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Euler–Maruyama scheme for Caputo stochastic fractional differential equations

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ABSTRACT

In this paper, we first construct a Euler–Maruyama type scheme for Caputo stochastic fractional differential equations (for short Caputo SFDE) of order $\alpha \in (\frac{1}{2}, 1)$ whose coefficients satisfy a standard Lipschitz and a linear growth bound condition. The strong convergence rate of this scheme is established. In particular, it is $\alpha - \frac{1}{2}$ when the coefficients of the SFDE are independent of time. Finally, we establish results on the convergence and stability of an exponential Euler–Maruyama scheme for bilinear scalar Caputo SFDEs

1. Introduction

In this paper, we study Caputo fractional differential equations in noisy environment of the form

$$ {}^{C}D^{\alpha}_{0+}X(t) = b(t, X(t)) + \sigma(t, X(t))\,\frac{dW_t}{dt}. \tag{1}$$

This type of systems is a natural type of fractional systems whose coefficients are random and thus has been received an increasing interest due to the fact that fractional systems appear in many models in mechanics, physics, electrical engineering, control theory, etc. see [1,2].

As far as we are aware, the main achieved results for (1) are limited to problem of the existence of strong solution [3,4] and mild solution [5]. A proof of coincidence of strong and mild solution of (1) under some natural assumptions on the coefficients has recently been proved in [6].

Our first aim in this paper is to establish a Euler–Maruyama numerical method for (1). Secondly, we are interested in the stability of numerical scheme for bilinear scalar Caputo SFDE. Note that in comparison to the bilinear scalar stochastic differential equations, there is no explicit formula of solutions of bilinear scalar Caputo SFDE. Then, it is hard to obtain a similar result as in [7, Section 4, Eq. (22)] about stability of Euler–Maruyama method for bilinear scalar Caputo SFDE. Here, we develop an exponential Euler–Maruyama method for bilinear scalar Caputo SFDE that can be considered as a

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natural extension of the exponential Euler–Maruyama method of stochastic differential equations to this setting. Then, we analysis the convergence and stability of this method.

The paper is organized as follows: In Section 2, we give a setting of the problem and state the main results of the paper (Theorems 1 and 3). Sections 3 and 4 are devoted to prove the main results. In Section 5, we study several examples to illustrate the numerical result. Precisely, a simple scalar system was studied to point out that the convergence rate in Theorem 1 is optimal.

Notations. Let $(W_t)_{t\in[0,\infty)}$ denote a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F}:=\{\mathcal{F}_t\}_{t\in[0,\infty)},\mathbb{P})$. For each $t\in[0,\infty)$, let $\mathfrak{X}_t:=\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P})$ denote the space of all $\mathcal{F}_t$-measurable, mean square integrable functions $f=(f_1,\ldots,f_d)^{\mathrm{T}}:\Omega\to\mathbb{R}^d$ with

$$\|f\|_{\mathrm{ms}}:=\sqrt{\sum_{1\le i\le d}\mathbb{E}(|f_i|^2)},$$

where $\mathbb{R}^d$ is endowed with the standard Euclidean norm. A process $X:[0,\infty)\to\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$ is said to be $\mathbb{F}$-adapted if $X(t)\in\mathfrak{X}_t$ for all $t\ge 0$.

For $\alpha,\beta\in(0,1)$, the Mittag-Leffler functions $E_{\alpha,\beta},E_\alpha:\mathbb{R}\to\mathbb{R}$ are defined by

$$E_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad E_\alpha(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

where Γ is the Gamma function, i.e. $\Gamma(x):=\int_0^\infty s^{x-1}e^{-s}\,ds$.

2. Preliminaries and the statement of the main results

2.1. Setting

Let $T>0$ be arbitrary and consider a Caputo SFDE of order $\alpha\in(\frac{1}{2},1)$ on the interval $[0,T]$ of the following form

$$^{C}D^{\alpha}_{0+}X(t)=b(t,X(t))+\sigma(t,X(t))\,\frac{dW_t}{dt},\tag{2}$$

where $b,\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ are measurable and satisfy the following conditions:

(H1) *Global Lipschitz continuity in $\mathbb{R}^d$ of the drift and diffusion*: There exists $L>0$ such that for all $x,y\in\mathbb{R}^d$, $t\in[0,T]$,

$$\|b(t,x)-b(t,y)\|\vee\|\sigma(t,x)-\sigma(t,y)\|\le L\|x-y\|.$$

(H2) *Hölder continuity in $[0,T]$ of the drift and diffusion*: There exist $L_1,L_2>0$ and $\beta,\gamma\in[0,1]$ such that for all $x\in\mathbb{R}^d$, $t,s\in[0,T]$

$$\|b(t,x)-b(s,x)\|\le L_1|t-s|^\beta,\quad \|\sigma(t,x)-\sigma(s,x)\|\le L_2|t-s|^\gamma.$$

(H3) *Linear growth bound*: There exists $K>0$ such that for $t\in[0,T]$, $x\in\mathbb{R}^d$

$$\|b(t,x)\|\vee\|\sigma(t,x)\|\le K(1+\|x\|).$$

For each $\eta\in\mathfrak{X}_0$, a $\mathbb{F}$-adapted process X is called a solution of (2) on the interval $[0,T]$ with the initial condition $X(0)=\eta$ if the following equality holds for $t\in[0,T]$

$$X(t)=\eta+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{b(s,X(s))}{(t-s)^{1-\alpha}}\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\sigma(s,X(s))}{(t-s)^{1-\alpha}}\,dW_s,\tag{3}$$

see [4, p. 209]. Thanks to [3, Theorem 1], for each initial value $\eta\in\mathfrak{X}_0$ system (2) has a unique solution on $[0,T]$ denoted by $X(t,\eta)$.[1]

2.2. Euler–Maruyama type scheme for Caputo SFDEs

An important task in applications is to realize (2) on computers, that is, to construct a discretized approximation. It is clear that the kernel of (3), the function $(t-s)^{\alpha-1}$, becomes infinity at point $s=t$. This brings us an obvious difficulty to discretize (2). To avoid touching the singular point, we introduce the following discretized scheme which is a *Euler–Maruyama type scheme for Caputo SFDEs*:

[1] By [3, Theorem 1], for the existence and uniqueness solution we only require the assumptions (H1) and (H3).

For each $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of positive integer numbers, the approximated solution $X^{(n)}(\cdot, \eta)$ is defined by $X^{(n)}(0, \eta) := \eta$ and for $t \in (0, T]$

$$X^{(n)}(t, \eta) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{b(\tau_n(s), X^{(n)}(\tau_n(s), \eta))}{(t-s)^{1-\alpha}}\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s), \eta))}{(\rho_n(t) - \tau_n(s))^{1-\alpha}}\, dW_s, \tag{4}$$

where $\tau_n(s) = \frac{kT}{n} =: \tau_k^{(n)}$ and $\rho_n(s) = \frac{(k+1)T}{n} =: \rho_k^{(n)}$ for $s \in \left(\frac{kT}{n}, \frac{(k+1)T}{n}\right]$.

This equation can be solved step by step on each interval $\left(\frac{Tk}{n}, \frac{T(k+1)}{n}\right]$, $k = 0, 1, \ldots, n-1$. Our first main result in this paper is to give an estimate on the mean-square distance between the numerical solution $X^{(n)}(t, \eta)$ and the exact solution $X(t, \eta)$.

Theorem 1 (*Strong Convergence of the Euler–Maruyama Scheme for Caputo SFDE*). *Let $\kappa := \min\{\beta, \gamma, \alpha - \frac{1}{2}\}$. Then, there exists a constant C depending only on $T, L, L_1, L_2, \alpha, \beta, \gamma, K$ such that*

$$\sup_{0 \le t \le T} \|X^{(n)}(t, \eta) - X(t, \eta)\|_{\text{ms}}^2 \le \frac{C}{n^{2\kappa}}. \tag{5}$$

Remark 2. (i) When the coefficients are independent of time, then the convergence rate of Euler–Maruyama scheme for (2) is $\alpha - \frac{1}{2}$.

(ii) When $\alpha = 1$, i.e. Eq. (2) becomes a stochastic differential equation, the convergent rate of the scheme in Theorem 1 coincides with the well-known convergent rate of the classical Euler–Maruyama, see [8].

(iii) It is clear that there exist connections between the result in this paper to the existing result about Euler scheme for general stochastic Volterra equations with singular kernels, see e.g. [9,10]. Since the kernels in our systems are explicit then we obtain explicit and optimal rate of convergence of Euler scheme. This rate is better than the restriction of results in [9,10] to fractional setting.

