Journal of Global Optimization
https://doi.org/10.1007/s10898-020-00927-7

Geoffrion's proper efficiency in linear fractional vector optimization with unbounded constraint sets

N. T. T. Huong[1] · J.-C. Yao[2] · N. D. Yen[3]

Received: 8 July 2019 / Accepted: 13 July 2020

Abstract

Choo (Oper Res 32:216–220, 1984) has proved that any efficient solution of a linear fractional vector optimization problem with a *bounded* constraint set is properly efficient in the sense of Geoffrion. This paper studies Geoffrion's properness of the efficient solutions of linear fractional vector optimization problems with *unbounded* constraint sets. By examples, we show that there exist linear fractional vector optimization problems with the efficient solution set being a proper subset of the unbounded constraint set, which have improperly efficient solutions. Then, we establish verifiable sufficient conditions for an efficient solution of a linear fractional vector optimization to be a Geoffrion properly efficient solution by using the recession cone of the constraint set. For bicriteria problems, it is enough to employ a system of two regularity conditions. If the number of criteria exceeds two, a third regularity condition must be added to the system. The obtained results complement the above-mentioned remarkable theorem of Choo and are analyzed through several interesting examples, including those given by Hoa et al. (J Ind Manag Optim 1:477–486, 2005).

Keywords Linear fractional vector optimization · Efficient solution · Gain-to-loss ratio · Geoffrion's properly efficient solution · Unbounded constraint set · Direction of recession · Regularity condition

Mathematics Subject Classification 90C29 · 90C32 · 49K30 · 49N60

1 Introduction

Linear fractional vector optimization problems (LFVOPs) have many applications in management science. Due to their remarkable properties and theoretical importance, these problems

The authors were supported by National Foundation for Science and Technology Development (Vietnam) under Grant No. 101.01-2018.306, Le Quy Don Technical University (Vietnam), Grant MOST 105-2115-M-039-002-MY3 (Taiwan), and the Vietnam Institute for Advanced Study in Mathematics (VIASM, Vietnam).

✉ N. D. Yen
ndyen@math.ac.vn

Extended author information available on the last page of the article

have been studied intensively. Steuer [30, p. 337] has observed that linear fractional (ratio) criteria are frequently encountered in finance: min{debt-to-equity ratio}, max{return to investment}, max{output per employee}, min{actual cost to standard cost} (in Corporate Planning); min{risk-assets to capital}, max{actual capital to required capital}, min{foreign loans to total loans}, min{residental mortgages to total mortgages} (in Bank Balance Sheet Management). Fractional objectives also occur in other areas of management: transportation management, education management, and medicine management (see, e.g., [30]).

LFVOPs constitute a special class of multiobjective fractional optimization problems. Note that, among the 520 articles cited by Stancu-Minasian in his ninth bibliography of fractional programming [29], 38 papers are devoted to multiobjective fractional programming.

Connections of LFVOPs with *monotone affine vector variational inequalities* were firstly recognized by Yen and Phuong [32].

Topological properties of the solution sets of LFVOPs and monotone affine vector variational inequalities have been studied by Choo and Atkins [8,9], Benoist [1,2], Yen and Phuong [32], Hoa et al. [17–19], Huong et al. [20,21], and other authors. In particular, the authors of [17] showed that *for any positive integer m, there exists a LFVOP with m objective criteria which has exactly m connected components in the Pareto efficient solution set and in the weak efficient solution set.* Recently, based on some theorems from real algebraic geometry [4], we have been able to prove in [21] that *the number of the connected components in the Pareto efficient solution set (resp., in the weak efficient solution set) of any LFVOP is finite.* From a fundamental theorem of Robinson [26, Theorem 2] on stability of monotone affine variational inequalities, Yen and Yao [34, Section 5] have derived several results on solution stability and topological properties of the solution sets of LFVOPs. Numerical methods for solving LFVOPs can be found in Steuer [30] and Malivert [25]. The interested readers are referred to the survey paper of Yen [31] for more information about linear fractional and convex quadratic vector optimization problems.

Very recently, Yen and Yang [33] have initiated a study on infinite-dimensional LFVOPs and infinite-dimensional convex quadratic vector optimization problems via affine variational inequalities on normed spaces.

Open questions concerning qualitative properties of finite-dimensional LFVOPs remain. For instance, *one still cannot prove or disprove the conjecture saying that the number of the connected components in the Pareto efficient solution set (resp., in the weak efficient solution set) of a LFVOP with m objective criteria does not exceed m.*

Considering a general vector optimization problem, Geoffrion [13] noticed that certain efficient solutions show unfavorable properties concerning *the ratio of the rate of gain in one cost over the corresponding rate of loss in another cost*. To clarify this kind of pathological behavior of the efficient solutions, he introduced [13, pp. 618–619] the notion *properly efficient solution*. Later, several extensions of Geoffrion's concept of proper efficiency were given by Borwein [5], Benson [3], Henig [16], and other scholars. Discussions on the relationships among different types of properly efficient solutions can be found in the papers of Benson [3] and of Guerraggio et al. [15].

Probably the paper by Choo [7] is the first work addressing Geoffrion's concept of proper efficiency for LFVOPs. We are indebted to one referee of an earlier version of the present paper for making us familiar with the significant work [7], from which we have learned that any efficient solution of a linear fractional vector optimization with *bounded* constraint set is properly efficient in the sense of Geoffrion. The proof of this remarkable theorem is enough complicated. To be more precise, Choo has used several technical lemmas and original arguments based on the necessary and sufficient conditions for a feasible point of a LFVOP to be an efficient solution, which will be recalled in Sect. 2 below.

Chew and Choo [6, Example 3.6] showed that there exits a LFVOP with an unbounded constraint set where none of the efficient solutions is properly efficient in the sense of Geoffrion. Note that the efficient solution set of the problem coincides with the constraint set. Apart from the just-cited example, up to now one still does not have any further information about the difference between the efficient solution set and the Geoffrion properly efficient solution set of a LFVOP with an unbounded constraint set.

By constructing suitable examples, we will show that there exist LFVOPs whose efficient solution sets do not coincide with the unbounded constraint sets, which have improperly efficient solutions (see Examples 2.6 and 4.7 below). Our main results give sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to belong to its properly efficient solution set. These conditions require certain regularity conditions, where the recession cone of the constraint set appears in a natural way. For bicriteria problems, it is enough to employ a system of two regularity conditions. If the number of criteria exceeds two, a third regularity condition must be added to the system. The obtained results complement the above-mentioned remarkable theorem of Choo and are analyzed through several interesting examples, including those given by Hoa et al. [17].

The variational inequality characterization of the efficient solutions of LFVOPs, which can be interpreted [32] as the approach to LFVOPs via the concept of *vector variational inequality* (VVI for brevity), is the starting point of this research. In Proposition 4.2 below, by using this characterization, we are able to prove that if the denominators of the objective functions are all the same, then every efficient solution is a Geoffrion's properly efficient solution. The relations between the Pareto solutions of a VVI and the proper efficiency have been explored by Crespi [10]. Since the Klinger properly efficient solution set of a LFVOP coincides with its efficient solution set, from Theorems 5–7 of [10] one recovers the known result [32, Remark 2] saying that the efficient solution set of LFVOP coincides with the Pareto solution set of the corresponding affine VVI. Furthermore, from [13, Theorem 1] and [10, Theorem 1] it follows that the Hurwicz properly efficient solution set of a LFVOP is a subset of its Geoffrion properly efficient solution set. Applied to a LFVOP, Theorem 11 of [10] says that if all the objective functions of the problem are convex on the whole space (this happens if and only if all the functions are affine), then the Benson properly efficient solution set of the given LFVOP coincides with its Hurwicz properly efficient solution set. As the former solution set is equal to the Geoffrion properly efficient solution set, this result follows from [13, Theorem 2]. Thus, the results of the present paper cannot be derived from the results of Crespi [10].

The contents of the rest of the paper are as follows. Section 2 recalls some definitions, auxiliary results, and presents a pathological bicriteria LFVOP which has improperly efficient solutions. Section 3 establishes the main results. They are analyzed in Sect. 4 where, among other things, a three-criteria LFVOP having improperly efficient solutions is given. The last section is devoted to some remarks and open questions.

