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Some results on linear nabla Riemann-Liouville fractional difference equations

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In this paper, we establish some criteria for boundedness, stability properties, and separation of solutions of autonomous nonlinear nabla Riemann-Liouville scalar fractional difference equations. To derive these results, we prove the variation of constants formula for nabla Riemann-Liouville fractional difference equations.

KEYWORDS
fractional difference equations, Lyapunov exponent, nabla Riemann-Liouville difference

MSC CLASSIFICATION
39A13; 93D05; 39A30

1 | INTRODUCTION

Recently, the theory of fractional calculus became an object of intensive research, and its development is very fast; see, eg, previous studies[1-4] and the references therein. This is accompanied by many interesting applications of fractional calculus to modelling of various problems via fractional differential and difference equations. These applications arise in various areas such as control theory, signal processing, and the theory of viscoelasticity; see, eg, previous studies.[5-11]

In the discrete-time framework, four types of fractional differences are considered: forward/backward Caputo and forward/backward Riemann-Liouville operators.[12-14] This paper is devoted to the study of discrete-time fractional systems with backward Riemann-Liouville differences, which is also called nabla Riemann-Liouville difference equation.

A main result of this paper is the variation of constants formula for nonlinear nabla Riemann-Liouville fractional difference equations. Using this result, we study some asymptotic properties of scalar equations. In particular, we provide some conditions for boundedness and stability of nonlinear equations as well as conditions for separation of solutions.

2 | BASIC NOTIONS

We recall some notions concerning fractional summation and fractional differences. Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geq 0}$ the set of natural numbers $\{0, 1, 2, \ldots\}$ including 0, and by

$$\mathbb{Z}_{\leq 0} := \{0, -1, -2, \ldots\}$$

the set of nonpositive integers. For $a \in \mathbb{R}$, we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \ldots\}$. By $\Gamma : \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{R}$, we denote the Euler Gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, which is well defined, since the limit exists (see, eg, Krantz[15, p156]), and

$$\Gamma(\alpha) = \begin{cases} \int_0^\infty x^{\alpha-1} e^{-x} dx & \text{if } \alpha > 0, \\ \dfrac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}. \end{cases}$$

Note that $\Gamma(\alpha) > 0$ for all $\alpha > 0$.

For $s \in \mathbb{R}$ with $s+1, s+1+\alpha \notin \mathbb{Z}_{\leq 0}$, the raising factorial power $(s)^{\overline{(\alpha)}}$ is defined by

$$(s)^{\overline{(\alpha)}} := \frac{\Gamma(s+\alpha)}{\Gamma(s)}$$

for $s \in (\mathbb{R} \setminus \mathbb{Z}_{\leq -1}) \cap (\mathbb{R} \setminus (-\alpha + \mathbb{Z}_{\leq -1}))$.

By

$$\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\},$$

we denote the least integer greater or equal to x.

Binomial coefficients $\binom{r}{m}$ can be defined for any $r, m \in \mathbb{C}$ as described in Graham et al[16] (section 5.5, formula (5.90)). For $r \in \mathbb{R}$ and $m \in \mathbb{Z}$, the binomial coefficient satisfies the following (Graham et al, section 5.1, formula (5.1))[16]:

$$\binom{r}{m} = \begin{cases} \frac{r(r-1)\ldots(r-m+1)}{m!} & \text{if } m \in \mathbb{Z}_{\geq 1}, \\ 1 & \text{if } m = 0, \\ 0 & \text{if } m \in \mathbb{Z}_{\leq -1}. \end{cases}$$

In this paper, we consider linear inhomogeneous fractional difference systems of the form

$$({}_{\text{R-L}}\Delta^\alpha x)(n) = Ax(n) + f(n), \tag{1}$$

where $x : \mathbb{N}_1 \to \mathbb{R}^d$, ${}_{\text{R-L}}\Delta^\alpha$ is Riemann-Liouville difference operator of a real order $\alpha \in (0, 1)$, $f : \mathbb{N}_1 \to \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$.

