p = 0).

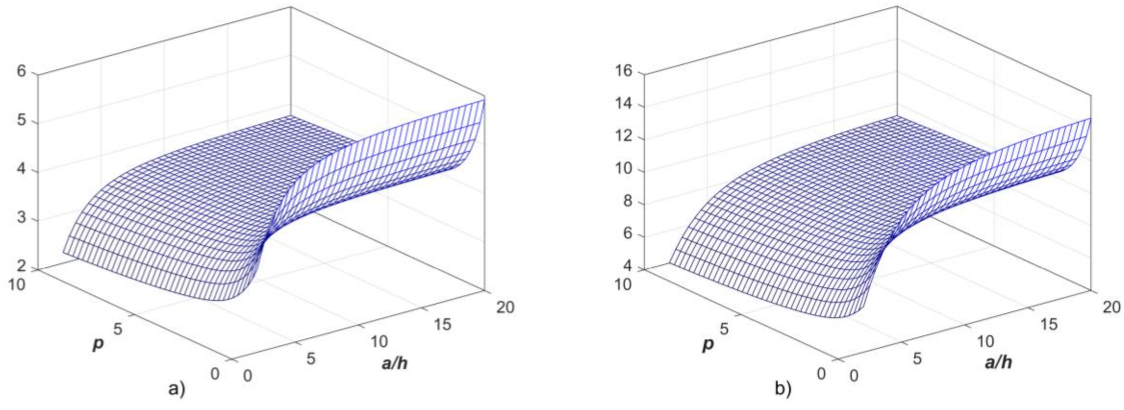


Figure 2.The influence of *a/h* ratio and material parameter on first two frequencies: a) fundamental non-dimensional frequencies, b) second non-dimensional frequencies.

Secondly, this subsection shows the dependence of dimensionless frequencies of rectangular FGM plates on the aspect ratio. The numerical results about the effects of aspect ratio in four cases of side-to-thickness ratios are shown in Fig. 3. It can see clearly that, when the aspect ratio increases, the fundamental dimensionless frequencies of the plate decrease. Not to mention, the fundamental dimensionless frequencies decrease rapidly when the aspect ratio varies in the range of 1 to 4, and then the fundamental non-dimensional frequencies decrease leisurely. Also, frequencies of the ceramic-rick plates are greater than those of metal-rick plates. The reason is that Young's modulus of ceramic is greater than it of metal, so the ceramic plates are stiffer than metallic plates. The maximum values of frequencies of FGM plates are achieved for ceramic plates and the minimum values of frequencies of FGM plates are achieved for metallic plates.