2 Preliminaries

For x^1, x^2, x in $\mathbb{R}^n$, the scalar product of the first two vectors and the Euclidean norm of the third one are denoted, respectively, by $\langle x^1, x^2\rangle$ and $\|x\|$. Vectors in $\mathbb{R}^n$ are interpreted as columns of real numbers in matrix calculations but, for simplicity, sometimes they will be described by rows of real numbers. The transpose of a matrix A is denoted by A^T. By $\mathbb{N}$ we denote the set of the positive integers.

Consider *linear fractional functions* $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, of the form

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i},$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let K be a *polyhedral convex set*, i.e., there exist $p \in \mathbb{N}$, a matrix $C = (c_{ij}) \in \mathbb{R}^{p\times n}$, and a vector $d = (d_i) \in \mathbb{R}^p$ such that $K = \{x \in \mathbb{R}^n : Cx \leq d\}$. Our standing condition is that $b_i^T x + \beta_i > 0$ for all $i \in I$ and $x \in K$, where $I := \{1, \ldots, m\}$. Put $f(x) = (f_1(x), \ldots, f_m(x))$ and let

$$\Omega = \{x \in \mathbb{R}^n : b_i^T x + \beta_i > 0, \ \forall i \in I\}.$$

Clearly, Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. The *linear fractional vector optimization problem* (LFVOP) given by f and K is formally written as

$$\text{(VP)} \qquad \text{Minimize } f(x) \text{ subject to } x \in K.$$

Definition 2.1 A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathbb{R}^m_+\setminus\{0\}\big) = \emptyset$, where $\mathbb{R}^m_+$ denotes the nonnegative orthant in $\mathbb{R}^m$. One calls $x \in K$ a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\operatorname{int}\mathbb{R}^m_+\big) = \emptyset$, where $\operatorname{int}\mathbb{R}^m_+$ abbreviates the topological interior of $\mathbb{R}^m_+$.

The efficient solution set (resp., the weakly efficient solution set) of (VP) are denoted, respectively, by E and E^w. According to [8,25] (see also [23, Theorem 8.1]), *for any $x \in K$, one has $x \in E$ (resp., $x \in E^w$) if and only if there exists a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{int}\mathbb{R}^m_+$ (resp., $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+\setminus\{0\}$) such that*

$$\left\langle \sum_{i=1}^m \xi_i \left[(b_i^T x + \beta_i)a_i - (a_i^T x + \alpha_i)b_i \right], y - x \right\rangle \geq 0, \quad \forall y \in K. \tag{2.1}$$

If $b_i = 0$ and $\beta_i = 1$ for all $i \in I$, then (VP) coincides with the classical *multiobjective linear optimization problem* (see Luc [24] and the references therein). By the above optimality conditions, for any $x \in K$, one has $x \in E$ (resp., $x \in E^w$) if and only if there exists a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{int}\mathbb{R}^m_+$ (resp., a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+\setminus\{0\}$) such that

$$\left\langle \sum_{i=1}^m \xi_i a_i, y - x \right\rangle \geq 0, \quad \forall y \in K. \tag{2.2}$$

During the last four decades, LFVOPs have attracted a lot of attention from researchers; see [7–9,17–22,25,30–34], Chapter 8 of [23], and the references therein. By the results of Choo and Atkins [9], Benoist [1], Yen and Phuong [32], one knows that if K bounded, then both solution sets E and E^w of (VP) are connected. Later, Hoa et al. [17] showed that if K is noncompact, then both solution sets E and E^w of (VP) where $m = n$ can have m connected components. Stability properties of (VP) with f and K being subject to perturbations can be found in [34]. Recently, by using some tools from real algebraic geometry, Huong et al. [21, Theorem 4.5] have proved that both E and E^w are semi-algebraic sets in $\mathbb{R}^n$; hence they have finitely many connected components. However, as far as we know, the proper efficiency of the solutions of (VP) has been studied just in the papers by Choo [7], Chew and Choo [6].

There are several notions of proper efficiency in vector optimization. The most fundamental ones have been suggested by Geoffrion [13], Borwein [5], Benson [3], and Henig [16]. It is worthy to stress that the notion of properly efficient solution proposed by Geoffrion [13] in 1968 has a clear practical meaning in the form of a *gain-to-loss ratio* [13, p. 624] and it is the starting point for all subsequent studies on proper efficiency. Applied to (VP), Geoffrion's definition of properly efficient solution can be formulated as follows.

Definition 2.2 (See [13, p. 618]) One says that $\bar{x} \in E$ is a *Geoffrion's properly efficient solution* of *(VP)* if there exists a scalar $M > 0$ such that, for each $i \in I$, whenever $x \in K$ and $f_i(x) < f_i(\bar{x})$ one can find an index $j \in I$ such that $f_j(x) > f_j(\bar{x})$ and $A_{i,j}(\bar{x}, x) \leq M$ with $A_{i,j}(\bar{x}, x) := \dfrac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})}$.

Geoffrion's efficient solution set of (VP) is denoted by E^{Ge}. Geoffrion [13, p. 619] noticed the following: *For any* $\bar{x} \in E$, $\bar{x} \notin E^{Ge}$ *if and only if for every scalar* $M > 0$ *there exist* $x \in K$ *and* $i \in I$ *with* $f_i(x) < f_i(\bar{x})$ *such that, for all* $j \in I$ *satisfying* $f_j(x) > f_j(\bar{x})$, *one has* $A_{i,j}(\bar{x}, x) > M$. This observation was made in [13] for general vector optimization problems which may not belong to the class of LFVOPs.

Remark 2.3 The quantity $A_{i,j}(\bar{x}, x)$ given in Definition 2.2 is a gain-to-loss ratio. Namely, it is the ratio of *the gain* $f_i(\bar{x}) - f_i(x) > 0$ of the plan x over the plan $\bar{x}$ in the ith criterion by *the loss* $f_j(x) - f_j(\bar{x}) > 0$ of x over $\bar{x}$ in the jth criterion.

Remark 2.4 If (VP) is a *linear vector optimization problem* (that is, $b_i = 0$ and $\beta_i = 1$ for all $i \in I$), then $E^{Ge} = E$. This well-known result can be proved easily by using Theorem 1 from [13] and the optimality condition (2.2). Indeed, if $x \in E$, then there exists a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{int} \mathbb{R}^m_+$ such that (2.2) holds. So,

$$\left\langle \sum_{i=1}^{m} \xi_i a_i, y \right\rangle \geq \left\langle \sum_{i=1}^{m} \xi_i a_i, x \right\rangle, \quad \forall y \in K.$$

Since $f_i(x) = a_i^T x + \alpha_i$ for all $i \in I$, this implies that

$$\sum_{i=1}^{m} \xi_i f_i(y) \geq \sum_{i=1}^{m} \xi_i f_i(x), \quad \forall y \in K.$$

Therefore, by [13, Theorem 1] one has $x \in E^{Ge}$. As the inclusion $E^{Ge} \subset E$ follows from Definition 2.2, we have thus proved that $E^{Ge} = E$.

As shown in the following example and some examples in Sect. 4, the equality $E^{Ge} = E$ still hods for many LFVOPs, which do not belong to the class of linear vector optimization problems.

Example 2.5 Consider problem (VP) with $m = 2$, $n = 1$, $K = \left\{x \in \mathbb{R} \,:\, x \geq \frac{1}{2}\right\}$, $f_1(x) = x^{-1}$ and $f_2(x) = x$ for all $x \in K$. It is easy to verify that $E = \left[\frac{1}{2}, +\infty\right)$. We will show that $E^{Ge} = E$. Take an arbitrary point $\bar{x} \in E$ and put $M = \max\left\{\bar{x}^{-2}, \bar{x}^2\right\}$. If x belongs to the set $\{x \in K \,:\, x > \bar{x}\}$, then for $i = 1$ and $j = 2$ one has

$$A_{i,j}(\bar{x}, x) = \frac{\bar{x}^{-1} - x^{-1}}{x - \bar{x}} = \frac{1}{x\bar{x}} \leq \frac{1}{\bar{x}^2} \leq M.$$

Similarly, if x belongs to the set $\{x \in K \,:\, x < \bar{x}\}$, then for $i = 2$ and $j = 1$ one has

$$A_{i,j}(\bar{x}, x) = \frac{\bar{x} - x}{x^{-1} - \bar{x}^{-1}} = x\bar{x} \leq \bar{x}^2 \leq M.$$

This proves that $\bar{x} \in E^{Ge}$; hence $E^{Ge} = E$.