2.3. Exponential Euler–Maruyama scheme for bilinear scalar Caputo SFDEs

We are interested in investigating the stability of numerical method on the test systems, bilinear scalar Caputo SFDE. More precisely, we consider systems of the form

$${}^C D_{0+}^{\alpha} X(t) = \lambda X(t) + \mu X(t) \frac{dW_t}{dt}. \tag{6}$$

Note that the problem in determining λ, μ for which system (6) is mean-square asymptotically stable is not trivial due to the fact that there is no explicit form of solutions of (6). It has been recently proved in [11, Proposition 11] that system (6) is mean-square asymptotically stable if and only if

$$\lambda < 0 \quad \text{and} \quad \mu^2 \int_0^\infty s^{2\alpha-2} \left(E_{\alpha,\alpha}(\lambda s^\alpha)\right)^2 ds < 1. \tag{7}$$

The main ingredient in the proof of the preceding result is to use the variation of constants formula developed in [6, Theorem 2.3], i.e. the integral form of (6) is given by

$$X(t) = E_\alpha(\lambda t^\alpha) X(0) + \mu \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) X(s)\, dW_s. \tag{8}$$

For a fixed step-size $h > 0$, the *exponential Euler–Maruyama scheme* for the above integral equation is given by

$$\widehat{X}_h(t) = E_\alpha(\lambda t^\alpha) X(0) + \mu \int_0^t (t - \tau_h(s))^{\alpha-1} E_{\alpha,\alpha}(\lambda (t - \tau_h(s))^\alpha) \widehat{X}_h(\tau_h(s))\, dW_s, \tag{9}$$

where $\tau_h : (0, \infty) \to [0, \infty)$ is defined by

$$\tau_h(s) = kh \quad \text{for } s \in (kh, (k+1)h],\ k = 0, 1, 2, \ldots. \tag{10}$$

We now state the result about convergence and stability of the above Euler–Maruyama scheme for bilinear scalar Caputo SFDE.

Theorem 3 (*Convergence and Stability of Exponential Euler–Maruyama Method for Bilinear Scalar Caputo SFDE*). *(i) For any $T > 0$, there exists a constant C depending on T, λ and μ such that*

$$\sup_{t \in [0,T]} \|\widehat{X}_h(t) - X(t)\|_{\text{ms}} \le C h^{\alpha - \frac{1}{2}}.$$

(ii) Suppose that the condition (7) holds. For any step-size $h > 0$, there exists $K > 0$ such that the solution $\widehat{X}_h$ of (9) satisfies

$$\|\widehat{X}_h(t)\|_{\mathrm{ms}} \le K\|X(0)\|_{\mathrm{ms}} \qquad \text{for all } t \ge 0$$

and furthermore for any $\delta \in (0, \alpha)$ we have $\lim_{t\to\infty} t^{\delta}\|\widehat{X}_h(t)\|_{\mathrm{ms}} = 0$. Consequently, the numerical solution remains asymptotically stable.

3. Proof of the strong convergence of the Euler–Maruyama method for Caputo SFDE

Before going to the proof of the main result, we need some preparatory lemmas. Firstly, we show in the following lemma a bound on $\sup_{0\le t\le T}\|X^n(t,\eta)\|_{\mathrm{ms}}$.

Lemma 4. *Let*

$$C_1 := (1 + 3\|\eta\|^2_{\mathrm{ms}})E_{2\alpha-1}\left(\frac{(6T+6)K^2T^{2\alpha-1}\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}\right). \tag{11}$$

Then, for all $n \in \mathbb{N}^*$ we have

$$\sup_{0\le t\le T}\|X^{(n)}(t,\eta)\|^2_{\mathrm{ms}} \le C_1.$$

Proof. From (4) and the inequality $\|x+y+z\|^2 \le 3(\|x\|^2+\|y\|^2+\|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, we derive

$$\mathbb{E}\left(\|X^{(n)}(t,\eta)\|^2\right) \le 3\mathbb{E}\|\eta\|^2 + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\left(\left\|\int_0^t \frac{b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(t-s)^{1-\alpha}}\,ds\right\|^2\right)$$
$$+ \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\left(\left\|\int_0^t \frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(\rho_n(t)-\tau_n(s))^{1-\alpha}}\,dW_s\right\|^2\right).$$

Using the Hölder inequality, Ito's isometry, we obtain

$$\|X^{(n)}(t,\eta)\|^2_{\mathrm{ms}} \le 3\|\eta\|^2_{\mathrm{ms}} + \frac{3t}{\Gamma^2(\alpha)}\int_0^t \frac{\mathbb{E}\|b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|^2}{(t-s)^{2-2\alpha}}\,ds$$
$$+ \frac{3}{\Gamma^2(\alpha)}\int_0^t \frac{\mathbb{E}\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|^2}{(\rho_n(t)-\tau_n(s))^{2-2\alpha}}\,ds.$$

This together with the fact that $|\rho_n(t) - \tau_n(s)| \ge |t-s|$ and the linear growth condition (H3) implies that

$$\|X^{(n)}(t,\eta)\|^2_{\mathrm{ms}} \le 3\|\eta\|^2_{\mathrm{ms}} + \frac{3t}{\Gamma^2(\alpha)}\int_0^t \frac{2K^2(1+\mathbb{E}\|X^{(n)}(\tau_n(s),\eta)\|^2)}{(t-s)^{2-2\alpha}}\,ds$$
$$+ \frac{3}{\Gamma^2(\alpha)}\int_0^t \frac{2K^2(1+\mathbb{E}\|X^{(n)}(\tau_n(s),\eta)\|^2)}{(t-s)^{2-2\alpha}}\,ds$$
$$= 3\|\eta\|^2_{\mathrm{ms}} + \frac{(6t+6)K^2}{\Gamma^2(\alpha)}\int_0^t \frac{1+\mathbb{E}\|X^{(n)}(\tau_n(s),\eta)\|^2}{(t-s)^{2-2\alpha}}\,ds.$$

Let $m_t := 1 + \sup_{0\le s\le t}\|X^{(n)}(s,\eta)\|^2_{\mathrm{ms}}$. Then,

$$m_t \le (1 + 3\|\eta\|^2_{\mathrm{ms}}) + \frac{(6T+6)K^2}{\Gamma(\alpha)^2}\int_0^t \frac{m_s}{(t-s)^{2-2\alpha}}\,ds.$$

Applying the Gronwall's inequality for fractional differential equations, see e.g. [1, Lemma 6.19], we arrive at

$$m_t \le (1 + 3\|\eta\|^2_{\mathrm{ms}})E_{2\alpha-1}\left(\frac{(6T+6)K^2}{\Gamma(\alpha)^2}t^{2\alpha-1}\Gamma(2\alpha-1)\right),$$

which completes the proof. □

Finally, we establish an upper bound on $\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|^2_{\mathrm{ms}}$ in terms of $|t-\widetilde{t}|^{2\alpha-1}$ and $\frac{1}{n^{2\alpha-1}}$.

Lemma 5. *Let*

$$C_2 := \frac{8K^2(1+C_1)\,T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}, \quad C_3 := \frac{8K^2(1+C_1)(T+2)}{(2\alpha-1)\Gamma^2(\alpha)}, \tag{12}$$

where C_1 is given as in (11). Then, for all $n \in \mathbb{N}^*$ and $t, \widetilde{t} \in [0, T]$ we have

$$\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|^2_{\mathrm{ms}} \le \frac{C_2}{n^{2\alpha-1}} + C_3|t-\widetilde{t}|^{2\alpha-1}.$$

Proof. Choose and fix $t, \widetilde{t} \in [0, T]$ with $t > \widetilde{t}$. By (4), we have

$$\begin{aligned}
&\Gamma(\alpha)\left(X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\right)\\
&= \int_{\widetilde{t}}^{t} \frac{b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(t-s)^{1-\alpha}}\,ds + \int_{\widetilde{t}}^{t} \frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(\rho_n(t)-\tau_n(s))^{1-\alpha}}\,dW_s\\
&\quad + \int_0^{\widetilde{t}} \Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(\widetilde{t}-s)^{1-\alpha}}\Big) b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\,ds\\
&\quad + \int_0^{\widetilde{t}} \Big(\frac{1}{(\rho_n(t)-\tau_n(s))^{1-\alpha}} - \frac{1}{(\rho_n(\widetilde{t})-\tau_n(s))^{1-\alpha}}\Big) \sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\,dW_s.
\end{aligned}$$