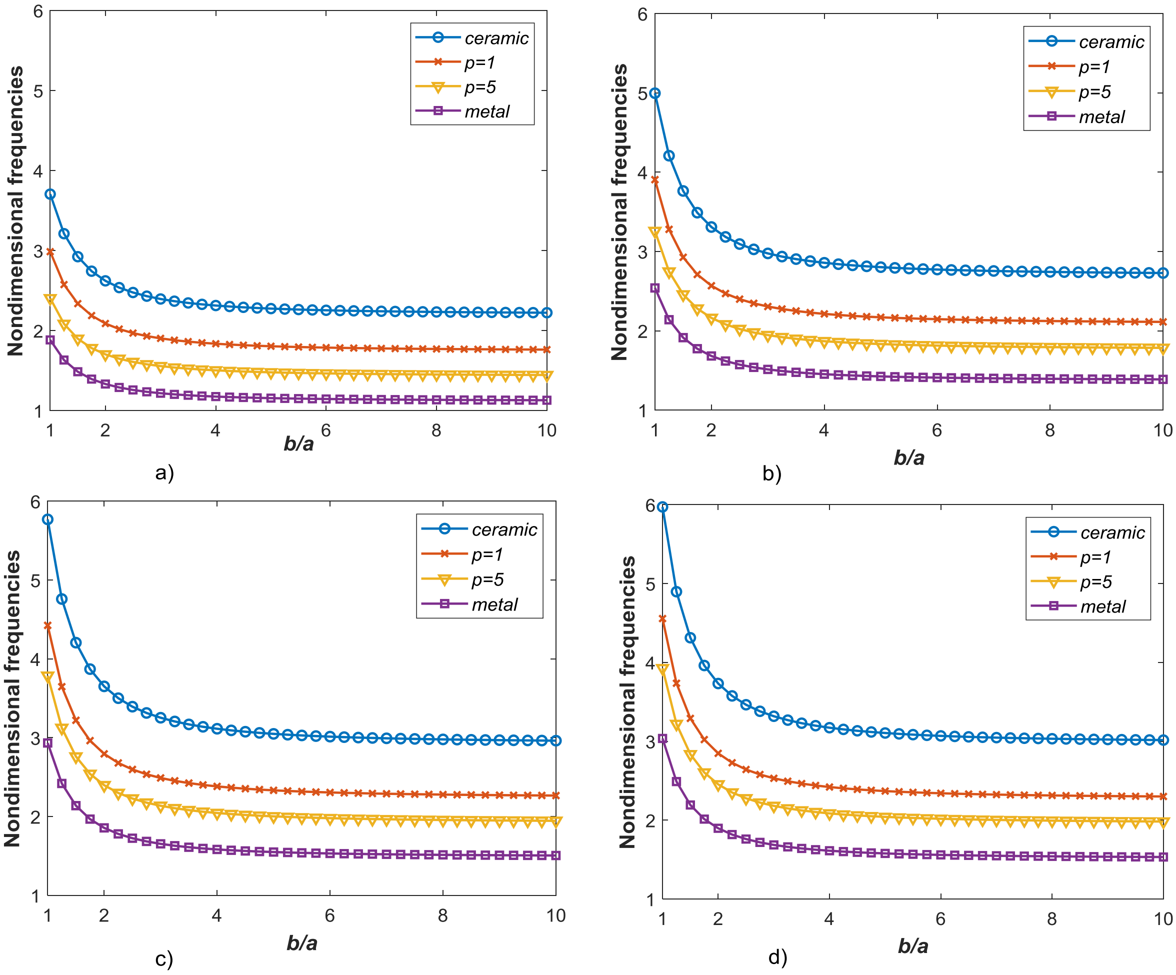


Figure 3. The influence of *b/a* ratio on free vibration of a rectangular FGM plate:
a) a/h = 2, b) a/h = 4, c) a/h = 10, d) a/h = 100.

Finally, the influences of aspect ratio on many higher frequencies of FGM plates are presented in Fig. 4. According to this figure, the change of aspect ratio leads to the change of frequencies of FGM plates. Once again, when *b/a* ratio increases, the frequencies decrease. Also, in the case of a square plate, there are many couples of similar mode shapes that occur while there are few similar mode shapes that occur in the case of a rectangular plate. This feature is also exhibited in Fig. 5 where the first eight mode shapes have been displayed. The mode shapes of the rectangular plate differ from the mode shapes of a square plate, this phenomenon is caused by the difference of the aspect ratio of the plates.