In addition to the example of Chew and Choo [6] cited in Sect. 1, the next example signifies the existence of pathological LFVOPs which have Geoffrion's improperly efficient solutions.

Example 2.6 Consider the problem (VP) with

$$\begin{aligned} K &= \left\{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \ge 0,\ x_2 \ge 0\right\}, \\ f_1(x) &= -x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}. \end{aligned}$$

Using the necessary and sufficient condition for the efficiency in (2.1), one can show that $E = \{(x_1, 0) \,:\, x_1 \ge 0\}$. Select any vector $\bar{x} = (\alpha, 0)$, $\alpha \ge 0$, from E. We will have $\bar{x} \notin E^{Ge}$, if *for every scalar* $M > 0$ *there exist* $x \in K$ and $i \in I$ *with* $f_i(x) < f_i(\bar{x})$ *such that, for all* $j \in I$ *satisfying* $f_j(x) > f_j(\bar{x})$,

$$A_{i,j}(\bar{x}, x) = \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} > M.$$

Given any $M > 0$, we choose $i = 1$ and select a point $x = (x_1, x_2) \in K$ with $x_1 \ge 0$, $x_2 > 0$, $x_1 + x_2 + 1 > M$. Then, $f_i(x) < f_i(\bar{x})$ and the unique index $j \in I$ satisfying $f_j(x) > f_j(\bar{x})$ is $j = 2$. We have

$$\begin{aligned} A_{1,2}(\bar{x}, x) &= \frac{0 - (-x_2)}{\dfrac{x_2}{x_1 + x_2 + 1} - \dfrac{0}{\alpha + 1}} \\ &= x_1 + x_2 + 1 > M. \end{aligned}$$

Thus, all the efficient points are improperly efficient in the sense of Geoffrion. In other words, $E^{Ge} = \emptyset$, while E is unbounded.

In the sequel, to establish verifiable sufficient conditions for a point $\bar{x} \in E$ to belong to E^{Ge}, we will need the forthcoming lemma.

Lemma 2.7 (See, e.g., [25] and [23, Lemma 8.1]) *Let* $\varphi(x) = \dfrac{a^T x + \alpha}{b^T x + \beta}$ *be a linear fractional function defined by* $a, b \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$. *Suppose that* $b^T x + \beta \neq 0$ *for every* $x \in K_0$, *where* $K_0 \subset \mathbb{R}^n$ *is an arbitrary convex set. Then, one has*

$$\varphi(y) - \varphi(x) = \frac{b^T x + \beta}{b^T y + \beta} \langle \nabla\varphi(x), y - x \rangle, \tag{2.3}$$

for any $x, y \in K_0$, *where* $\nabla\varphi(x)$ *denotes the Fréchet derivative of* φ *at* x.

The connectedness of K_0 and the condition $b^T x + \beta \neq 0$ for every $x \in K_0$ imply that either $b^T x + \beta > 0$ for all $x \in K_0$, or $b^T x + \beta < 0$ for all $x \in K_0$. Hence, for any $x, y \in K_0$, one has $\dfrac{b^T x + \beta}{b^T y + \beta} > 0$. Given vectors $x, y \in K_0$ with $x \neq y$, we consider two points from the line segment $[x, y]$:

$$z_t = x + t(y - x), \quad z_{t'} = x + t'(y - x) \quad (t \in [0, 1],\ t' \in [0, t)).$$

By (2.3) we can assert that

(i) If $\langle \nabla\varphi(x), y - x \rangle > 0$, then $\varphi(z_{t'}) < \varphi(z_t)$ for every $t' \in [0, t)$;
(ii) If $\langle \nabla\varphi(x), y - x \rangle < 0$, then $\varphi(z_{t'}) > \varphi(z_t)$ for every $t' \in [0, t)$;
(iii) If $\langle \nabla\varphi(x), y - x \rangle = 0$, then $\varphi(z_{t'}) = \varphi(z_t)$ for every $t' \in [0, t)$.

This shows that φ is monotonic on every line segment or ray contained in K_0.

Remark 2.8 The monotonicity of a linear fractional function on every line segment or ray contained in its effective domain has some connection with the concept of increasing convex-along-rays functions [28, Definition 2.1], which is defined for functions from $\mathbb{R}^n_+$ to $\mathbb{R}$. This concept is useful for global optimality conditions for a class of nonconvex optimization problems [28, Section 4]. Note that a linear fractional function defined on $\mathbb{R}^n_+$ need not to be an increasing convex-along-rays function. Crepsi et al. [11] showed that star-shaped set and increasingly along rays functions have interesting applications in Minty variational inequalities. In a subsequent paper [12, Definition 4.1], they showed how the concept of increasingly along rays function [11, Definition 3] can be extended for vector functions.

To deal with LFVOPs having unbounded constraint sets, we will use the notion of recession cone. Recall [27, p. 61] that a nonzero vector $v \in \mathbb{R}^n$ is said to be a *direction of recession* of a nonempty convex set $D \subset \mathbb{R}^n$ if $x + tv \in D$ for every $t \geq 0$ and every $x \in D$. The set composed by $0 \in \mathbb{R}^n$ and all the directions $v \in \mathbb{R}^n \setminus \{0\}$ satisfying the last condition, is called the *recession cone* of D and denoted by 0^+D. If D is closed and convex, then

$$0^+D = \{v \in \mathbb{R}^n \ : \ \exists x \in \Omega \text{ s.t. } x + tv \in D \text{ for all } t > 0\}. \tag{2.4}$$

The recession cone of a polyhedral convex set can be easily computed by using the next lemma, which can be proved by a direct verification.

Lemma 2.9 *If* $K = \left\{x \in \mathbb{R}^n \ : \ Cx \leq d\right\}$ *with* $C \in \mathbb{R}^{p \times n}$ *and* $d \in \mathbb{R}^p$, *and* K *is nonempty, then* $0^+K = \left\{v \in \mathbb{R}^n \ : \ Cv \leq 0\right\}$.

We will need the following lemma, whose elementary proof is given for the sake of completeness.

Lemma 2.10 *Let* C *be a closed and convex set in* $\mathbb{R}^n$, $\bar{x} \in C$, *and let* $\{x^k\}$ *be a sequence in* $C \setminus \{\bar{x}\}$ *with* $\lim\limits_{k\to\infty} \|x^k\| = +\infty$. *If* $\lim\limits_{k\to\infty} \dfrac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = v$, *then* $v \in 0^+C$.

Proof By (2.4), to obtain the desired conclusion we need to show that $\bar{x} + tv \in C$ for an arbitrarily given $t > 0$. Choose k_1 such that $\|x^k - \bar{x}\| > t$ for all $k \geq k_1$. Then, by the convexity of C, vector

$$\bar{x} + t\,\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = \frac{\|x^k - \bar{x}\| - t}{\|x^k - \bar{x}\|}\,\bar{x} + \frac{t}{\|x^k - \bar{x}\|}\,x^k$$

belongs to C for every $k \geq k_1$. Passing the expression $\bar{x} + t\,\dfrac{x^k - \bar{x}}{\|x^k - \bar{x}\|}$ to limit as $k \to \infty$ and using the closedness of C, one gets $\bar{x} + tv \in C$. $\square$

3 Sufficient conditions for Geoffrion's proper efficiency

First, we study Geoffrion's proper efficiency for bicriteria LFVOPs.

Theorem 3.1 *Suppose that* $m = 2$ *and* $\bar{x} \in E$. *If* the first regularity condition

$$\begin{cases} \textit{there exist no } (i, j) \in I^2,\ j \neq i,\ \textit{and } v \in (0^+K)\setminus\{0\} \textit{ with} \\ \langle \nabla f_i(\bar{x}), v\rangle = 0 \textit{ and } \left\langle \nabla f_j(\bar{x}), v\right\rangle = 0 \end{cases} \tag{3.1}$$

and the second regularity condition

$$\begin{cases} \textit{there exist no } (i,j) \in I^2,\ j \neq i,\ \textit{and } v \in (0^+K)\setminus\{0\} \textit{ such that} \\ b_i^T v = 0,\quad \langle \nabla f_i(\bar{x}), v\rangle \leq 0,\quad \left\langle \nabla f_j(\bar{x}), v\right\rangle > 0 \end{cases} \tag{3.2}$$

are satisfied, then $\bar{x} \in E^{Ge}$.