Using the inequality $\|x+y+z+w\|^2 \le 4(\|x\|^2+\|y\|^2+\|z\|^2+\|w\|^2)$ for all $x, y, z, w \in \mathbb{R}^d$ and Ito's isometry, we derive that

$$\begin{aligned}
&\frac{\Gamma^2(\alpha)}{4}\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|_{\mathrm{ms}}^2\\
&\le \mathbb{E}\left(\left\|\int_{\widetilde{t}}^{t} \frac{b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(t-s)^{1-\alpha}}\,ds\right\|^2\right) + \int_{\widetilde{t}}^{t} \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}^2}{(\rho_n(t)-\tau_n(s))^{2-2\alpha}}\,ds\\
&\quad + \mathbb{E}\left(\left\|\int_0^{\widetilde{t}} \Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(\widetilde{t}-s)^{1-\alpha}}\Big) b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\,ds\right\|^2\right)\\
&\quad + \int_0^{\widetilde{t}} \Big(\frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(\rho_n(t)-\tau_n(s))^{1-\alpha}} - \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(\rho_n(\widetilde{t})-\tau_n(s))^{1-\alpha}}\Big)^2\,ds.
\end{aligned}$$

By Hölder inequality and the fact that $|\rho_n(t) - \tau_n(s)| \ge |t-s|$, we have

$$\begin{aligned}
&\frac{\Gamma^2(\alpha)}{4}\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|_{\mathrm{ms}}^2\\
&\le (t-\widetilde{t})\int_{\widetilde{t}}^{t} \frac{\|b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}}\,ds + \int_{\widetilde{t}}^{t} \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}}\,ds\\
&\quad + \widetilde{t}\int_0^{\widetilde{t}} \Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(\widetilde{t}-s)^{1-\alpha}}\Big)^2 \|b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}^2\,ds\\
&\quad + \int_0^{\widetilde{t}} \Big(\frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(\rho_n(t)-\tau_n(s))^{1-\alpha}} - \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(\rho_n(\widetilde{t})-\tau_n(s))^{1-\alpha}}\Big)^2\,ds.
\end{aligned}$$

This together with the inequality

$$\Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(\widetilde{t}-s)^{1-\alpha}}\Big)^2 < \frac{1}{(\widetilde{t}-s)^{2-2\alpha}} - \frac{1}{(t-s)^{2-2\alpha}},$$

and[2]

$$\frac{1}{(\rho_n(\widetilde{t})-\tau_n(s))^{2-2\alpha}} - \frac{1}{(\rho_n(t)-\tau_n(s))^{2-2\alpha}} \le \frac{1}{(\rho_n(\widetilde{t})-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-s)^{2-2\alpha}}$$

implies that

$$\begin{aligned}
&\frac{\Gamma^2(\alpha)}{8K^2(1+C_1)}\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|_{\mathrm{ms}}^2\\
&\le \int_{\widetilde{t}}^{t} \frac{t-\widetilde{t}+1}{(t-s)^{2-2\alpha}}\,ds + \int_0^{\widetilde{t}} \left(\frac{\widetilde{t}}{(\widetilde{t}-s)^{2-2\alpha}} - \frac{\widetilde{t}}{(t-s)^{2-2\alpha}}\right) ds\\
&\quad + \int_0^{\widetilde{t}} \left(\frac{1}{(\rho_n(\widetilde{t})-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-s)^{2-2\alpha}}\right) ds. \qquad (13)
\end{aligned}$$

[2] To gain this inequality, we define $h(x) := \frac{1}{(\rho_n(\widetilde{t})-x)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-x)^{2-2\alpha}}$ for $x < \rho_n(\widetilde{t}) < \rho_n(t)$. Then, $h'(x) = (2-2\alpha)((\rho_n(\widetilde{t})-x)^{2\alpha-3} - (\rho_n(t)-x)^{2\alpha-3}) > 0$. It together with $\tau_n(s) < s$ implies that $h(\tau_n(s)) < h(s)$.

A direct computation yields that

$$\begin{aligned}
&\int_{\widetilde{t}}^{t} \frac{t-\widetilde{t}+1}{(t-s)^{2-2\alpha}}\,ds + \int_0^{\widetilde{t}} \left(\frac{\widetilde{t}}{(\widetilde{t}-s)^{2-2\alpha}} - \frac{\widetilde{t}}{(t-s)^{2-2\alpha}} \right) ds \\
&= -(t-\widetilde{t}+1)\frac{(t-s)^{2\alpha-1}}{2\alpha-1}\Big|_{\widetilde{t}}^{t} + \widetilde{t}\left(-\frac{(\widetilde{t}-s)^{2\alpha-1}}{2\alpha-1} + \frac{(t-s)^{2\alpha-1}}{2\alpha-1} \right)\Big|_0^{\widetilde{t}} \\
&\leq \frac{(t+1)(t-\widetilde{t})^{2\alpha-1}}{2\alpha-1}.
\end{aligned} \tag{14}$$

Furthermore, we also have

$$\begin{aligned}
&\int_0^{\widetilde{t}} \left(\frac{1}{(\rho_n(\widetilde{t})-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-s)^{2-2\alpha}} \right) ds \\
&= \left(-\frac{(\rho_n(\widetilde{t})-s)^{2\alpha-1}}{2\alpha-1} + \frac{(\rho_n(t)-s)^{2\alpha-1}}{2\alpha-1} \right)\Big|_0^{\widetilde{t}} \\
&\leq \frac{(\rho_n(t)-\widetilde{t})^{2\alpha-1} - (\rho_n(\widetilde{t})-\widetilde{t})^{2\alpha-1}}{2\alpha-1}.
\end{aligned}$$

Since $0 < 2\alpha - 1 < 1$ and using the inequality $|x+y|^{2\alpha-1} \leq |x|^{2\alpha-1} + |y|^{2\alpha-1}$, we obtain that

$$\int_0^{\widetilde{t}} \left(\frac{1}{(\rho_n(\widetilde{t})-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-s)^{2-2\alpha}} \right) ds \leq \frac{(\rho_n(t)-\rho_n(\widetilde{t}))^{2\alpha-1}}{2\alpha-1}.$$

By definition of ρ_n, we also have $\rho_n(t) - \rho_n(\widetilde{t}) \leq t - \widetilde{t} + \frac{T}{n}$. Thus,

$$\int_0^{\widetilde{t}} \left(\frac{1}{(\rho_n(\widetilde{t})-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t)-s)^{2-2\alpha}} \right) ds \leq \frac{(t-\widetilde{t})^{2\alpha-1} + \frac{T^{2\alpha-1}}{n^{2\alpha-1}}}{2\alpha-1}.$$

This together with (13) and (14) implies that

$$\|X^{(n)}(t,\eta) - X^{(n)}(\widetilde{t},\eta)\|_{\mathrm{ms}}^2 \leq \frac{8K^2(1+C_1)}{(2\alpha-1)\Gamma^2(\alpha)} \left((T+2)(t-\widetilde{t})^{2\alpha-1} + \frac{T^{2\alpha-1}}{n^{2\alpha-1}} \right).$$

The proof is complete. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Choose and fix $\eta \in \mathfrak{X}_0$. From (3) and (4) we have

$$\begin{aligned}
&X^{(n)}(t,\eta) - X(t,\eta) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{b(\tau_n(s), X^{(n)}(\tau_n(s),\eta)) - b(s, X(s,\eta))}{(t-s)^{1-\alpha}}\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta)) - \sigma(s, X(s,\eta))}{(t-s)^{1-\alpha}}\,dW_s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(\rho_n(t)-\tau_n(s))^{1-\alpha}} - \frac{\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(t-s)^{1-\alpha}} \right) dW_s.
\end{aligned}$$

Using the inequality $\|x+y+z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, Hölder inequality and Ito's isometry, we derive that

$$\begin{aligned}
&\frac{\Gamma^2(\alpha)}{3} \|X^{(n)}(t,\eta) - X(t,\eta)\|_{\mathrm{ms}}^2 \\
&\leq t \int_0^t \frac{\|b(\tau_n(s), X^{(n)}(\tau_n(s),\eta)) - b(s, X(s,\eta))\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}}\,ds \\
&+ \int_0^t \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta)) - \sigma(s, X(s,\eta))\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}}\,ds \\
&+ \int_0^t \left(\frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(\rho_n(t)-\tau_n(s))^{1-\alpha}} - \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s),\eta))\|_{\mathrm{ms}}}{(t-s)^{1-\alpha}} \right)^2 ds.
\end{aligned} \tag{15}$$