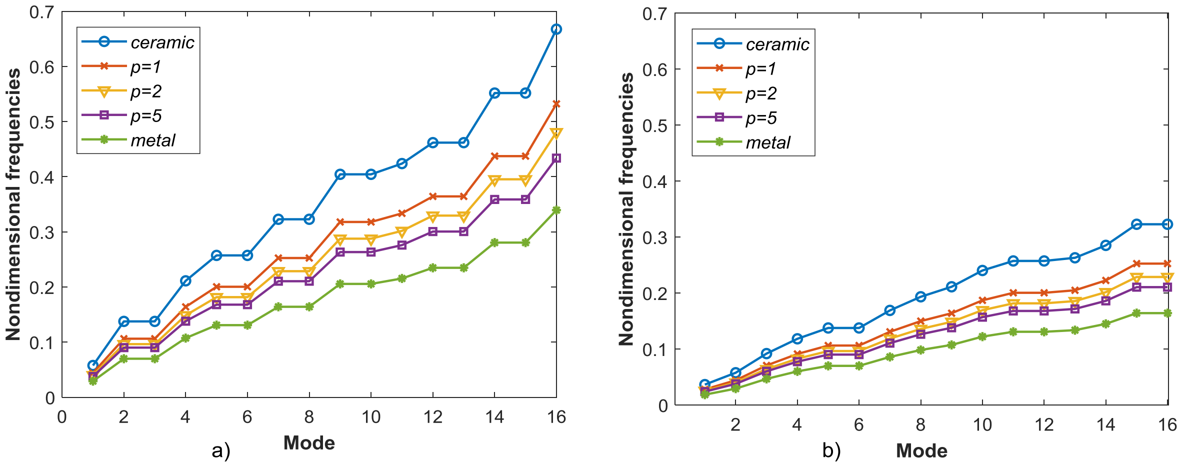
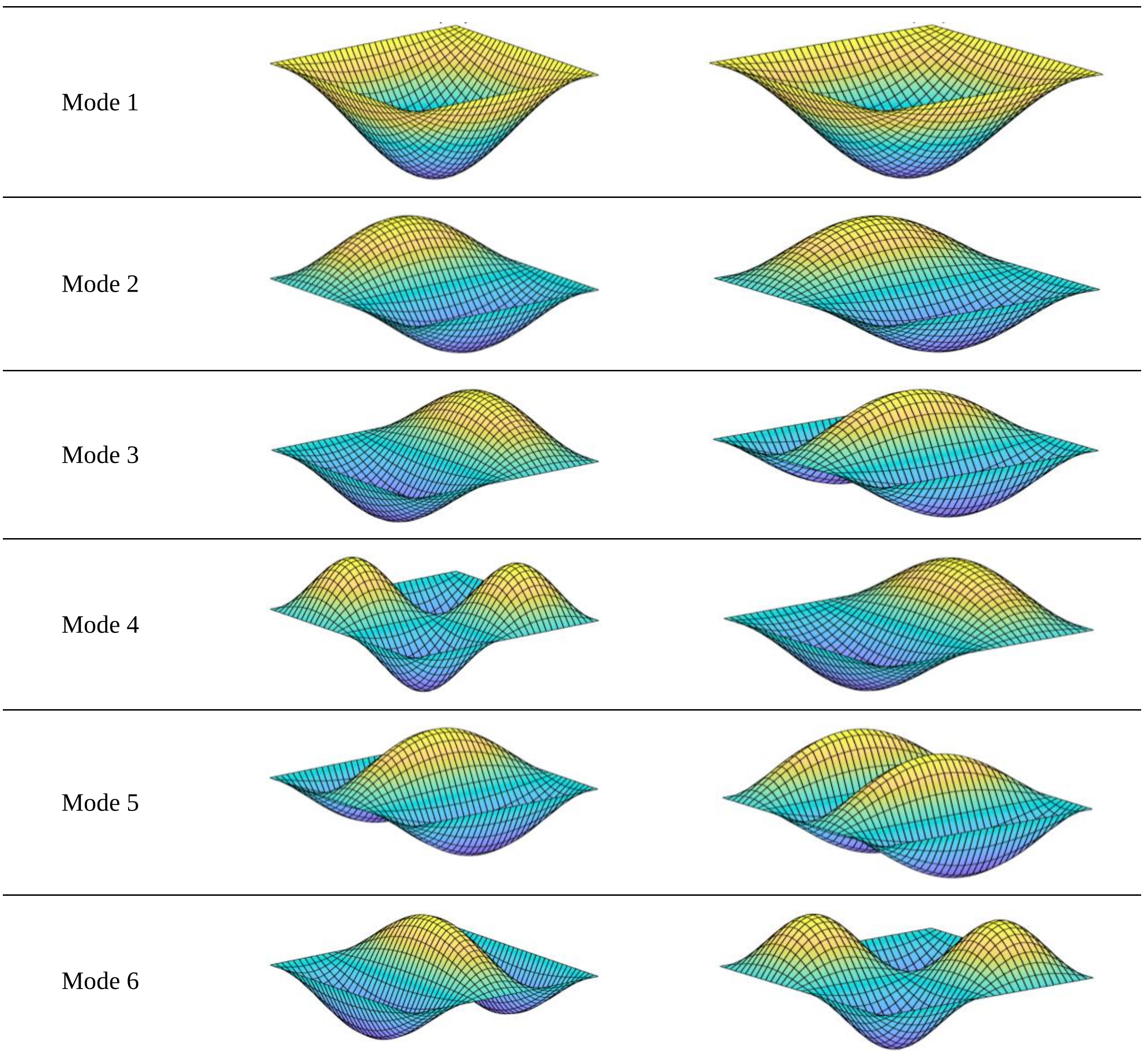