For $\nu \in \mathbb{R}_{\geq 0}$ and a function $x : \mathbb{N}_1 \to \mathbb{R}$, the νth nabla fractional sum (starting from 0) $\Delta^{-\nu} x : \mathbb{N}_1 \to \mathbb{R}^d$ of order ν of x is defined as

$$(\Delta_0^{-\nu} x)(n) := \frac{1}{\Gamma(\nu)} \sum_{k=1}^{n} (n-k+1)^{\overline{(\nu-1)}} x(k) \tag{2}$$

for $n \in \mathbb{N}_1$.

Let $\alpha \in (0,1)$ and $x : \mathbb{N}_1 \to \mathbb{R}^d$. The nabla Riemann-Liouville difference ${}_{\text{R-L}}\Delta^{\alpha}x : \mathbb{N}_1 \to \mathbb{R}$ of x of order α is defined as

$$_{\text{R-L}}\Delta^{\alpha} := \Delta \circ \Delta^{-(1-\alpha)},$$

ie,

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n) := (\Delta\Delta^{-(1-\alpha)}x)(n) \tag{3}$$

for $n \in \mathbb{N}_2$, where Δ is a backward difference operator, ie,

$$\Delta x(n) = x(n) - x(n-1).$$

The formula (3) does not cover the case $n = 1$. When $n = 1$, (1) can be written as

$$(\Delta^{-(1-\alpha)}x)(1) - (\Delta^{-(1-\alpha)}x)(0) = Ax(1) + f(1). \tag{4}$$

Note that the symbol $(\Delta^{-(1-\alpha)}x)(0)$ is not defined by (2) because $0 \notin \mathbb{N}_1$. Therefore, we formulate the initial value problem for (1) as follows: For a given $x_1 \in \mathbb{R}^d$, find a sequence $x : \mathbb{N} \to \mathbb{R}^d$ such that

$$x(0) = (I - A)x_1 - f(1), \quad x(1) = x_1 \tag{5}$$

and (1) is satisfied for all $n \in \mathbb{N}_2$. Hence, from now on, we assume invertibility of $I - A$ to ensure that there is a unique $x(1)$.

Expanding (3), we can write

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n) = \sum_{k=1}^{n}(-1)^{n-k}\binom{\alpha}{n-k}x(k)$$

for $n \in \mathbb{N}_2$. Hence, (1) is equivalent to

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{\alpha}{n-k}x(k) = Ax(n) + f(n)$$

for $n \in \mathbb{N}_2$. Therefore,

$$x(n) = -(I-A)^{-1}\sum_{k=1}^{n-1}(-1)^{n-k}\binom{\alpha}{n-k}x(k) + (I-A)^{-1}f(n)$$

for $n \in \mathbb{N}_2$, and we receive the existence and uniqueness of solutions of (1).

Next, the backward Laplace transform ($\mathcal{L}$ transform) of a sequence x is defined by

$$\mathcal{L}(x)(s) = \sum_{k=1}^{\infty}x(k)(1-s)^{k-1}$$

for all $s \in \mathbb{C}$ for which the series converges. From Čermák et al,[17, Remark 12] the backward Laplace transform is given by a power series with the centre at $s_0 = 1$. And, if the series converges at some $s \neq 1$, then there exists $r_x > 0$ such that the series converges locally uniformly (and absolutely) in the open disc $B(s_0, r_x) := \{z \in \mathbb{C} : |z - s_0| < r_x\}$. Moreover, the series expansion is determined uniquely, and $\mathcal{L}(x)(s)$ is an analytic function on $B(1, r_x)$.

Recall that for two sequences $x, y : \mathbb{N}_1 \to \mathbb{C}$, the convolution $x * y : \mathbb{N}_1 \to \mathbb{C}$ is defined as

$$(x * y)(n) = \sum_{k=1}^{n}x(n-k+1)y(k) \tag{6}$$

for $n \in \mathbb{N}_1$.