Moreover, in light of (H1) and (H2) it is easily seen that

$$\begin{aligned}&\|b(\tau_n(s), X^{(n)}(\tau_n(s), \eta)) - b(s, X(s, \eta))\|_{\mathrm{ms}}^2\\ \leq\ & 2L^2\|X^{(n)}(\tau_n(s), \eta) - X(s, \eta)\|_{\mathrm{ms}}^2 + 2L_1^2|\tau_n(s) - s|^{2\beta},\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s), \eta)) - \sigma(s, X(s, \eta))\|_{\mathrm{ms}}^2\\ \leq\ & 2L^2\|X^{(n)}(\tau_n(s), \eta) - X(s, \eta)\|_{\mathrm{ms}}^2 + 2L_2^2|\tau_n(s) - s|^{2\gamma}.\end{aligned} \tag{17}$$

By (H3), Lemma 4 and the inequality

$$\begin{aligned}\left(\frac{1}{(\rho_n(t) - \tau_n(s))^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\right)^2 &\leq \frac{1}{(t-s)^{2-2\alpha}} - \frac{1}{(\rho_n(t) - \tau_n(s))^{2-2\alpha}}\\ &\leq \frac{1}{(t-s)^{2-2\alpha}} - \frac{1}{(\frac{2T}{n} + t - s)^{2-2\alpha}},\end{aligned}$$

we have

$$\begin{aligned}&\int_0^t \left(\frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s), \eta))\|_{\mathrm{ms}}}{(\rho_n(t) - \tau_n(s))^{1-\alpha}} - \frac{\|\sigma(\tau_n(s), X^{(n)}(\tau_n(s), \eta))\|_{\mathrm{ms}}}{(t-s)^{1-\alpha}}\right)^2 ds\\ \leq\ & 2K^2(1 + C_1)\int_0^t \left(\frac{1}{(t-s)^{2-2\alpha}} - \frac{1}{(\frac{2T}{n} + t - s)^{2-2\alpha}}\right) ds\\ \leq\ & \frac{2K^2(1 + C_1)(2T)^{2\alpha-1}}{(2\alpha - 1)} \frac{1}{n^{2\alpha-1}}.\end{aligned}$$

This together with (15)–(17) implies that

$$\begin{aligned}&\|X^{(n)}(t, \eta) - X(t, \eta)\|_{\mathrm{ms}}^2\\ \leq\ & \frac{6L^2(t+1)}{\Gamma^2(\alpha)}\int_0^t \frac{\|X^{(n)}(\tau_n(s), \eta) - X(s, \eta)\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}} ds\\ &+ \frac{6L_1^2 t}{\Gamma^2(\alpha)}\int_0^t \frac{|\tau_n(s) - s|^{2\beta}}{(t-s)^{2-2\alpha}} ds + \frac{6L_2^2}{\Gamma^2(\alpha)}\int_0^t \frac{|\tau_n(s) - s|^{2\gamma}}{(t-s)^{2-2\alpha}} ds\\ &+ \frac{6K^2(1 + C_1)(2T)^{2\alpha-1}}{(2\alpha - 1)\Gamma^2(\alpha)} \frac{1}{n^{2\alpha-1}}.\end{aligned} \tag{18}$$

By definition of τ_n, we have $|\tau_n(s) - s| \leq \frac{T}{n}$ for $s \in [0, T]$. Hence, a direct computation yields that

$$\begin{aligned}&\frac{6L_1^2 t}{\Gamma^2(\alpha)}\int_0^t \frac{|\tau_n(s) - s|^{2\beta}}{(t-s)^{2-2\alpha}} ds + \frac{6L_2^2}{\Gamma^2(\alpha)}\int_0^t \frac{|\tau_n(s) - s|^{2\gamma}}{(t-s)^{2-2\alpha}} ds\\ \leq\ & \frac{6L_1^2 T^{2\alpha+2\beta}}{(2\alpha - 1)\Gamma^2(\alpha)} \frac{1}{n^{2\beta}} + \frac{6L_2^2 T^{2\alpha+2\gamma-1}}{(2\alpha - 1)\Gamma^2(\alpha)} \frac{1}{n^{2\gamma}}.\end{aligned} \tag{19}$$

On the other hand, by virtue of Lemma 5 we have

$$\begin{aligned}&\|X^{(n)}(\tau_n(s), \eta) - X(s, \eta)\|_{\mathrm{ms}}^2\\ \leq\ & 2\|X^{(n)}(\tau_n(s), \eta) - X^{(n)}(s, \eta)\|_{\mathrm{ms}}^2 + 2\|X^{(n)}(s, \eta) - X(s, \eta)\|_{\mathrm{ms}}^2\\ \leq\ & \frac{2C_2}{n^{2\alpha-1}} + 2C_3|\tau_n(s) - s|^{2\alpha-1} + 2\|X^{(n)}(s, \eta) - X(s, \eta)\|_{\mathrm{ms}}^2\\ \leq\ & \frac{2C_2 + 2T^{2\alpha-1}C_3}{n^{2\alpha-1}} + 2\|X^{(n)}(s, \eta) - X(s, \eta)\|_{\mathrm{ms}}^2,\end{aligned}$$

where C_2 and C_3 are given as in (12). This together with (18) and (19) gives that

$$\begin{aligned}&\sup_{0\leq s\leq t}\|X^{(n)}(s, \eta) - X(s, \eta)\|_{\mathrm{ms}}^2\\ \leq\ & \frac{12L^2(T+1)}{\Gamma^2(\alpha)}\int_0^t \frac{\sup_{0\leq r\leq s}\|X^{(n)}(r, \eta) - X(r, \eta)\|_{\mathrm{ms}}^2}{(t-s)^{2-2\alpha}} ds\\ &+ D_1\frac{1}{n^{2\beta}} + D_2\frac{1}{n^{2\gamma}} + D_3\frac{1}{n^{2\alpha-1}},\end{aligned}$$

where $D_1 := \frac{6L_1^2 T^{2\alpha+2\beta}}{(2\alpha-1)\Gamma^2(\alpha)}$, $D_2 := \frac{6L_2^2 T^{2\alpha+2\gamma-1}}{(2\alpha-1)\Gamma^2(\alpha)}$ and

$$D_3 := \frac{6K^2(1+C_1)(2T)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} + (2C_2 + 2T^{2\alpha-1}C_3)\frac{6L^2(T+1)}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{2\alpha-1}.$$

Applying the Gronwall's inequality for fractional differential equations, we arrive at

$$\sup_{0\le t\le T} \|X^{(n)}(t,\eta) - X(t,\eta)\|_{\mathrm{ms}}^2 \le \left(\frac{D_1}{n^{2\beta}} + \frac{D_2}{n^{2\gamma}} + \frac{D_3}{n^{2\alpha-1}}\right) E_{2\alpha-1}\left(\frac{12L^2(T+1)T^{2\alpha-1}\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}\right).$$

Hence, inequality (5) holds for

$$C := (D_1 + D_2 + D_3)E_{2\alpha-1}\left(\frac{12L^2(T+1)T^{2\alpha-1}\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}\right).$$

The proof is complete. □

The main result of this paper can be extended to more general Caputo SFDEs with vector-valued noise. Precisely, for $\alpha \in (\frac{1}{2}, 1)$ we consider the following system of Caputo SFDEs on $[0, T]$

$$^{C}D_{0+}^{\alpha} X(t) = b(t, X(t)) + \sum_{i=1}^{N} \sigma_i(t, X(t))\,\frac{dW_t^i}{dt}, \tag{20}$$

where drift function b and the diffusion functions $\sigma_i, i = 1, \dots N$, are measurable and satisfy the same assumptions as in (H1), (H2) and (H3) with the same constants β, γ. For the initial value condition $X(0) = \eta$, the corresponding integral form of (20) is

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{b(s, X(s))}{(t-s)^{1-\alpha}}\,ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{N}\int_0^t \frac{\sigma_i(s, X(s))}{(t-s)^{1-\alpha}}\,dW_s^i.$$

The Euler–Maruyama scheme is now written as

$$X^{(n)}(t,\eta) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{b(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(t-s)^{1-\alpha}}\,ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{N}\int_0^t \frac{\sigma_i(\tau_n(s), X^{(n)}(\tau_n(s),\eta))}{(\rho_n(t)-\tau_n(s))^{1-\alpha}}\,dW_s^i,$$

where $\tau_n(s) = \frac{kT}{n} =: \tau_k^{(n)}$ and $\rho_n(s) = \frac{(k+1)T}{n} =: \rho_k^{(n)}$ for $s \in (\frac{kT}{n}, \frac{(k+1)T}{n}]$. By simply adapting the proof of Theorem 1, we also have the following result about the convergence rate of $\|X^{(n)}(t,\eta) - X(t,\eta)\|_{\mathrm{ms}}^2$.