Figure 4.The influences of aspect ratio on first sixteen frequencies of FGM plates:
a) square plates b/a = 1, b) rectangular plates b/a = 2.



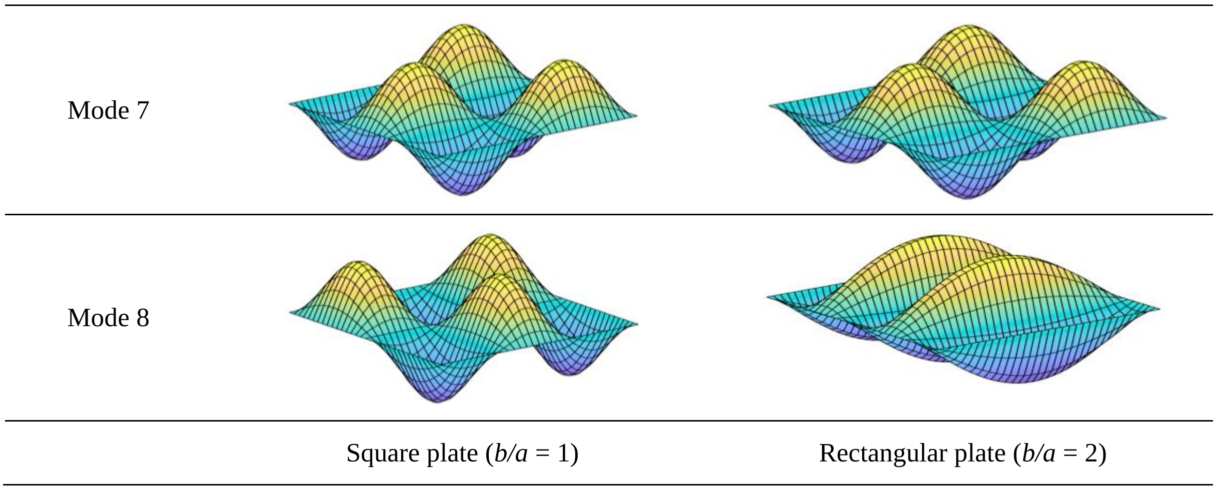


Figure 5. First eight mode shapes of square and rectangular FGM plate.

6. Conclusions

In this study, a modified single variable shear deformation plate theory has been developed based on RPT, and then the authors have been employed the proposed plate theory to study the free vibration of the FGM plates. The modified single variable plate theory consists of only one unknown variable in its displacement field as well as its governing differential equation and can be applied to analyze heterogeneous plates such as FGM plates. The modified single variable plate theory is verified thank to several comparison studies. The free vibration of the FGM plates and the influence of some parameters are investigated carefully. In conclusion, the modified single variable plate theory can be employed to research the FGM plates in an effective and simplistic manner.

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