Then, by Čermák et al,[17, Lemma 14(i),(v)]

$$\mathcal{L}(x * y)(s) = \mathcal{L}(x)(s)\mathcal{L}(y)(s) \tag{7}$$

on $B(1, r_*)$, where $r_* = \min\{r_x, r_y\}$. And, if $0 < \alpha < 1$, then

$$\mathcal{L}({}_{\text{R-L}}\Delta^{\alpha}x)(s) = s^{\alpha}\mathcal{L}(x)(s) - {}_{\text{R-L}}\Delta^{-(1-\alpha)}x(0)$$

on $B(1,1) \cap B(1, r_x)$.

The following definition can be found, eg, in Čermák et al[17] (see also Nechvátal[18]).

Definition 1. (Nabla discrete-time Mittag-Leffler functions). For $\lambda \in \mathbb{R}$, $|\lambda| < 1$, and $\alpha, \beta, z \in \mathbb{C}$ with $\text{Re}\alpha > 0$, the nabla discrete-time Mittag-Leffler type function is defined by

$$E_{\overline{(\alpha,\beta)}}(\lambda, z) := \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k + \beta)}.$$

Remark 1. Because

$$\frac{n^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k+\beta)} = \frac{\Gamma(n+k\alpha+\beta-1)}{\Gamma(n)\Gamma(\alpha k+\beta)} = \binom{n+k\alpha+\beta-2}{n-1} = \binom{n-1+k\alpha+\beta-1}{n-1} = (-1)^{n-1}\binom{-k\alpha-\beta}{n-1},$$

we have

$$E_{\overline{(\alpha,\beta)}}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k (-1)^{n-1}\binom{-k\alpha-\beta}{n-1}$$

for $n \in \mathbb{N}_1$. We use the convention that $0^0 = 1$, and let $\lambda = 0$,

$$E_{\overline{(\alpha,\beta)}}(0, n) = (-1)^{n-1}\binom{-\beta}{n-1}.$$

Next, we present the Laplace transform of the nabla discrete-time Mittag-Leffler function.

Proposition 1. *(Čermák et al)[17], proposition 17 It holds that*

$$\mathcal{L}\{E_{\overline{(\alpha,\beta)}}(A, n)\}(s) = s^{\alpha-\beta}(s^{\alpha}I - A)^{-1}$$

for $\alpha, \beta \in \mathbb{R}$.

For $p > 0$, consider the following spaces of real-valued sequences:

$$\ell_p := \left\{ (a_n)_{n\in\mathbb{N}_1} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

and

$$\ell_\infty := \left\{ (a_n)_{n\in\mathbb{N}_1} : \sup_{n\in\mathbb{N}_1} |a_n| < \infty \right\}.$$

We have that for $p \geq 1$, ℓ_p is a Banach space with norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}},$$

and ℓ_∞ is a Banach space with norm $\|x\|_\infty = \sup_{n\in\mathbb{N}_1}|a_n|$. The following result is called the Young convolution inequality. Suppose that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

with $p, q, r \geq 1$ and $x \in \ell_p, y \in \ell_q$; then, $\|x * y\|_r \leq \|x\|_p\|y\|_q$. Using this inequality, we conclude that $x * y \in \ell_r$ for $x \in \ell_p$ and $y \in \ell_q$.

We define the asymptotic stability of (1).

Definition 2. (Čermák et al)[17, Definition 5] The fractional difference Equation (1) is said to be asymptotically stable if and only if for any $x_1 \in \mathbb{R}^d$, the solution x of (1), with initial condition $x(1) = x_1$, satisfies $\|x(n)\| \to 0$ as $n \to \infty$.