Theorem 6. *Let $\kappa := \min\{\beta, \gamma, \alpha - \frac{1}{2}\}$. Then, there exists a constant C such that*

$$\sup_{0\le t\le T} \|X^{(n)}(t,\eta) - X(t,\eta)\|_{\mathrm{ms}}^2 \le \frac{C}{n^{2\kappa}}. \tag{21}$$

4. Proof of the convergence and stability of exponential Euler–Maruyama method for bilinear scalar Caputo SFDE

Before going to the proof of Theorem 3, we need some preparatory lemmas. Lemmas 7 and 8 are useful in proving the part (i) of Theorem 3 and there is no requirement on λ, μ. Meanwhile, Lemmas 9 and 10 are useful in proving the part (ii) of Theorem 3 and here we assume that λ, μ satisfy condition (7).

Firstly, we show in the following lemma a bound on $\sup_{0\le t\le T}\|\widehat{X}_h(t)\|_{\mathrm{ms}}$.

Lemma 7. *Let $M_1 := \max_{0\le t\le T}(E_\alpha(\lambda t^\alpha))^2$, $M_2 := \max_{0\le t\le T}(E_{\alpha,\alpha}(\lambda t^\alpha))^2$ and*

$$C_4 := M_1 E_{2\alpha-1}\left(\mu^2 M_2 \Gamma(2\alpha-1)T^{2\alpha-1}\right)\|X(0)\|_{\mathrm{ms}}^2. \tag{22}$$

Then, for all $h > 0$ we have

$$\sup_{0\le t\le T}\|\widehat{X}_h(t)\|_{\mathrm{ms}}^2 \le C_4.$$

Proof. By (9), we arrive at

$$(\widehat{X}_h(t))^2 = \left((E_\alpha(\lambda t^\alpha))\, X(0) + \mu \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(t-\tau_h(s))^{1-\alpha}} \widehat{X}_h(\tau_h(s))\, dW_s \right)^2.$$

Taking the expectation of the both sides of the above equality and using the Ito's isometry, we obtain that

$$\begin{aligned}\|\widehat{X}_h(t)\|^2_{\mathrm{ms}} &= (E_\alpha(\lambda t^\alpha))^2\, \|X(0)\|^2_{\mathrm{ms}} \\ &\quad + \mu^2 \int_0^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)\right)^2}{(t-\tau_h(s))^{2-2\alpha}} \|\widehat{X}_h(\tau_h(s))\|^2_{\mathrm{ms}}\, ds.\end{aligned}$$

Note that $\tau_h(s) \le s$ and using the monotonically decreasing of the function $E_{\alpha,\alpha}(\cdot)$ on $\mathbb{R}_-$ (see e.g. [12]), we derive

$$\begin{aligned}\|\widehat{X}_h(t)\|^2_{\mathrm{ms}} &\le (E_\alpha(\lambda t^\alpha))^2\, \|X(0)\|^2_{\mathrm{ms}} \\ &\quad + \mu^2 \int_0^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right)^2}{(t-s)^{2-2\alpha}} \|\widehat{X}_h(\tau_h(s))\|^2_{\mathrm{ms}}\, ds \\ &\le M_1 \|X(0)\|^2_{\mathrm{ms}} + \mu^2 M_2 \int_0^t \frac{\sup_{0\le r\le s} \|\widehat{X}_h(r)\|^2_{\mathrm{ms}}}{(t-s)^{2-2\alpha}}\, ds.\end{aligned}$$

Let $m_t := \sup_{0\le s\le t} \|\widehat{X}_h(s)\|^2_{\mathrm{ms}}$. Then,

$$m_t \le M_1 \|X(0)\|^2_{\mathrm{ms}} + \mu^2 M_2 \int_0^t \frac{m_s}{(t-s)^{2-2\alpha}}\, ds.$$

Applying the Gronwall's inequality for fractional differential equations, see e.g. [1, Lemma 6.19], we arrive at

$$\sup_{0\le s\le t} \|\widehat{X}_h(s)\|^2_{\mathrm{ms}} \le M_1 \|X(0)\|^2_{\mathrm{ms}} E_{2\alpha-1}\left(\mu^2 M_2 \Gamma(2\alpha-1) t^{2\alpha-1}\right),$$

which completes the proof. □

Secondly, we establish an upper bound on the difference $\|\widehat{X}_h(t) - \widehat{X}_h(\widetilde{t})\|^2_{\mathrm{ms}}$ in terms of $|t-\widetilde{t}|^{2\alpha-1}$.

Lemma 8. *Let*

$$M_3 := \left(\max_{x\in[0,T^\alpha]} \partial_x E_\alpha(\lambda x)\right)^2, M_4 := \left(\max_{x\in[0,T^\alpha]} \partial_x E_{\alpha,\alpha}(\lambda x)\right)^2$$

and

$$C_5 := 4M_3 T \|X(0)\|^2_{\mathrm{ms}} + \frac{8\mu^2 M_2 C_4 + 4\mu^2 M_4 C_4 T^{2\alpha-1}}{2\alpha-1}, \tag{23}$$

where C_4 is given as in (22). Then, for all $h>0$ and $t, \widetilde{t} \in [0,T]$ we have

$$\|\widehat{X}_h(t) - \widehat{X}_h(\widetilde{t})\|^2_{\mathrm{ms}} \le C_5 |t-\widetilde{t}|^{2\alpha-1}.$$

Proof. Choose and fix $t, \widetilde{t} \in [0,T]$ with $t > \widetilde{t}$. By (9), we have

$$\begin{aligned}&\widehat{X}_h(t) - \widehat{X}_h(\widetilde{t}) \\ &= (E_\alpha(\lambda t^\alpha) - E_\alpha(\lambda \widetilde{t}^\alpha))X(0) + \mu \int_{\widetilde{t}}^t \frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(t-\tau_h(s))^{1-\alpha}} \widehat{X}_h(\tau_h(s))\, dW_s \\ &\quad + \mu \int_0^{\widetilde{t}} \left(\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(t-\tau_h(s))^{1-\alpha}} - \frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}} \right) \widehat{X}_h(\tau_h(s))\, dW_s \\ &\quad + \mu \int_0^{\widetilde{t}} \left(\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}} - \frac{E_{\alpha,\alpha}(\lambda(\widetilde{t}-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}} \right) \widehat{X}_h(\tau_h(s))\, dW_s.\end{aligned}$$

Using the inequality $\|x+y+z+w\|^2 \le 4(\|x\|^2 + \|y\|^2 + \|z\|^2 + \|w\|^2)$ for all $x, y, z, w \in \mathbb{R}^d$ and Ito's isometry, we derive that

$$\begin{aligned}&\|\widehat{X}_h(t) - \widehat{X}_h(\widetilde{t})\|^2_{\mathrm{ms}} \\ &\le 4|E_\alpha(\lambda t^\alpha) - E_\alpha(\lambda \widetilde{t}^\alpha)|^2 \|X(0)\|^2_{\mathrm{ms}} \\ &\quad + 4\mu^2 \int_{\widetilde{t}}^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)\right)^2}{(t-\tau_h(s))^{2-2\alpha}} \|\widehat{X}_h(\tau_h(s))\|^2_{\mathrm{ms}}\, ds\end{aligned}$$

$$+4\mu^2\int_0^{\widetilde{t}}\Big(\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(t-\tau_h(s))^{1-\alpha}}-\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}}\Big)^2\|\widehat{X}_h(\tau_h(s))\|^2_{\rm ms}\,ds$$
$$+4\mu^2\int_0^{\widetilde{t}}\Big(\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}}-\frac{E_{\alpha,\alpha}(\lambda(\widetilde{t}-\tau_h(s))^\alpha)}{(\widetilde{t}-\tau_h(s))^{1-\alpha}}\Big)^2\|\widehat{X}_h(\tau_h(s))\|^2_{\rm ms}\,ds.$$