Proof Suppose that (3.1) and (3.2) hold at a point $\bar{x} \in E$, but $\bar{x} \notin E^{Ge}$. Then, for every $p \in \mathbb{N}$, there exist $x^p \in K$ and $i(p) \in I$ with $f_{i(p)}(x^p) < f_{i(p)}(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x^p) > f_j(\bar{x})$, one has $A_{i(p),j}(\bar{x}, x^p) > p$ (see the observation after Definition 2.2). Since the sequence $\{i(p)\}$ has values in the finite set I, by choosing a subsequence, we may assume that $i(p) = i$ for all p, where $i \in I$ is a fixed index. For each p, as $\bar{x} \in E$ and $f_i(x^p) < f_i(\bar{x})$, there must exist an index $j(p) \in I\setminus\{i\}$ satisfying $f_{j(p)}(x^p) > f_{j(p)}(\bar{x})$. Since the sequence $\{j(p)\}$ has values in the finite set $I\setminus\{i\}$, by considering a subsequence, we may assume that $j(p) = j$ for all p, where $j \in I\setminus\{i\}$ is a fixed index. Thus,

$$A_{i,j}(\bar{x}, x^p) = \frac{f_i(\bar{x}) - f_i(x^p)}{f_j(x^p) - f_j(\bar{x})} > p \quad (\forall p \in \mathbb{N}). \tag{3.3}$$

If $\{x^p\}$ is bounded, then we can obtain a contradiction by arguing as in the proof of the main theorem of Choo [7, p. 220].

If $\{x^p\}$ is unbounded then, by selecting a subsequence (if necessary), we may assume that $\lim_{p\to\infty} \|x^p\| = +\infty$. Let $v^p = \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|}$. Without loss of generality, we can suppose that $\lim_{p\to\infty} v^p = v$, where v is a unit vector. Since K is closed and convex, applying Lemma 2.10 gives $v \in (0^+K)\setminus\{0\}$. By Lemma 2.7,

$$\begin{aligned} f_i(\bar{x}) - f_i(x^p) &= -\frac{b_i^T\bar{x} + \beta_i}{b_i^T x^p + \beta_i}\left\langle \nabla f_i(\bar{x}), x^p - \bar{x}\right\rangle \\ &= -\|x^p - \bar{x}\|\frac{b_i^T\bar{x} + \beta_i}{b_i^T x^p + \beta_i}\left\langle \nabla f_i(\bar{x}), v^p\right\rangle. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned} f_j(x^p) - f_j(\bar{x}) &= \frac{b_j^T\bar{x} + \beta_j}{b_j^T x^p + \beta_j}\left\langle \nabla f_j(\bar{x}), x^p - \bar{x}\right\rangle \\ &= \|x^p - \bar{x}\|\frac{b_j^T\bar{x} + \beta_j}{b_j^T x^p + \beta_j}\left\langle \nabla f_j(\bar{x}), v^p\right\rangle. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} A_{i,j}(\bar{x}, x^p) &= -\frac{f_i(x^p) - f_i(\bar{x})}{f_j(x^p) - f_j(\bar{x})} \\ &= -\frac{b_i^T\bar{x} + \beta_i}{b_i^T x^p + \beta_i}\cdot\frac{b_j^T x^p + \beta_j}{b_j^T\bar{x} + \beta_j}\cdot\frac{\langle \nabla f_i(\bar{x}), v^p\rangle}{\left\langle \nabla f_j(\bar{x}), v^p\right\rangle}. \end{aligned}$$

Since $b_k^T\bar{x} + \beta_k > 0$ for every $k \in I$, one sees that $A_{i,j}(\bar{x}, x^p)$ tends to $+\infty$ if and only if the quantity

$$\bar{A}_{i,j}(\bar{x}, x^p) := -\frac{b_j^T x^p + \beta_j}{b_i^T x^p + \beta_i}\cdot\frac{\langle \nabla f_i(\bar{x}), v^p\rangle}{\left\langle \nabla f_j(\bar{x}), v^p\right\rangle}$$

tends to $+\infty$ as $p \to \infty$. Note that

$$\frac{b_j^T x^p + \beta_j}{\|x^p - \bar{x}\|} = \frac{b_j^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \frac{\beta_j}{\|x^p - \bar{x}\|} + \frac{b_j^T \bar{x}}{\|x^p - \bar{x}\|} \tag{3.6}$$

and

$$\frac{b_i^T x^p + \beta_i}{\|x^p - \bar{x}\|} = \frac{b_i^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \frac{\beta_i}{\|x^p - \bar{x}\|} + \frac{b_i^T \bar{x}}{\|x^p - \bar{x}\|}. \tag{3.7}$$

Therefore,

$$\bar{A}_{i,j}(\bar{x}, x^p) = -\frac{\dfrac{b_j^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \dfrac{\beta_j}{\|x^p - \bar{x}\|} + \dfrac{b_j^T \bar{x}}{\|x^p - \bar{x}\|}}{\dfrac{b_i^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \dfrac{\beta_i}{\|x^p - \bar{x}\|} + \dfrac{b_i^T \bar{x}}{\|x^p - \bar{x}\|}} \cdot \frac{\langle \nabla f_i(\bar{x}), v^p \rangle}{\langle \nabla f_j(\bar{x}), v^p \rangle}.$$

Since $\lim\limits_{p\to\infty} \|x^p\| = +\infty$, one has $\lim\limits_{p\to\infty} \|x^p - \bar{x}\| = +\infty$. Thus, if $(b_i^T v)\langle \nabla f_j(\bar{x}), v \rangle \neq 0$, then

$$\lim_{p\to\infty} \bar{A}_{i,j}(\bar{x}, x^p) = -\frac{(b_j^T v)\, \langle \nabla f_i(\bar{x}), v \rangle}{(b_i^T v)\langle \nabla f_j(\bar{x}), v \rangle}. \tag{3.8}$$

This contradicts the condition $\lim\limits_{p\to\infty} \bar{A}_{i,j}(\bar{x}, x^p) = +\infty$. So, the denominator of the fraction in (3.8) must be 0, i.e.,

$$(b_i^T v)\langle \nabla f_j(\bar{x}), v \rangle = 0, \tag{3.9}$$

where $v \in (0^+ K)\backslash\{0\}$.

Since $b_k^T x + \beta_k > 0$ for all $k \in I$ and $x \in K$, one has $\dfrac{b_j^T x^p + \beta_j}{\|x^p - \bar{x}\|} > 0$ for all $p \in \mathbb{N}$. Hence, passing (3.6) to limit as $p \to \infty$, one obtains $b_j^T v \geq 0$. As $f_i(\bar{x}) - f_i(x^p) > 0$ for every $p \in \mathbb{N}$, from (3.4) it follows that $\langle \nabla f_i(\bar{x}), v^p \rangle < 0$ for all $p \in \mathbb{N}$. Hence,

$$b_j^T v \geq 0 \quad \text{and} \quad \langle \nabla f_i(\bar{x}), v \rangle \leq 0.$$

Analogously, from (3.7) and (3.5) one can deduce that

$$b_i^T v \geq 0 \quad \text{and} \quad \langle \nabla f_j(\bar{x}), v \rangle \geq 0.$$

Since the *denominator* of the fraction in (3.8) equals to 0 by (3.9), one of the following situations must occur:

(d_1) $b_i^T v = 0$ and $\langle \nabla f_j(\bar{x}), v \rangle = 0$;
(d_2) $b_i^T v > 0$ and $\langle \nabla f_j(\bar{x}), v \rangle = 0$;
(d_3) $b_i^T v = 0$ and $\langle \nabla f_j(\bar{x}), v \rangle > 0$.

As the *numerator* of (3.8) can be either zero or a negative number, there are four possibilities:

(n_1) $b_j^T v = 0$ and $\langle \nabla f_i(\bar{x}), v \rangle = 0$;
(n_2) $b_j^T v = 0$ and $\langle \nabla f_i(\bar{x}), v \rangle < 0$;
(n_3) $b_j^T v > 0$ and $\langle \nabla f_i(\bar{x}), v \rangle = 0$;
(n_4) $b_j^T v > 0$ and $\langle \nabla f_i(\bar{x}), v \rangle < 0$.