3 | VARIATION OF CONSTANTS FORMULA

The initial value problem (5) for Equation (1) has a unique solution $x : \mathbb{N} \to \mathbb{R}^d$, which satisfies the initial condition

$$(\Delta^{-(1-\alpha)}x)(0) = x(0) = x_0.$$

We denote the solution x by $\varphi_{\text{R-L}}(\cdot, x_0)$. In the following main theorem of this paper, we establish a variation of constants formula.

Theorem 1. *(Variation of constants formula for nabla Riemann-Liouville fractional difference equations). Let $\alpha \in (0, 1)$ and $I - A$ be invertible. Consider Equation (1) with initial condition $x(0) = x_0$. The solution of the nabla Riemann-Liouville Equation (1) is given by*

$$\varphi_{R\text{-}L}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(A, n)x_0 + \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(A, n - k + 1)f(k) \tag{8}$$

for all $n \in \mathbb{N}_2$ and

$$\varphi_{R\text{-}L}(1, x_0) = (I - A)^{-1}(x_0 + f(1)).$$

Proof. Applying the Laplace transform to both sides of (1) and assuming that $n \in \mathbb{N}_2$, we have

$$s^\alpha \mathcal{L}(x)(s) - {}_{\text{R-L}}\Delta^{-(1-\alpha)}x(0) = A\mathcal{L}(x)(s) + \mathcal{L}(f)(s).$$

Therefore,

$$\mathcal{L}(x)(s) = (s^\alpha I - A)^{-1}x_0 + (s^\alpha I - A)^{-1}\mathcal{L}(f)(s) = \mathcal{L}(E_{\overline{(\alpha,\alpha)}}(A, \cdot)x_0 + \mathcal{L}(E_{\overline{(\alpha,\alpha)}}(A, \cdot)\mathcal{L}(f)(s). \tag{9}$$

Hence, by using Proposition 1 and inverse Laplace transform, we obtain

$$\varphi_{\text{R-L}}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(A, n)x_0 + \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(A, n - k + 1)f(k) \tag{10}$$

for all $n \in \mathbb{N}_2$. □

As a corollary, for the linear homogeneous case

$${}_{\text{R-L}}\Delta^\alpha x(n) = Ax(n) \quad (n \in \mathbb{N}_1), \tag{11}$$

we recall the following theorem for explicit solutions.

Theorem 2. *(Čermák et al)[17], theorem 18 Assume that $I - A$ is invertible; then, the solution of (11) is*

$$\varphi_{R\text{-}L}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(A, n)x_0 \quad (n \in \mathbb{N}_2).$$

Theorem 1 can be applied to a nonlinear equation yielding an implicit solution representation by the variation of constants formula. Let $x : \mathbb{N} \to \mathbb{R}^d$ be a solution of the nonlinear nabla Riemann-Liouville fractional difference equation

$$({}_{\text{R-L}}\Delta^\alpha x)(n) = Ax(n) + g(x(n)) \tag{12}$$

for $n \in \mathbb{N}_1$, where ${}_{\text{R-L}}\Delta^\alpha$ is the nabla Riemann-Liouville difference operator of order α , $g : \mathbb{R}^d \to \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$. The initial value problem of the above equation is formulated as for (1), ie, for a given $x_1 \in \mathbb{R}^d$, find a sequence $x : \mathbb{N} \to \mathbb{R}^d$ such that

$$x(0) = (I - A)x_1 - g(x(1))$$

and (12) is satisfied for all $n \in \mathbb{N}_2$. But now, the existence of the solution to this problem is not as obvious as in the case of Equation (1). Then, x is also a solution of the (nonautonomous) linear fractional difference Equation (1) with

$$f : \mathbb{N} \to \mathbb{R}^d, \quad n \mapsto g(x(n)).$$

By Theorem 1, if the solution x exists, it satisfies the implicit equation

$$\varphi_{\text{R-L}}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(A, n)x_0 + \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(A, n-k+1)g(x(k)). \tag{13}$$