By Mean Value Theorem, Lemma 7 and the fact that $(\frac{1}{(t-\tau_h(s))^{1-\alpha}}-\frac{1}{(\widetilde{t}-\tau_h(s))^{1-\alpha}})^2\le\frac{1}{(\widetilde{t}-s)^{2-2\alpha}}-\frac{1}{(t-s)^{2-2\alpha}}$, we arrive at

$$\begin{aligned}&\|\widehat{X}_h(t)-\widehat{X}_h(\widetilde{t})\|^2_{\rm ms}\\ \le\;& 4M_3|t^\alpha-(\widetilde{t})^\alpha|^2\|X(0)\|^2_{\rm ms}+\int_{\widetilde{t}}^t\frac{4\mu^2M_2C_4}{(t-s)^{2-2\alpha}}\,ds\\ &+4\mu^2M_2C_4\int_0^{\widetilde{t}}\Big(\frac{1}{(\widetilde{t}-s)^{2-2\alpha}}-\frac{1}{(t-s)^{2-2\alpha}}\Big)\,ds\\ &+4\mu^2C_4\int_0^{\widetilde{t}}\frac{M_4\big((t-\tau_h(s))^\alpha-(\widetilde{t}-\tau_h(s))^\alpha\big)^2}{(\widetilde{t}-s)^{2-2\alpha}}\,ds.\end{aligned}$$

Since $0<\alpha<1$ it follows that $|x+y|^\alpha\le|x|^\alpha+|y|^\alpha$ for $x,y>0$. Thus we obtain that

$$\begin{aligned}&\|\widehat{X}_h(t)-\widehat{X}_h(\widetilde{t})\|^2_{\rm ms}\\ \le\;& 4M_3(t-\widetilde{t})^{2\alpha}\|X(0)\|^2_{\rm ms}+\frac{4\mu^2M_2C_4}{2\alpha-1}(t-\widetilde{t})^{2\alpha-1}\\ &+\frac{4\mu^2M_2C_4}{2\alpha-1}(t-\widetilde{t})^{2\alpha-1}+\frac{4\mu^2M_4C_4(\widetilde{t})^{2\alpha-1}}{2\alpha-1}(t-\widetilde{t})^{2\alpha}\\ \le\;&\Big(4M_3T\|X(0)\|^2_{\rm ms}+\frac{8\mu^2M_2C_4+4\mu^2M_4C_4T^{2\alpha-1}}{2\alpha-1}\Big)(t-\widetilde{t})^{2\alpha-1}.\end{aligned}$$

The proof is complete. □

To proof Theorem 3(ii), we recall the following result about an estimate of the Mittag-Leffler function.

Lemma 9. *Suppose that $\lambda>0$. Then, there exists $M(\lambda,\alpha)$ depending on λ and α such that*

$$E_\alpha(\lambda t^\alpha)\le\frac{M(\lambda,\alpha)}{\max\{1,t^\alpha\}},\quad E_{\alpha,\alpha}(\lambda t^\alpha)\le\frac{M(\lambda,\alpha)}{\max\{1,t^{2\alpha}\}}\qquad\text{for all } t\ge0. \tag{24}$$

Proof. See [2, Theorem 1.4] or [13, Theorem 2]. □

Next, we need the following preparatory lemma.

Lemma 10. *Suppose that $\mu^2\int_0^\infty\frac{(E_{\alpha,\alpha}(\lambda s^\alpha))^2}{s^{2-2\alpha}}\,ds<1$. Let $\delta\in(0,\alpha)$ be arbitrary. Then,*

$$\limsup_{t\to\infty}\mu^2\int_0^t\frac{\big(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\big)^2}{(t-s)^{2-2\alpha}}\frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}\,ds<1.$$

Proof. Since $\mu^2\int_0^\infty\frac{(E_{\alpha,\alpha}(\lambda s^\alpha))^2}{s^{2-2\alpha}}\,ds<1$, there exists $\eta\in(0,1)$ such that

$$\frac{\mu^2}{\eta^{2\delta}}\int_0^\infty\frac{\big(E_{\alpha,\alpha}(\lambda s^\alpha)\big)^2}{s^{2-2\alpha}}\,ds<1.$$

Choose and fix such a η satisfying the preceding inequality. Then,

$$\begin{aligned}&\limsup_{t\to\infty}\mu^2\int_{\eta t}^t\frac{\big(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\big)^2}{(t-s)^{2-2\alpha}}\frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}\,ds\\ \le\;&\limsup_{t\to\infty}\frac{\mu^2}{\eta^{2\delta}}\int_{\eta t}^t\frac{\big(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\big)^2}{(t-s)^{2-2\alpha}}\,ds\\ <\;&\frac{\mu^2}{\eta^{2\delta}}\int_0^\infty\frac{\big(E_{\alpha,\alpha}(\lambda u^\alpha)\big)^2}{u^{2-2\alpha}}\,ds<1.\end{aligned} \tag{25}$$

On the other hand, by virtue of Lemma 9 we have

$$\int_0^{\eta t} \frac{\left(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right)^2}{(t-s)^{2-2\alpha}} \frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}\, ds \le M(\alpha,\lambda)^2 \int_0^{\eta t} \frac{\max\{1,t^{2\delta}\}}{(t-s)^{2+2\alpha}}\, ds$$

A direct estimation yields that

$$\limsup_{t\to\infty} t^{2\delta} \int_0^{\eta t} \frac{1}{(t-s)^{2+2\alpha}}\, ds \le \limsup_{t\to\infty} t^{2\delta} \frac{\eta t}{(t-\eta t)^{2+2\alpha}} = 0,$$

which implies that

$$\limsup_{t\to\infty} \int_0^{\eta t} \frac{\left(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right)^2}{(t-s)^{2-2\alpha}} \frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}\, ds = 0.$$

This together with (25) completes the proof. □

Finally, we are now in a position to prove Theorem 3.

Proof of Theorem 3. (i) From (8) and (9) we have

$$\begin{aligned}
&\widehat{X}_h(t) - X(t)\\
&= \mu \int_0^t \Big(\frac{1}{(t-\tau_h(s))^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\Big) E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)\widehat{X}_h(\tau_h(s))\, dW_s\\
&\quad + \mu \int_0^t \Big(\frac{E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)}{(t-s)^{1-\alpha}} - \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}\Big)\widehat{X}_h(\tau_h(s))\, dW_s\\
&\quad + \mu \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}} (\widehat{X}_h(\tau_h(s)) - X(s))\, dW_s.
\end{aligned}$$

Using the inequality $\|x+y+z\|^2 \le 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$ and Ito's isometry, we derive that

$$\begin{aligned}
&\|\widehat{X}_h(t) - X(t)\|_{\rm ms}^2\\
&\le 3\mu^2 \int_0^t \Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(t-\tau_h(s))^{1-\alpha}}\Big)^2 |E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)|^2 \|\widehat{X}_h(\tau_h(s))\|_{\rm ms}^2\, ds\\
&\quad + 3\mu^2 \int_0^t \frac{1}{(t-s)^{2-2\alpha}} \Big|E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha) - E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\Big|^2 \|\widehat{X}_h(\tau_h(s))\|_{\rm ms}^2\, ds\\
&\quad + 3\mu^2 \int_0^t \frac{|E_{\alpha,\alpha}(\lambda(t-s)^\alpha)|^2}{(t-s)^{2-2\alpha}} \|\widehat{X}_h(\tau_h(s)) - X(s)\|_{\rm ms}^2\, ds.
\end{aligned}$$

Moreover, using the inequality $\left(\frac{1}{(t-\tau_n(s))^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\right)^2 \le \frac{1}{(t-s)^{2-2\alpha}} - \frac{1}{(t-\tau_n(s))^{2-2\alpha}}$ and $t - \tau_n(s) \le t + h - s$, we obtain that

$$\begin{aligned}
\int_0^t \left(\frac{1}{(t-\tau_n(s))^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\right)^2 ds &\le \int_0^t \Big(\frac{1}{(t-s)^{2-2\alpha}} - \frac{1}{(h+t-s)^{2-2\alpha}}\Big)\, ds\\
&\le \frac{h^{2\alpha-1}}{2\alpha-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&3\mu^2 \int_0^t \Big(\frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(t-\tau_h(s))^{1-\alpha}}\Big)^2 |E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)|^2 \|\widehat{X}_h(\tau_h(s))\|_{\rm ms}^2\, ds\\
&\le \frac{3\mu^2 M_2 C_4}{2\alpha-1} h^{2\alpha-1}.
\end{aligned} \tag{26}$$

By the Mean Value Theorem and Lemma 7, we arrive at

$$\begin{aligned}
&3\mu^2 \int_0^t \frac{1}{(t-s)^{2-2\alpha}} \Big|E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha) - E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\Big|^2 \|\widehat{X}_h(\tau_h(s))\|_{\rm ms}^2\, ds\\
&\le 3\mu^2 M_4 C_4 \int_0^t \frac{|(t-\tau_h(s))^\alpha - (t-s)^\alpha|^2}{(t-s)^{2-2\alpha}}\, ds\\
&\le 3\mu^2 M_4 C_4 \int_0^t \frac{|s-\tau_h(s)|^{2\alpha}}{(t-s)^{2-2\alpha}}\, ds \le \frac{3\mu^2 M_4 C_4 T^{2\alpha-1}}{2\alpha-1} h^{2\alpha}.
\end{aligned} \tag{27}$$