If (d_1) and (n_1) occur, then one has $\langle \nabla f_j(\bar{x}), v \rangle = 0$ and $\langle \nabla f_i(\bar{x}), v \rangle = 0$. This contradicts condition (3.1), because $v \in (0^+ K)\backslash\{0\}$.

If (d_1) and (n_2) take place, then one has $\langle \nabla f_j(\bar{x}), v \rangle = 0$ and $\langle \nabla f_i(\bar{x}), v \rangle < 0$. Since $v \in (0^+K)\setminus\{0\}$, $\bar{x} + tv \in K$ for every $t \geq 0$. Therefore, thanks to Lemma 2.7, we have $f_j(\bar{x} + tv) = f_j(\bar{x})$ and $f_i(\bar{x} + tv) < f_i(\bar{x})$ for every $t > 0$. As $m = 2$, this contradicts the assumption $\bar{x} \in E$.

The case where (d_1) and (n_3) (resp., (d_1) and (n_4)) occur can be treated similarly as that where (d_1) and (n_1) (resp., (d_1) and (n_2)) occur.

It is clear that the situations (d_2)–(n_1), (d_2)–(n_2), (d_2)–(n_3), and (d_2)–(n_4) also lead to a conflict either with the first regularity condition or with the inclusion $\bar{x} \in E$.

Finally, observe that each one of the situations (d_3)–(n_1), (d_3)–(n_2), (d_3)–(n_3), and (d_3)–(n_4) leads to the system

$$b_i^T v = 0, \quad \langle \nabla f_i(\bar{x}), v \rangle \leq 0, \quad \langle \nabla f_j(\bar{x}), v \rangle > 0,$$

where $v \in (0^+K)\setminus\{0\}$. This obviously contradicts the regularity condition in (3.2).

We have thus established the inclusion $\bar{x} \in E^{Ge}$ and completed the proof. □

Consider the problem in Example 2.6. For every efficient solution $\bar{x} = (\bar{x}_1, 0)$, we have

$$\nabla f_1(\bar{x}) = (0, -1)^T, \quad \nabla f_2(\bar{x}) = \begin{pmatrix} 0 \\ \frac{1}{\bar{x}_1 + 1} \end{pmatrix}.$$

Hence, (3.1) is violated if one choses $i = 1$, $j = 2$ and $v = (1, 0) \in (0^+K)\setminus\{0\}$. Moreover, condition (3.2) is also violated for $i = 1$, $j = 2$ and $v = (v_1, v_2)$ with $v_1 \geq 0$ and $v_2 > 0$. Here, the violation of the regularity conditions of Theorem 3.1 is a reason for $\bar{x} \notin E^{Ge}$.

Now, we study Geoffrion's proper efficiency for LFVOPs having more than two objective functions.

Theorem 3.2 *In the case $m \geq 3$, suppose that $\bar{x} \in E$. If (3.1), (3.2), and* the third regularity condition

$$\begin{cases} \textit{there exist no triplet } (i, j, k) \in I^3, \textit{ where } i, j, k \textit{ are pairwise distinct}, \\ \textit{and } v \in (0^+K)\setminus\{0\} \textit{ with } \langle \nabla f_i(\bar{x}), v \rangle < 0, \ \langle \nabla f_j(\bar{x}), v \rangle = 0, \ \langle \nabla f_k(\bar{x}), v \rangle > 0 \end{cases} \tag{3.10}$$

are satisfied, then $\bar{x} \in E^{Ge}$.

Proof Suppose to the contrary that $\bar{x} \in E$ and (3.1), (3.10), and (3.2) are satisfied, but $\bar{x} \notin E^{Ge}$. Then, as it has been shown in the proof of Theorem 3.1, there exist a sequence $\{x^p\} \subset K\setminus\{\bar{x}\}$ and a pair $(i, j) \in I^2$, $i \neq j$, such that (3.3) holds.

If $\{x^p\}$ is bounded, then the arguments used for proving the main theorem of Choo [7, p. 220] lead us to a contradiction.

If $\{x^p\}$ is unbounded then, we may assume that $\lim\limits_{p \to \infty} \|x^p\| = +\infty$. Arguing as in the proof of Theorem 3.1, we can find a vector $v \in 0^+K$ with $\|v\| = 1$, such that the *denominator* of the fraction in (3.8) equals to 0. Hence, one of the three situations (d_1)–(d_3) must happen. As the *numerator* of (3.8) can be either zero or a negative number, one of the four possibilities (n_1)–(n_4) occurs.

The situations (d_1)–(n_1), (d_1)–(n_3), (d_2)–(n_1), (d_2)–(n_3) are prohibited by condition (3.1).

The situations (d_3)–(n_1), (d_3)–(n_2), (d_3)–(n_3), (d_3)–(n_4) are excluded by condition (3.2). Indeed, in each one of these situations, one has $b_i^T v = 0$, $\langle \nabla f_i(\bar{x}), v \rangle \leq 0$, and $\langle \nabla f_j(\bar{x}), v \rangle > 0$.

Consider the situation (d_1)–(n_2). As $v \in (0^+K)\setminus\{0\}$, one has $\bar{x} + tv \in K$ for every $t \geq 0$. Since $\langle \nabla f_i(\bar{x}), v \rangle < 0$ and $\langle \nabla f_j(\bar{x}), v \rangle = 0$, by Lemma 2.7 we can assert that $f_i(\bar{x} + tv) < f_i(\bar{x})$ and $f_j(\bar{x} + tv) = f_j(\bar{x})$ for every $t > 0$. Hence, noting that $0^+K\setminus\{0\}$, from (3.10) we have $\langle \nabla f_k(\bar{x}), v\rangle \le 0$ for every $k \in I\backslash\{i, j\}$. Therefore, according to Lemma 2.7, $f_k(\bar{x} + tv) \le f_k(\bar{x})$ for any $k \in I\backslash\{i, j\}$ and $t > 0$. Picking any $t \in (0, +\infty)$, we observe that the inequalities $f_k(\bar{x} + tv) \le f_k(\bar{x})$ for all $k \in I\backslash\{i\}$ together with the strict inequality $f_i(\bar{x} + tv) < f_i(\bar{x})$ contradict the inclusion $\bar{x} \in E$. If one of the situations (d_2)–(n_2), (d_2)–(n_4), and (d_1)–(n_4) occurs, we can obtain a contradiction by arguing similarly.

The proof is complete. □

4 Further analysis

4.1 Choo and Atkins' example

The sufficient conditions for Geoffrion's proper efficiency given in Sect. 3 are verifiable. To justify this assertion, we consider the well-known LFVOP having disconnected solution sets, which was given by Choo and Atkins [9].

Example 4.1 (See [9, Example 2]) Consider problem (VP) with

$$K = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \ge 2,\ 0 \le x_2 \le 4\right\},$$
$$f_1(x) = \frac{-x_1}{x_1 + x_2 - 1}, \quad f_2(x) = \frac{-x_1}{x_1 - x_2 + 3}.$$

Using the criteria for $x \in E$ and $x \in E^w$ recalled in (2.1), it is easy to show that

$$E = E^w = \left\{(x_1, 0) \,:\, x_1 \ge 2\} \cup \{(x_1, 4) \,:\, x_1 \ge 2\right\}.$$

Here we have $b_1 = (1, 1)$, $b_2 = (1, -1)$, and $0^+K = \{v = (v_1, 0) \,:\, v_1 \ge 0\}$. For every $\bar{x} \in \{(\bar{x}_1, 0) \,:\, \bar{x}_1 \ge 2\}$,

$$\nabla f_1(\bar{x}) = \begin{pmatrix} \dfrac{1}{(\bar{x}_1 - 1)^2} \\ \dfrac{\bar{x}_1}{(\bar{x}_1 - 1)^2} \end{pmatrix}, \quad \nabla f_2(\bar{x}) = \begin{pmatrix} \dfrac{-3}{(\bar{x}_1 + 3)^2} \\ \dfrac{-\bar{x}_1}{(\bar{x}_1 + 3)^2} \end{pmatrix}.$$

Hence, for any vector $v = (v_1, v_2)$,

$$\begin{cases} \langle \nabla f_1(\bar{x}), v\rangle = 0 \\ \langle \nabla f_2(\bar{x}), v\rangle = 0 \end{cases} \iff \begin{cases} v_1 + \bar{x}_1 v_2 = 0 \\ -3v_1 + \bar{x}_1 v_2 = 0 \end{cases} \iff \begin{cases} v_1 = 0 \\ v_2 = 0. \end{cases}$$

So, condition (3.1) is satisfied. Similarly, (3.1) holds for every $\bar{x} \in \left\{(x_1, 4) \,:\, x_1 \ge 2\right\}$. In addition, for every $v = (v_1, 0) \in (0^+K)\backslash\{0\}$, one has $b_1^T v = b_2^T v = v_1 > 0$. Thus, condition (3.2) is satisfied. Therefore,

$$E^{Ge} = E = E^w = \left\{(x_1, 0) \,:\, x_1 \ge 2\} \cup \{(x_1, 4) \,:\, x_1 \ge 2\right\}.$$

4.2 LFVOPs having more than two connected components in the solution sets

To proceed furthermore, we need the following specific result.