4 | APPLICATIONS

Consider a linear inhomogeneous fractional difference equation of the form

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n) = \lambda x(n) + f(n) \tag{14}$$

for $n \in \mathbb{N}_1$, which satisfies the initial condition

$$(\Delta^{-(1-\alpha)}x)(0) = x(0) = x_0,$$

where $x : \mathbb{N}_1 \to \mathbb{R}$, ${}_{\text{R-L}}\Delta^{\alpha}$ is the Riemann-Liouville difference operator of a real order $\alpha \in (0, 1)$, $f : \mathbb{N}_1 \to \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Corollary 1. *(Čermák et al)*[17]*, corollary 23 If $\lambda \in \mathbb{R}$ with $\lambda < 0$ or $\lambda > 2^{\alpha}$, then*

$${}_{R\text{-}L}\Delta^{\alpha}x(n) = \lambda x(n) \quad (n \in \mathbb{N}_1), \tag{15}$$

is asymptotically stable. Moreover, if $-1 < \lambda < 0$, then

$$x(n) = O(n^{-(1+\alpha)})$$

as $n \to \infty$ for all solutions x of (15).

Remark 2. By Corollary 1 and Theorem 2, if $-1 < \lambda < 0$, then

$$E_{\overline{(\alpha,\alpha)}}(\lambda, n) = O(n^{-(1+\alpha)})$$

as $n \to \infty$.

4.1 | Boundedness of solutions of scalar autonomous nonlinear nabla Riemann-Liouville difference equations

The next theorem describes spaces of sequences f and a range of the values of the parameter λ for which the solutions of (14) are bounded.

Theorem 3. *If $-1 < \lambda < 0$ and $f \in \ell_{\infty}$ or $f \in \ell_p$ for $0 < p < 1$, then the solutions of equation (14) are bounded.*

Proof. By the variation of constants formula (Theorem 1), Equation (14) has an unique solution given by the following formula:

$$\varphi_{\text{R-L}}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(\lambda, n)x_0 + \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(\lambda, n-k+1)f(k). \tag{16}$$

From Remark 2, we get

$$\lim_{n\to\infty} E_{\overline{(\alpha,\alpha)}}(\lambda, n)x_0 = 0.$$

Because $f \in \ell_\infty$ or $f \in \ell_p$ for $0 < p < 1$, there exists $M > 0$ such that $|f(n)| < M$ for all n. Then,

$$\left| \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(\lambda, k) f(k) \right| \leq M \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(\lambda, k).$$

Combining this with

$$E_{\overline{(\alpha,\alpha)}}(\lambda, n) = O(n^{-(1+\alpha)})$$

as $n \to \infty$ (see Remark 2), we conclude that the solution of (14) is bounded. □

4.2 | Stability of scalar autonomous nonlinear nabla Riemann-Liouville difference equations

If we consider Equation (14) with sequences f tending to zero fast enough, then the equation will be asymptotically stable for certain values of the parameter λ. This is described in the next result.

Theorem 4. *If $-1 < \lambda < 0$ and $f \in \ell_q$ with $q \geq 1$, then Equation (14) is asymptotically stable.*

Proof. By the variation of constants formula (Theorem 1), Equation (14) has an unique solution given by

$$\varphi_{\text{R-L}}(n, x_0) = E_{\overline{(\alpha,\alpha)}}(\lambda, n) x_0 + \left(E_{\overline{(\alpha,\alpha)}}(\lambda, \cdot) * f(\cdot) \right)(n). \tag{17}$$