Moreover, in light of Lemma 8

$$\begin{aligned}\|\widehat{X}_h(\tau_h(s)) - X(s)\|^2_{\text{ms}} &\le 2\|\widehat{X}_h(\tau_h(s)) - \widehat{X}_h(s)\|^2_{\text{ms}} + 2\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}\\ &\le 2C_5|\tau_h(s) - s|^{2\alpha-1} + 2\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}\\ &\le 2C_5 h^{2\alpha-1} + 2\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}.\end{aligned}$$

Thus,

$$\begin{aligned}&3\mu^2\int_0^t \frac{|E_{\alpha,\alpha}(\lambda(t-s)^\alpha)|^2}{(t-s)^{2-2\alpha}}\|\widehat{X}_h(\tau_h(s)) - X(s)\|^2_{\text{ms}}\,ds\\ &\le 3\mu^2 M_2\left(\int_0^t \frac{2C_5h^{2\alpha-1}}{(t-s)^{2-2\alpha}}\,ds + \int_0^t \frac{2\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}}{(t-s)^{2-2\alpha}}\,ds\right)\\ &\le \frac{3\mu^2 M_2 C_5 T^{2\alpha-1}}{2\alpha-1}h^{2\alpha-1} + 6\mu^2 M_2\int_0^t \frac{\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}}{(t-s)^{2-2\alpha}}\,ds,\end{aligned}$$

which together with (26) and (27) implies that

$$\begin{aligned}\|\widehat{X}_h(t) - X(t)\|^2_{\text{ms}} \le &\left(\frac{3\mu^2M_2C_4}{2\alpha-1} + \frac{3\mu^2M_4C_4T^{2\alpha-1}h}{2\alpha-1} + \frac{3\mu^2M_2C_5T^{2\alpha-1}}{2\alpha-1}\right)\\ &\times h^{2\alpha-1} + 6\mu^2M_2\int_0^t \frac{\|\widehat{X}_h(s) - X(s)\|^2_{\text{ms}}}{(t-s)^{2-2\alpha}}\,ds.\end{aligned}$$

Applying the Gronwall's inequality for fractional differential equations, see e.g. [1, Lemma 6.19], we arrive at

$$\sup_{0\le t\le T}\|\widehat{X}_h(t) - X(t)\|^2_{\text{ms}} \le Ch^{2\alpha-1},$$

where

$$C := \left(\frac{3\mu^2M_2C_4}{2\alpha-1} + \frac{3\mu^2M_4C_4T^{2\alpha}}{2\alpha-1} + \frac{3\mu^2M_2C_5T^{2\alpha-1}}{2\alpha-1}\right)E_{2\alpha-1}(6\mu^2M_2\Gamma(2\alpha-1)T^{2\alpha-1}).$$

The proof is complete.

(ii) By (9) and using the Ito's isometry, we arrive at

$$\begin{aligned}\|\widehat{X}_h(t)\|^2_{\text{ms}} = &\,(E_\alpha(\lambda t^\alpha))^2\,\|X(0)\|^2_{\text{ms}}\\ &+\mu^2\int_0^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-\tau_h(s))^\alpha)\right)^2}{(t-\tau_h(s))^{2-2\alpha}}\|\widehat{X}_h(\tau_h(s))\|^2_{\text{ms}}\,ds.\end{aligned}$$

Note that $\tau_h(s) \le s$ and using the monotonically decreasing of the function $E_{\alpha,\alpha}(\cdot)$ on $\mathbb{R}_-$ (see e.g. [12]), we obtain that

$$\begin{aligned}\|\widehat{X}_h(t)\|^2_{\text{ms}} \le &\,(E_\alpha(\lambda t^\alpha))^2\,\|X(0)\|^2_{\text{ms}}\\ &+\mu^2\int_0^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right)^2}{(t-s)^{2-2\alpha}}\|\widehat{X}_h(\tau_h(s))\|^2_{\text{ms}}\,ds.\end{aligned}$$

By virtue of Lemma 9, there exists $M(\alpha,\lambda) > 0$ such that for any $X(0) \neq 0$

$$\frac{\|\widehat{X}_h(t)\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}} \le \frac{M(\alpha,\lambda)}{\max\{1, t^{2\alpha}\}} + \mu^2\int_0^t \frac{\left(E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right)^2}{(t-s)^{2-2\alpha}}\frac{\|\widehat{X}_h(\tau_h(s))\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}}\,ds. \tag{28}$$

Now, let

$$K := \frac{M(\alpha,\lambda)}{1-\mu^2\int_0^\infty \frac{\left(E_{\alpha,\alpha}(\lambda s^\alpha)\right)^2}{s^{2-2\alpha}}\,ds}. \tag{29}$$

Thanks to (7), $K > 0$ and we are now proving $\sup_{t\ge 0}\frac{\|\widehat{X}_h(t)\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}} < K$ by contradiction, i.e. there exists $T > 0$ being the first time for which $\frac{\|\widehat{X}_h(t)\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}} \ge K$, i.e.

$$\frac{\|\widehat{X}_h(T)\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}} = K, \quad \frac{\|\widehat{X}_h(t)\|^2_{\text{ms}}}{\|X(0)\|^2_{\text{ms}}} < K \quad \text{for } t \in [0, T) \tag{30}$$

Thus, replacing $t = T$ in (28) yields that

$$K \leq M(\alpha,\lambda) + \mu^2 K \int_0^T \frac{\left(E_{\alpha,\alpha}(\lambda(T-s)^\alpha)\right)^2}{(T-s)^{2-2\alpha}}\,ds$$
$$< M(\alpha,\lambda) + \mu^2 K \int_0^\infty \frac{\left(E_{\alpha,\alpha}(\lambda u^\alpha)\right)^2}{u^{2-2\alpha}}\,du,$$

which contradicts to the definition of K as in (29). Let $\delta \in (0,\alpha)$ be arbitrary and then to conclude the proof we need to show that $\lim_{t\to\infty} t^\delta \|\widehat{X}_h(t)\|_{\rm ms} = 0$. In fact, choose and fix an arbitrary $\widehat{\delta} \in (\delta,\alpha)$ it is sufficient to show that

$$\limsup_{t\to\infty} t^{2\widehat{\delta}} \frac{\|\widehat{X}_h(t)\|^2_{\rm ms}}{\|X(0)\|^2_{\rm ms}} < \infty.$$

Suppose the contrary. Then, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that $\gamma_n := \max\{1, t_n^{2\widehat{\delta}}\} \frac{\|\widehat{X}_h(t_n)\|^2_{\rm ms}}{\|X(0)\|^2_{\rm ms}}$ satisfies

$$\gamma_n = \max\left\{\max\{1,t^{2\widehat{\delta}}\}\frac{\|\widehat{X}_h(t)\|^2_{\rm ms}}{\|X(0)\|^2_{\rm ms}} : t\in[0,t_n]\right\} \quad \text{for } n \in \mathbb{N} \tag{31}$$

and $\lim_{n\to\infty}\gamma_n = \infty$. Replacing $t = t_n$ in (28) yields that

$$\frac{\|\widehat{X}_h(t_n)\|^2_{\rm ms}}{\|X(0)\|^2_{\rm ms}} \leq \frac{M(\alpha,\lambda)}{\max\{1,t_n^{2\alpha}\}} + \mu^2 \int_0^{t_n} \frac{\left(E_{\alpha,\alpha}(\lambda(t_n-s)^\alpha)\right)^2}{(t_n-s)^{2-2\alpha}} \frac{\|\widehat{X}_h(\tau_h(s))\|^2_{\rm ms}}{\|X(0)\|^2_{\rm ms}}\,ds,$$

which together with (31) implies that

$$\gamma_n \leq \frac{M(\alpha,\lambda)\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,t_n^{2\alpha}\}} + \gamma_n\mu^2 \int_0^{t_n} \frac{\left(E_{\alpha,\alpha}(\lambda(t_n-s)^\alpha)\right)^2}{(t_n-s)^{2-2\alpha}} \frac{\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,s^{2\widehat{\delta}}\}}\,ds.$$

Thus,

$$\gamma_n\left(1-\mu^2\int_0^{t_n} \frac{\left(E_{\alpha,\alpha}(\lambda(t_n-s)^\alpha)\right)^2}{(t_n-s)^{2-2\alpha}} \frac{\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,s^{2\widehat{\delta}}\}}\,ds\right) \leq \frac{M(\alpha,\lambda)\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,t_n^{2\alpha}\}}.$$