Proposition 4.2 *For a LFVOP in the form* (VP), *if the denominators of the functions f_i are all the same, then one has $E^{Ge} = E$.*

Proof Suppose that the denominators of f_i coincide. To show that $E = E^{Ge}$, we fix any $\bar{x} \in E$. By the necessary and sufficient optimality condition recalled in (2.1), there is a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{int} \mathbb{R}^m_+$ such that

$$\left\langle \sum_{i=1}^{m} \xi_i \left[\left(b_i^T \bar{x} + \beta_i\right) a_i - \left(a_i^T \bar{x} + \alpha_i\right) b_i \right], y - \bar{x} \right\rangle \geq 0, \quad \forall y \in K. \tag{4.1}$$

For each $i \in I$, one has

$$\left\langle \nabla f_i(\bar{x}), y - \bar{x} \right\rangle = \frac{1}{p(\bar{x})} \left\langle \left(b_i^T \bar{x} + \beta_i\right) a_i - \left(a_i^T \bar{x} + \alpha_i\right) b_i, y - \bar{x} \right\rangle,$$

where $p(\bar{x}) := \left(b_1^T \bar{x} + \beta_1\right)^2 = \cdots = \left(b_m^T \bar{x} + \beta_m\right)^2$ (see the first equality in [23, p. 147]). Hence, we can rewrite (4.1) equivalently as

$$\left\langle \sum_{i=1}^{m} \xi_i p(\bar{x}) \nabla f_i(\bar{x}), y - \bar{x} \right\rangle \geq 0, \quad \forall y \in K.$$

Putting $\bar{\xi}_i = \xi_i p(\bar{x})$ and noting that $\bar{\xi}_i > 0$ for $i \in I$, we get

$$\left\langle \sum_{i=1}^{m} \bar{\xi}_i \nabla f_i(\bar{x}), y - \bar{x} \right\rangle \geq 0, \quad \forall y \in K. \tag{4.2}$$

Clearly, (4.2) shows that $\bar{x}$ is satisfies the first-order necessary optimality condition for the following scalar optimization problem

$$(P_0) \qquad \text{Minimize } \varphi(x) := \sum_{i=1}^{m} \bar{\xi}_i f_i(x) \ \text{ subject to } \ x \in K.$$

It is easy to verify that, in this example, $\varphi(x)$ *is a linear fractional function*. Since $\nabla\varphi(\bar{x}) = \sum_{i=1}^{m} \bar{\xi}_i \nabla f_i(\bar{x})$, combining (4.2) with (2.3) yields $\varphi(y) \geq \varphi(\bar{x})$ for all $y \in K$. This means that $\bar{x}$ is a global solution of (P_0). Then, by [13, Theorem 1] we can assert that $\bar{x} \in E^{Ge}$.

The proof is complete. □

The efficient solution set of the problem considered in Example 4.1 has two connected components. We now analyze a LFVOP whose efficient solution set has three connected components.

Example 4.3 (See [17, p. 483]) Consider problem (VP) where $n = m = 3$,

$$K = \Big\{x \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 \leq 1,\ x_1 - 2x_2 + x_3 \leq 1, \\ -2x_1 + x_2 + x_3 \leq 1,\ x_1 + x_2 + x_3 \geq 1\Big\},$$

and

$$f_i(x) = \frac{-x_i + \frac{1}{2}}{x_1 + x_2 + x_3 - \frac{3}{4}} \qquad (i = 1, 2, 3).$$

Here we have

$$a_1 = (-1, 0, 0),\ a_2 = (0, -1, 0),\ a_3 = (0, 0, -1),\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2},$$

and

$$b_1 = b_2 = b_3 = (1, 1, 1), \;\; \beta_1 = \beta_2 = \beta_3 = -\frac{3}{4}.$$

Setting

$$C = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix},$$

we see that $K = \{x \in \mathbb{R}^3 \; : \; Cx \leq d\}$. As it has been shown in [17],

$$\begin{aligned} E = E^w = &\{(x_1, x_2, x_3) \; : \; x_1 \geq 1, \; x_3 = x_2 = x_1 - 1\} \\ &\cup\{(x_1, x_2, x_3) \; : \; x_2 \geq 1, \; x_3 = x_1 = x_2 - 1\} \\ &\cup\{(x_1, x_2, x_3) \; : \; x_3 \geq 1, \; x_2 = x_1 = x_3 - 1\}. \end{aligned} \tag{4.3}$$

Setting $p(x) = \big(x_1 + x_2 + x_3 - \frac{3}{4}\big)^2$, one has

$$\begin{aligned} \nabla f_1(x) &= \frac{1}{p(x)} \left(-x_2 - x_3 + \frac{1}{4}, x_1 - \frac{1}{2}, x_1 - \frac{1}{2}\right), \\ \nabla f_2(x) &= \frac{1}{p(x)} \left(x_2 - \frac{1}{2}, -x_1 - x_3 + \frac{1}{4}, x_2 - \frac{1}{2}\right), \\ \nabla f_3(x) &= \frac{1}{p(x)} \left(x_3 - \frac{1}{2}, x_3 - \frac{1}{2}, -x_1 - x_2 + \frac{1}{4}\right). \end{aligned}$$

By Lemma 2.9, we get $0^+K = \{v = (\tau, \tau, \tau) \in \mathbb{R}^3 \; : \; \tau \geq 0\}$. As $b_i^T v \neq 0$ for any $i \in I$ and $v \in 0^+K \backslash \{0\}$, the second regularity condition (3.2) is satisfied for every $\bar{x} \in E$. To check the first and the third regularity conditions in (3.1) and (3.10), fix any $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in E$ with $\bar{x}_1 \geq 1$ and $\bar{x}_2 = \bar{x}_3 = \bar{x}_1 - 1$. Note that

$$\begin{aligned} \nabla f_1(\bar{x}) &= \frac{1}{p(\bar{x})} \left(-2\bar{x}_1 + \frac{9}{4}, \bar{x}_1 - \frac{1}{2}, \bar{x}_1 - \frac{1}{2}\right), \\ \nabla f_2(\bar{x}) &= \frac{1}{p(\bar{x})} \left(\bar{x}_1 - \frac{3}{2}, -2\bar{x}_1 + \frac{5}{4}, \bar{x}_1 - \frac{3}{2}\right), \\ \nabla f_3(\bar{x}) &= \frac{1}{p(\bar{x})} \left(\bar{x}_1 - \frac{3}{2}, \bar{x}_1 - \frac{3}{2}, -2\bar{x}_1 + \frac{5}{4}\right). \end{aligned}$$

Hence, we have

$$\langle \nabla f_1(\bar{x}), v\rangle = \frac{5\tau}{4p(\bar{x})} > 0, \; \langle \nabla f_2(\bar{x}), v\rangle = -\frac{7\tau}{4p(\bar{x})} < 0, \; \langle \nabla f_3(\bar{x}), v\rangle = -\frac{7\tau}{4p(\bar{x})} < 0$$

for any $v = (\tau, \tau, \tau)$ with $\tau > 0$. Therefore, both conditions (3.1) and (3.10) are satisfied. Thus, by Theorem 3.2 we can assert that $\bar{x} \in E^{Ge}$. The fact that the last inclusion holds for any $\bar{x} \in E$ follows from the description of E in (4.3) and the symmetry of our problem (VP) w.r.t. the variables x_1, x_2, x_3. This means that $E = E^{Ge}$.

Remark 4.4 In Example 4.3, since the denominators of f_i are all the same for $i \in I$, by Proposition 4.2 we have $E^{Ge} = E = E^w$. This result fully agrees with the one obtained by using Theorem 3.2.