Because

$$E_{\overline{(\alpha,\alpha)}}(\lambda, n) = O(n^{-(1+\alpha)})$$

for $n \to \infty$, we have

$$E_{\overline{(\alpha,\alpha)}}(\lambda, \cdot) \in \ell_{\frac{1+\alpha}{1+\varepsilon}}$$

for all $0 < \varepsilon < \alpha$. Hence,

$$r := \frac{1}{\frac{\varepsilon-\alpha}{1+\alpha} + \frac{1}{q}} \geq 1$$

and

$$\frac{1}{\frac{1+\alpha}{1+\varepsilon}} + \frac{1}{q} = \frac{1}{r} + 1.$$

By the Young inequality for

$$E_{\overline{(\alpha,\alpha)}}(\lambda, \cdot) \in \ell_{\frac{1+\alpha}{1+\varepsilon}}$$

and $f \in \ell_q$, we have

$$E_{\overline{(\alpha,\alpha)}}(\lambda, \cdot) * f(\cdot) \in \ell_r$$

and conclude that (14) is asymptotically stable. □

4.3 | Scalar solution separation

Consider a scalar nonlinear fractional difference equation of the form

$$({}_{\text{R-L}}\Delta^{\alpha} x)(n) = \lambda x(n) + g(x(n)), \tag{18}$$

which satisfies the initial condition

$$(\Delta^{-(1-\alpha)} x)(0) = x(0) = x_0,$$

where $x_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, ie, there is a constant $L > 0$ such that

$$|g(x) - g(y)| \leq L|x - y| \tag{19}$$

for $x, y \in \mathbb{R}$. Moreover, we assume that

$$L < |1 - \lambda|. \tag{20}$$

Hence, the assumption (20) ensures the existence of a unique solution of (18) as a special case of Equation (1) with

$$f : \mathbb{N} \to \mathbb{R}^d, \quad n \mapsto g(x(n)).$$

The next theorem presents a lower bound on the separation between two solutions.

Theorem 5. *Consider Equation (18) with $\lambda \in (0,1)$ and assume that g satisfies (19) for*

$$L \in (0, \min\{\lambda, 1-\lambda\}).$$

Then, the unique solution of (18) satisfies the estimate

$$|\varphi_{R\text{-}L}(n,x) - \varphi_{R\text{-}L}(n,y)| \geq E_{\overline{(\alpha,\alpha)}}(\lambda - L, n)|x-y|$$

for $x, y \in \mathbb{R}, n \in \mathbb{N}$.

In the proof of the above theorem, we will use the following lemma on monotonicity with respect to the initial conditions of scalar equations.

Lemma 1. *Consider Equation (18) with $\lambda \in (0,1)$ and assume that g satisfies (19) for $L \in (0,\lambda)$. If $x \leq y$, then*

$$\varphi_{R\text{-}L}(n,x) \leq \varphi_{R\text{-}L}(n,y)$$

for $n \in \mathbb{N}$.

Proof. Define $h(x) := Lx + g(x)$. Then, Equation (18) can be rewritten as

$$({}_{\text{R-L}}\Delta^{\alpha}x)(n) = (\lambda - L)x(n) + h(x(n)) \tag{21}$$

for $n \in \mathbb{N}$. Moreover, for $x \leq y$,

$$\begin{aligned} h(y) - h(x) &= Ly + g(y) - (Lx + g(x)) \\ &= g(y) - g(x) + L(y-x) \\ &\geq -L(y-x) + L(y-x) \\ &= 0, \end{aligned}$$

ie, h is monotonically increasing. By Theorem 1, for $x, y \in \mathbb{R}$,

$$\varphi_{\text{R-L}}(n,y) - \varphi_{\text{R-L}}(n,x) = E_{\overline{(\alpha,\alpha)}}(\lambda - L, n)(y-x) + \sum_{k=1}^{n} E_{\overline{(\alpha,\alpha)}}(\lambda - L, n-k+1)(h(\varphi_{\text{R-L}}(k,y)) - h(\varphi_{\text{R-L}}(k,x))) \quad (n \in \mathbb{N}_1). \tag{22}$$

Since $\lambda - L > 0$, we have

$$E_{\overline{(\alpha,\alpha)}}(\lambda - L, n) > 0$$

for all $n \in \mathbb{N}_1$. Hence, $x \leq y$ implies

$$\varphi_{\text{R-L}}(n,x) \leq \varphi_{\text{R-L}}(n,y)$$

for $n \in \mathbb{N}_1$. □

We are now in a position to prove Theorem 5.