Since $\widehat{\delta} < \alpha$ it follows that

$$\limsup_{n\to\infty} \frac{M(\alpha,\lambda)\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,t_n^{2\alpha}\}} = 0.$$

However, by virtue of Lemma 10 and $\lim_{n\to\infty}\gamma_n = \infty$

$$\limsup_{n\to\infty} \gamma_n\left(1-\mu^2\int_0^{t_n} \frac{\left(E_{\alpha,\alpha}(\lambda(t_n-s)^\alpha)\right)^2}{(t_n-s)^{2-2\alpha}} \frac{\max\{1,t_n^{2\widehat{\delta}}\}}{\max\{1,s^{2\widehat{\delta}}\}}\,ds\right) = \infty,$$

which leads to a contradiction. The proof is complete. □

5. Examples

In this section, we study a simple Caputo SFDE with additive noise. For this kind of system, we have explicit formulas for the solution and the numerical solution by using the Euler–Maruyama scheme. Then, we arrive at that the convergence rate of the scheme is $\alpha - \frac{1}{2}$, i.e. the rate in Theorem 1 is optimal. Note that for the stochastic differential equation with additive noise the convergence rate of the Euler–Maruyama scheme is usually equal to 1 (see [8] and [14]). Then the convergence rate which we find in the following example indicates a new aspect in numerical computation of stochastic fractional systems.

Example 11. Consider a simple scalar Caputo SFDE on interval [0, 1] of the form

$${}^C D^\alpha_{0+} X(t) = 1\,\frac{dW_t}{dt}.$$

Then, the exact solution for $X(0) = 0$ is given by

$$X(1) = \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{1}{(1-s)^{1-\alpha}}\,dW_s.$$

Table 1
Rates of convergence for a simple scalar SFDE.

	$\alpha = 0.75$		$\alpha = 0.6$		$\alpha = 0.9$	
	$\sqrt{e_n}$	$\log_2 \frac{\sqrt{e_n}}{\sqrt{e_{2n}}}$	$\sqrt{e_n}$	$\log_2 \frac{\sqrt{e_n}}{\sqrt{e_{2n}}}$	$\sqrt{e_n}$	$\log_2 \frac{\sqrt{e_n}}{\sqrt{e_{2n}}}$
$n = 128$	0.1744	0.2499	1.0094	0.0999	0.0249	0.3998
$n = 256$	0.1467	0.2499	0.9418	0.1000	0.0188	0.3999
$n = 512$	0.1233	0.2499	0.8787	0.1000	0.0143	0.3999
$n = 1024$	0.1037	0.2499	0.8199	0.1000	0.0108	0.3999
$n = 2048$	0.0872	0.2500	0.7650	0.1000	0.0082	0.3999
$n = 4096$	0.0733	0.2500	0.7137	0.1000	0.0062	0.3999
$\alpha - 1/2$	0.25		0.1		0.4	

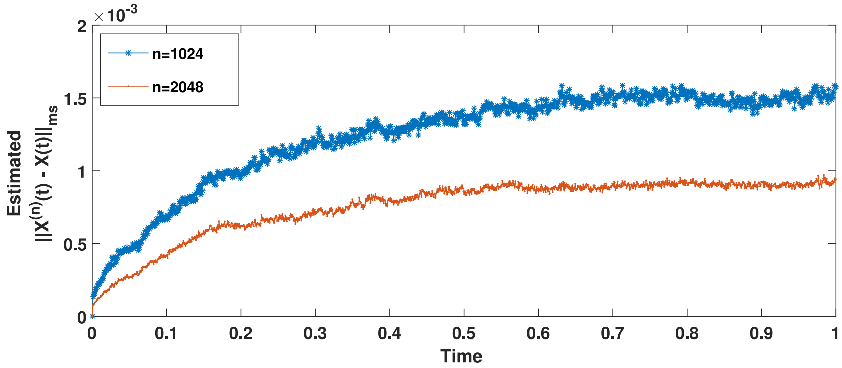

Fig. 1. Estimated errors for Eq. (32).

Meanwhile, by (4) the numerical solution $X^{(n)}$ is given by

$$X^{(n)}(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(1-\tau_n(s))^{1-\alpha}}\, dW_s.$$

Then, by Ito's isometry we arrive at

$$\|X(1) - X^{(n)}(1)\|_{\mathrm{ms}}^2 = \frac{1}{\Gamma^2(\alpha)} \int_0^1 \left(\frac{1}{(1-s)^{1-\alpha}} - \frac{1}{(1-\tau_n(s))^{1-\alpha}} \right)^2 ds.$$

Let us denote $e_n = \|X(1) - X^{(n)}(1)\|_{\mathrm{ms}}^2$. The rate of convergence of our scheme for this equation will then be estimated by $\log_2 \frac{\sqrt{e_n}}{\sqrt{e_{2n}}}$ for some large values of n. Table 1 reveals that those rates are $\alpha - 1/2$ for various values of α, i.e. $\sqrt{e_n} \sim n^{\alpha - \frac{1}{2}}$ as $n \to \infty$.

In order to get a further insight into the performance of the proposed method, we investigate a nonlinear scalar Caputo SFDE whose drift term and diffusion term satisfy the conditions (H1), (H2) and (H3).

Example 12. Consider the following autonomous Caputo SFDE with $\alpha = \frac{3}{4}$

$$^{C}D_{0+}^{\alpha} X(t) = \cos(X(t)) + \sin(X(t)) \frac{dW_t}{dt}. \tag{32}$$

The computation process is as follows. We first generate 1000 Brownian motions, for each motion, we use our scheme to compute the approximate solutions with $n, 2n, 4n$ discretized points for $n = 1024$. The errors of $X^{(n)}$ and $X^{(2n)}$ are estimated by $\|X^{(n)}(t) - X^{(4n)}(t)\|_{\mathrm{ms}}$ and $\|X^{(2n)}(t) - X^{(4n)}(t)\|_{\mathrm{ms}}$, respectively. These errors in turn are computed by taking the average of 1000 computed solution errors. The outcome of this process is presented in Fig. 1.

Conclusion

In this paper, we have established the convergence of the Euler–Maruyama type scheme for Caputo stochastic fractional differential equations. The convergence rate of this scheme is given explicitly. Next, we investigated the convergence and stability of an exponential Euler–Maruyama scheme for bilinear scalar Caputo SFDEs.

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References

[1] K. Diethelm, The Analysis of Fractional Differential Equations. An Application-Oriented Exposition using Differential Operators of Caputo Type, in: Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, 2010.
[2] I. Podlubny, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications, Academic Press, Inc., San Diego, CA, 1999.
[3] T.S. Doan, P.T. Huong, P.E. Kloeden, H.T. Tuan, Asymptotic separation between solutions of Caputo fractional stochastic differential equations, Stoch. Anal. Appl. 36 (4) (2018) 654–664.
[4] Y. Wang, X. Yejuan, Peter E. Kloeden, Asymptotic behavior of stochastic lattice systems with a Caputo fractional time derivative, Nonlinear Anal. 135 (2016) 205–222.
[5] R. Sakthivel, P. Revathi, Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations, Nonlinear Anal. Theory Methods Appl. 81 (2013) 70–86.
[6] P.T. Anh, T.S. Doan, P.T. Huong, A variation of constant formula for Caputo fractional stochastic differential equations, Statist. Probab. Lett. 145 (2019) 351–358.
[7] Y. Saito, T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal. 33 (6) (1996) 2254–2267.
[8] Peter E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, in: Applications of Mathematics, vol. 23, Springer-Verlag, New York, Berlin, 1992.
[9] X. Zhang, Euler schemes and large deviations for stochastic Volterra equations with singular kernels, J. Differential Equations 244 (2008) 2226–2250.
[10] Z. Wang, Existence and uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients, Statist. Probab. Lett. 78 (2008) 1062–1071.
[11] H.T. Tuan, On the asymptotic behavior of solutions to time-fractional elliptic equations driven a multiplicative white noise, arXiv:2002.06054.
[12] W.R. Schneider, Completely monotone generalized Mittag–Leffler functions, Expo. Math. 14 (1996) 3–16.
[13] N.D. Cong, T.S. Doan, H.T. Tuan, Asymptotic stability of linear fractional systems with constant coefficients and small time dependent perturbations, Vietnam J. Math. 46 (2018) 665–680.
[14] G.N. Milstein, M.V. Tretyakov, Stochastic Numerics for Mathematical Physics, Springer-Verlag Berlin Heidelberg, 2004.