Now, let us examine a LFVOP where the number of criteria can be any integer $m \geq 2$.

Example 4.5 (See [17, pp. 479–480]) We consider problem (VP) where $n = m$, $m \geq 2$,

$$K = \left\{x \in \mathbb{R}^m \ : \ x_1 \geq 0,\ x_2 \geq 0, \dots,\ x_m \geq 0,\ \sum_{k=1}^{m} x_k \geq 1\right\},$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{\sum_{k=1}^m x_k - \frac{3}{4}} \qquad (i = 1, \dots, m).$$

Here we have

$$\begin{aligned} &a_1 = (-1, 0, 0, \dots, 0),\ a_2 = (0, -1, 0, \dots, 0), \dots, \\ &a_m = (0, 0, 0, \dots, -1),\ \alpha_1 = \alpha_2 = \cdots = \alpha_m = \frac{1}{2}, \end{aligned}$$

and

$$b_1 = b_2 = \cdots = b_m = (1, 1, 1, \dots, 1),\ \ \beta_1 = \beta_2 = \cdots = \beta_m = -\frac{3}{4}.$$

Setting

$$C = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix},$$

we see that $K = \{x \in \mathbb{R}^n \ : \ Cx \leq d\}$. According to [17, p. 483], one has

$$\begin{aligned} E = E^w = &\{(x_1, 0, \dots, 0) \ : \ x_1 \geq 1\} \\ &\cup\{(0, x_2, \dots, 0) \ : \ x_2 \geq 1\} \\ &\dots\ \dots\ \dots \\ &\cup\{(0, \dots, 0, x_m) \ : \ x_m \geq 1\}. \end{aligned} \tag{4.4}$$

Since $b_i = (1, 1, 1, \dots, 1)^T$ for $i = 1, \dots, m$ and $0^+K = \mathbb{R}^m_+$, if $b_i^T v = 0$ for some $v \in 0^+K$, then $v = 0$. This implies that the second regularity in (3.2) is satisfied. Setting $q(x) = \left(\sum_{k=1}^m x_k - \dfrac{3}{4}\right)^2$, one has

$$\nabla f_i(x) = \frac{1}{q(x)}\left(x_i - \frac{1}{2}, \dots, -\sum_{k \neq i} x_k + \frac{1}{4}, \dots, x_i - \frac{1}{2}\right)$$

for any $x \in K$, where the expression $-\sum_{k\neq i} x_k + \frac{1}{4}$ is the ith component of $\nabla f_i(x)$. Especially, for any $\bar{x} \in E$, where $\bar{x} = (\bar{x}_1, 0, \ldots, 0)$ and $\bar{x}_1 \geq 1$, we get

$$\nabla f_1(\bar{x}) = \frac{1}{q(\bar{x})}\left(\frac{1}{4}, \bar{x}_1 - \frac{1}{2}, \ldots, \bar{x}_1 - \frac{1}{2}\right),$$
$$\nabla f_2(\bar{x}) = \frac{1}{q(\bar{x})}\left(-\frac{1}{2}, -\bar{x}_1 + \frac{1}{4}, -\frac{1}{2}, \ldots, -\frac{1}{2}\right),$$
$$\vdots$$
$$\nabla f_m(\bar{x}) = \frac{1}{q(\bar{x})}\left(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\bar{x}_1 + \frac{1}{4}\right).$$

Clearly, all the components of $\nabla f_1(\bar{x})$ are positive, while all the components of the vector $\nabla f_i(\bar{x})$, $i = 2, \ldots, m$, are negative. So, for every $v \in 0^+K\setminus\{0\}$, one has $\langle \nabla f_1(\bar{x}), v\rangle > 0$ and $\langle \nabla f_i(\bar{x}), v\rangle < 0$ for $i = 2, \ldots, m$. Therefore, the conditions (3.1) and (3.10) are satisfied for the efficient solution $\bar{x}$ in question. Hence, by Theorem 3.2, $\bar{x} \in E^{Ge}$. The fact that the last inclusion holds for any $\bar{x} \in E$ follows from the description of E in (4.4) and the symmetry of our problem (VP) w.r.t. the variables $x_1, \ldots, x_n$. We have thus shown that $E = E^{Ge}$.

Remark 4.6 In Example 4.5, since the denominators of f_i coincide for all $i \in I$, we have $E^{Ge} = E = E^w$ by Proposition 4.2.

4.3 More about the essentialness of the regularity conditions

The following counterexample shows that the regularity conditions in Theorem 3.2 cannot be skipped.

Example 4.7 Consider problem (VP) with $m = 3,\ n = 2$,

$$K = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \ :\ x_1 \geq 0,\ x_2 \geq 0\right\},$$
$$f_1(x) = -x_1 - x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}, \quad f_3(x) = x_1 - x_2.$$

Using the optimality conditions recalled in (2.1), one can find that

$$E = \left\{x = (x_1, x_2)\ :\ x_1 \geq 0,\ x_2 \geq 0,\ x_2 < x_1 + 1\right\}$$

and

$$E^w = \left\{x = (x_1, x_2)\ :\ x_1 \geq 0,\ x_2 \geq 0,\ x_2 \leq x_1 + 1\right\}.$$

Let us prove that every point $\bar{x} = (\bar{x}_1, 0) \in E$ is an improperly efficient solution in the sense of Geoffrion. Given any $M > 0$, we choose $i = 1$ and select a point $x = (x_1, x_2)$ from $K = \mathbb{R}^2_+$ satisfying the conditions $x_1 \geq \bar{x}_1$, $x_2 > x_1 - \bar{x}_1$, $x_1 + x_2 > M - 1$. Then, one has $f_i(x) < f_i(\bar{x})$. Moreover, since $f_3(x) = x_1 - x_2 < \bar{x}_1 = f_3(\bar{x})$, the unique index $j \in I$

having the property $f_j(x) > f_j(\bar{x})$ is $j = 2$. As

$$
\begin{aligned}
A_{i,j}(\bar{x}, x) &= \frac{f_1(\bar{x}) - f_1(x)}{f_2(x) - f_2(\bar{x})} \\
&= \frac{-\bar{x}_1 + x_1 + x_2}{\dfrac{x_2}{x_1 + x_2 + 1} - \dfrac{0}{\bar{x}_1 + 1}} \\
&= (x_1 + x_2 + 1)(-\bar{x}_1 + x_1 + x_2)\frac{1}{x_2} \\
&= (x_1 + x_2 + 1)\left(1 + \frac{x_1 - \bar{x}_1}{x_2}\right) > M,
\end{aligned}
$$

by the observation given after Definition 2.2 we have $\bar{x} \notin E^{Ge}$. Now, we check the regularity conditions in Theorem 3.2. For every $\bar{x} = (\bar{x}_1, 0) \in E$, we have

$$
\nabla f_1(\bar{x}) = (-1, -1), \;\; \nabla f_2(\bar{x}) = \left(0, \;\; \frac{1}{\bar{x}_1 + 1}\right), \;\; \nabla f_3(\bar{x}) = (1, -1).
$$

Hence, there exist no $(i, j) \in I^2, \;\; j \neq i$, and $v \in (0^+K)\backslash\{0\}$ with $\langle \nabla f_i(\bar{x}), v\rangle = 0$ and $\left\langle \nabla f_j(\bar{x}), v\right\rangle = 0$. In particular, condition (3.1) is fulfilled. For $(i, j, k) := (1, 2, 3) \in I^3$ and $v := (1, 0) \in (0^+K)\backslash\{0\}$, one has

$$
\langle \nabla f_i(\bar{x}), v\rangle < 0, \; \left\langle \nabla f_j(\bar{x}), v\right\rangle = 0, \; \langle \nabla f_k(\bar{x}), v\rangle > 0.
$$

Thus, condition (3.10) is violated. To see that (3.2) is not satisfied, it suffices to choose $(i, j) = (1, 3) \in I^2, \;\; v = (1, 0) \in (0^+K)\backslash\{0\}$, and note that

$$
b_i^T v = 0, \; \langle \nabla f_i(\bar{x}), v\rangle \leq 0, \; \left\langle \nabla f_j(\bar{x}), v\right\rangle > 0.
$$

Now, take any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$ with $\bar{x}_1 > 0$. It is easy to verify that condition (3.1) is satisfied. Since $\bar{x}_2 \leq \bar{x}_1 + 1$, for $(i, j, k) := (1, 2, 3)$ and $v = (v_1, v_2)$ with $v_1 > 0$ and $v_2 = \dfrac{\bar{x}_2}{\bar{x}_1 + 1} v_1$, we have $v \in (0^+K)\backslash\{0\}$ and

$$
\langle \nabla f_i(\bar{x}), v\rangle < 0, \; \left\langle \nabla f_j(\bar{x}), v\right\rangle = 0, \; \langle \nabla f_k(\bar{x}), v\rangle > 0. \tag{4.5}
$$

So, condition (3.10) is violated.