Proof. Assume that $x < y$. By Lemma 1, Equation (22), and the fact that h is monotonically increasing, we get

$$\varphi_{\text{R-L}}(n,y) - \varphi_{\text{R-L}}(n,x) \geq E_{\overline{(\alpha,\alpha)}}(\lambda - L, n)(y-x)$$

for $n \in \mathbb{N}$. □

As an application of Theorem 5 to Equation (18) with $y = 0$, we get that the Lyapunov exponent of nonzero solutions is nonnegative, where the Lyapunov exponent of function $f : \mathbb{N}_0 \to \mathbb{R}^d$ is defined by $\limsup_{n\to\infty} \frac{1}{n} \ln \|f(n)\|$.

Corollary 2. *Consider Equation (18) with $\lambda \in (0,1)$ and assume that f satisfies (19) with*

$$L \in [0, \min\{\lambda, 1-\lambda\}).$$

Then, for all nontrivial solutions of Equation (18), we have

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\varphi_{R\text{-}L}(n, x_0)| \geq 0 \tag{23}$$

for $0 < \lambda - L < 1$ *and* $x_0 \in \mathbb{R} \setminus \{0\}$.

Proof. Using Theorem 5 for $y = 0$, we have

$$|\varphi_{\text{R-L}}(n, x_0)| \geq E_{\overline{(\alpha,\alpha)}}(\lambda - L, n)|x_0|.$$

Recall from Graham et al[16], p165 and Čermák et al[19], p656 that for all $\alpha > 0$,

$$\begin{aligned}
&(-1)^{n-1}\binom{-(k\alpha+\alpha)}{n-1} \\
&= (-1)^{n-1}\frac{(-(k\alpha+\alpha))\ \dots\ (-(k\alpha+\alpha+n-2))}{1\cdot 2\ \dots\ (n-1)} \\
&= \frac{(k\alpha+\alpha)(k\alpha+\alpha+1)\ \dots\ (k\alpha+\alpha+n-2)}{1\cdot 2\ \dots\ (n-1)} > 0
\end{aligned}$$

for all $n \geq 2$. With

$$n_0 := \lceil \frac{1-\alpha}{\alpha} \rceil,$$

we have $k\alpha + \alpha \geq 1$ for all $k \geq n_0$. As a consequence, for $n > n_0$,

$$(-1)^{n-1}\binom{-(k\alpha+\alpha)}{n-1} < 1$$

for $k \in \{0, 1, \dots n_0 - 1\}$ and

$$(-1)^{n-1}\binom{-(k\alpha+\alpha)}{n-1} \geq 1$$

for $k \in \{n_0, n_0 + 1, \dots n\}$. Therefore,

$$|\varphi_{\text{R-L}}(n, x_0)| \geq \sum_{k=n_0}^{n} (\lambda - L)^k |x_0|.$$

Combining the last inequality with

$$\lim_{n\to\infty} \frac{1}{n} \ln \sum_{k=n_0}^{n} q^k = \begin{cases} \ln q & \text{if} q > 1, \\ 0 & \text{if} 0 < q \leq 1, \end{cases}$$

we obtain (23). □

5 | CONCLUSIONS

In this paper, we investigated properties of discrete nonlinear nabla Riemann-Liouville equations. We presented a variation of constants formula that expresses the solution of semilinear equations by the solution of its linear part, which is in turn given by the nabla discrete-time Mittag-Leffler function. On the basis of this formula, we obtained results about the asymptotic behaviour of one-dimensional equations such as boundedness conditions and separation of solutions. In particular, we showed that the Lyapunov exponent of certain nonlinear equations is positive for all initial conditions.

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FINANCIAL DISCLOSURE

None reported.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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