Finally, take any $\bar{x} = (0, \bar{x}_2) \in E$ with $\bar{x}_2 \in (0, 1)$. For $(i, j, k) := (1, 2, 3)$ and $v = (v_1, v_2)$ with $v_1 > 0$ and $v_2 = \bar{x}_2 v_1$, we see that $v \in (0^+K)\backslash\{0\}$ and (4.5) holds. Thus, condition (3.10) is not fulfilled.

We have shown that, for every $\bar{x} \in E$, at least one of the three regularity conditions in Theorem 3.2 is violated. So, the theorem cannot be used to assure that $\bar{x} \in E^{Ge}$.

5 Concluding remarks and open questions

By constructing two concrete examples (see Examples 2.6 and 4.7), we have proved that there exist linear fractional vector optimization problems with unbounded constraint sets whose Geoffrion's efficient solution set differs from the efficient solution set, and the latter is a proper subset of the constraint set. Our main results are two theorems giving suitable sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to be a properly efficient solution in the sense of Geoffrion.

In the notation of Sect. 2, letting A (resp., B) be the matrix consisting of the rows a_i^T (resp., b_i^T) and α (resp., β) be the column vector consisting of the numbers α_i (resp., β_i) for

$i \in I$, one can represent (VP) by the data point $(A, \alpha, B, \beta, C, d)$ in the Euclidean space $\mathbb{R}^s$ with $s := 2(m \times n + m) + p \times n + p$. In what follows, this space $\mathbb{R}^s$ is equipped with the Lebesgue measure.

Below are some open questions in this topic.

(Q1) *Is it possible to obtain necessary and sufficient conditions for the validity of the equality $E = E^{Ge}$, or not?*
(Q2) *Does a Geoffrion's improperly efficient solution in a LFVOP have any special properties?*
(Q3) *Is it true that, for almost all LFVOPs, one has $E = E^{Ge}$?*

Acknowledgements The authors are greatly indebted to the referees for making us familiar with the significant works of Choo [7], Chew and Choo [6], Stancu-Minasian [29], and for valuable suggestions. We thank Prof. Phan Quoc Khanh and Prof. Christiane Tammer for useful discussions on the first draft of this paper.

References

1. Benoist, J.: Connectedness of the efficient set for strictly quasiconcave sets. J. Optim. Theory Appl. **96**, 627–654 (1998)
2. Benoist, J.: Contractibility of the efficient set in strictly quasiconcave vector maximization. J. Optim. Theory Appl. **110**, 325–336 (2001)
3. Benson, B.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. **71**, 232–241 (1979)
4. Bochnak, R., Coste, M., Roy, M.F.: Real Algebraic Geometry. Spinger, Berlin (1998)
5. Borwein, J.M.: Proper efficient points for maximizations with respect to cones. SIAM J. Control Optim. **15**, 57–63 (1977)
6. Chew, K.L., Choo, E.U.: Pseudolinearity and efficiency. Math. Program. **28**, 226–239 (1984)
7. Choo, E.U.: Proper efficiency and the linear fractional vector maximum problem. Oper. Res. **32**, 216–220 (1984)
8. Choo, E.U., Atkins, D.R.: Bicriteria linear fractional programming. J. Optim. Theory Appl. **36**, 203–220 (1982)
9. Choo, E.U., Atkins, D.R.: Connectedness in multiple linear fractional programming. Manag. Sci. **29**, 250–255 (1983)
10. Crespi, G.P.: Proper efficiency and vector variational inequalities. J. Inform. Optim. Sci. **23**, 49–62 (2002)
11. Crespi, G.P., Ginchev, I., Rocca, M.: Existence of solutions and star-shapedness in Minty variational inequalities. J. Global Optim. **32**, 485–494 (2005)
12. Crespi, G.P., Ginchev, I., Rocca, M.: Increasing-along-rays property for vector functions. J. Nonlinear Convex Anal. **7**, 39–50 (2006)
13. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. **22**, 613–630 (1968)
14. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibria. Kluwer Academic Publishers, Dordrecht (2000)
15. Guerraggio, A., Molho, E., Zaffaroni, A.: On the notion of proper efficiency in vector optimization. J. Optim. Theory Appl. **82**, 1–21 (1994)
16. Henig, M.I.: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**, 387–407 (1982)
17. Hoa, T.N., Phuong, T.D., Yen, N.D.: Linear fractional vector optimization problems with many components in the solution sets. J. Ind. Manag. Optim. **1**, 477–486 (2005)
18. Hoa, T.N., Phuong, T.D., Yen, N.D.: On the parametric affine variational inequality approach to linear fractional vector optimization problems. Vietnam J. Math. **33**, 477–489 (2005)
19. Hoa, T.N., Huy, N.Q., Phuong, T.D., Yen, N.D.: Unbounded components in the solution sets of strictly quasiconcave vector maximization problems. J. Glob. Optim. **37**, 1–10 (2007)
20. Huong, N.T.T., Hoa, T.N., Phuong, T.D., Yen, N.D.: A property of bicriteria affine vector variational inequalities. Appl. Anal. **10**, 1867–1879 (2012)
21. Huong, N.T.T., Yao, J.-C., Yen, N.D.: Connectedness structure of the solution sets of vector variational inequalities. Optimization **66**, 889–901 (2017)
22. Huy, N.Q., Yen, N.D.: Remarks on a conjecture of. J. Benoist. Nonlinear Anal. Forum **9**, 109–117 (2004)

23. Lee, G.M., Tam, N.N., Yen, N.D.: Quadratic Programming and Affine Variational Inequalities: A Qualitative Study Series: Nonconvex Optimization and its Applications, vol. 78. Springer, New York (2005)
24. Luc, D.T.: Multiobjective Linear Programming: An Introduction. Springer, Berlin (2016)
25. Malivert C.: Multicriteria fractional programming. In: Sofonea, M., Corvellec, J.N. (eds.) Proceedings of the 2nd Catalan Days on Applied Mathematics. Presses Universitaires de Perpinan, pp. 189–198 (1995)
26. Robinson, S.M.: Generalized equations and their solutions. Part I: Basic theory. Math. Program. Study **10**, 128–141 (1979)
27. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
28. Rubinov, A.M., Glover, B.M.: Increasing convex-along-rays functions with applications to global optimization. J. Optim. Theory Appl. **102**, 615–642 (1999)
29. Stancu-Minasian, I.M.: A ninth bibliography of fractional programming. Optimization **68**, 2123–2167 (2019)
30. Steuer, R.E.: Multiple Criteria Optimization: Theory Computation and Application. Wiley, New York (1986)
31. Yen, N.D.: Linear fractional and convex quadratic vector optimization problems. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Developments in Vector Optimization, pp. 297–328. Springer, Berlin (2012)
32. Yen N.D., Phuong T.D.: Connectedness and stability of the solution set in linear fractional vector optimization problems. **38**, 479–489. https://link.springer.com/chapter/10.1007/978-1-4613-0299-5_29. (2000)
33. Yen, N.D., Yang, X.Q.: Affine variational inequalities on normed spaces. J. Optim. Theory Appl. **178**, 36–55 (2018)
34. Yen, N.D., Yao, J.-C.: Monotone affine vector variational inequalities. Optimization **60**, 53–68 (2011)

Affiliations

N. T. T. Huong[1] · J.-C. Yao[2] · N. D. Yen[3]

N. T. T. Huong
nguyenhuong2308.mta@gmail.com

J.-C. Yao
yaojc@mail.cmu.edu.tw

1 Department of Mathematics, Faculty of Information Technology, Le Quy Don Technical University, 236 Hoang Quoc Viet, Hanoi, Vietnam

2 Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung, Taiwan

3 Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam