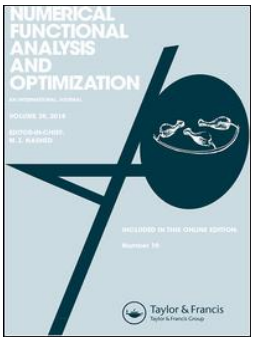

Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: https://www.tandfonline.com/loi/lnfa20

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NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
https://doi.org/10.1080/01630563.2020.1737937

The Ishikawa Subgradient Extragradient Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

Manh Hy Duc, Ha Nguyen Thi Thanh, Thanh Tran Thi Huyen, and Bui Van Dinh

Department of Mathematics, Le Quy Don Technical University, Hanoi, Vietnam

ABSTRACT

In this paper, we introduce a novel algorithm for finding a common point of the solution set of a class of equilibrium problems involving pseudo-monotone bifunctions and satisfying the Lipschitz-type condition and the set of fixed points of a quasi-nonexpansive in a real Hilbert space. This algorithm can be considered as a combination of the subgradient extragradient method for equilibrium problems and the Ishikawa method for fixed point problems. The strong convergence of the iterates generated by the proposed method is obtained under the main assumptions that the fixed-point mapping is demiclosed at 0 and Lipschitz-type constant of the cost bifunction is unknown. Some numerical examples are implemented to show the computational efficiency of the proposed algorithm.

ARTICLE HISTORY

Received 3 June 2019
Revised 19 February 2020
Accepted 29 February 2020

KEYWORDS

Pseudo-monotone; equilibria; Hilbert space; Lipschitz-type continuous; quasi-nonexpansive mapping; strong convergence

2010 MATHEMATICS SUBJECT CLASSIFICATION

90C25; 90C33; 65K10; 65K15

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Let C be a nonempty closed convex subset of $\mathbb{H}$ and $f : C \times C \to \mathbb{R}$ be a function satisfying $f(x, x) = 0$ for all $x \in C$. Such a function is called the equilibrium bifunction. The equilibrium problem (shortly EP(C, f)), in the sense of Blum, Muu and Oettli [1, 2], is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C. \tag{1.1}$$

The solution set of EP(C, f) is denoted by $Sol(C, f)$. Problem (1.1) is also well known as the Ky Fan inequality due to his contribution in this field [3]. Although problem EP(C, f) has a simple formulation, it is quite general in the sense that, it includes as special cases, many known mathematical models as: variational inequality problems, optimization problems, fixed point problems, saddle point problems, the Nash equilibrium problem in noncooperative game; see, for example, [1, 2, 4–6].

CONTACT Bui Van Dinh vandinhb@gmail.com Department of Mathematics, Le Quy Don Technical University, Hanoi, Vietnam.

The existence and solution methods for equilibrium problems have been intensively studied in cases the bifunction f is monotone or general monotone [4, 7–12], nonmonotone [13, 14] in Hilbert spaces or Banach spaces setting [15].

Let $T : C \to C$ be a mapping and $Fix(T)$ be the set of fixed points of T; that is, $Fix(T) = \{p \in C : Tp = p\}$. Recall that T is said to be nonexpansive if for all $x, y \in C$, $||Tx - Ty|| \le ||x - y||$. If $Fix(T)$ is nonempty and $||Tx - p|| \le ||x - p||, \forall x \in C, p \in Fix(T)$, then T is called quasi-nonexpansive. It is well-known that $Fix(T)$ is closed and convex when T is quasi-nonexpansive [16]. A mapping T is said to be pseudo contractive if for all $x, y \in C$ and $\tau > 0$ we have

$$||x - y|| \le ||(1 + \tau)(x - y) - \tau(Tx - Ty)||.$$

Most of the methods for finding a fixed point of mapping T are deduced from the Mann iteration [17]:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k)Tx^k, \end{cases} \tag{1.2}$$

where the parameter sequence $\{\alpha_k\} \subset (0, 1)$ and satisfies some certain conditions. It was proved that the sequence $\{x^k\}$ weakly converges to a fixed point of mapping T.

Another well-known method to find a fixed point of a Lipschitzian pseudo contractive map was proposed by Ishikawa [18] as follows

$$\begin{cases} x^0 \in C, \\ y^k = \alpha_k x^k + (1 - \alpha_k)Tx^k, \\ x^{k+1} = \beta_k x^k + (1 - \beta_k)Ty^k \end{cases} \tag{1.3}$$

where $0 \le \alpha_k \le \beta_k \le 1$ for all k. He proved that if $\lim_{k\to\infty} \beta_k = 1$, $\sum_{k=1}^{\infty}(1 - \alpha_k)(1 - \beta_k) = \infty$, then $\{x^k\}$ generated by (1.3) converges weakly to a fixed point of mapping T (see [18, 19]).

In order to get the strong convergence algorithm, the Halpern iteration was proposed as follows (see [20])

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)Tx^k, \end{cases} \tag{1.4}$$

where $\{\alpha_k\} \subset (0, 1)$, $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = +\infty$. It was showed that the sequence $\{x^k\}$ generated by (1.4) converges strongly to a fixed point of mapping T.

In recent years, many researchers studied the problem of finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive or demicontractive mapping; see, for instance, [21–25] and the references therein. The motivation for studying such a problem is its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or equilibrium problems. This happens, especially, in the practical problems as signal processing, network resource allocation, image recovery and Nash-Cournot oligopolistic equilibrium models in economy [15, 26–29].

For obtaining a common element of the set of solutions of EP(C, f) and fixed points of a κ-demicontractive mapping T, Hieu [30] proposed to modify the Halpern subgradient extragradient method for variational inequality problems [11, 31] to get the following algorithm:

Algorithm 1.1

Choose $x^0 \in C$ and parameters λ, $\{\alpha_k\}, \{\beta_k\}$ such that $0<\lambda<\min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. $0<\alpha_k<1$, $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = +\infty$; $0<a \le \beta_k \le \frac{1-\kappa}{2}$.

Step 1. *Solve two strongly convex optimization problems*

$$\begin{cases} y^k = \text{argmin}\{\lambda f(x^k, y) + \frac{1}{2}||y-x^k|| : y \in C\} \\ z^k = \text{argmin}\{\lambda f(y^k, y) + \frac{1}{2}||y-x^k|| : y \in H_k\}, \end{cases}$$

where $H_k = \{x \in \mathbb{H} : \langle x^k - \lambda w^k - y^k, x - y^k \rangle \le 0\}$ and $w^k \in \partial_2 f(x^k, y^k)$.

Step 2. *Compute* $t^k = \alpha_k x^0 + (1-\alpha_k) z^k$,

$$x^{k+1} = \beta_k T t^k + (1-\beta_k) t^k.$$

Set $k := k+1$ and go back to **Step 1**.

Where c_1, c_2 are Lipschitz-type constants of the bifunction f.

He proved in [30] that if the bifunction f is pseudomonotone, Lipschitz-type continuous, weakly continuous and convex with respect to the second variable and the mapping T is κ-demicontractive and demiclosed at zero such that $\Omega = Sol(C, f) \cap Fix(T) \neq \emptyset$. Then the sequences$\{x^k\}$ generated by **Algorithm 1.1** converges strongly to $x^* = P_\Omega(x^0)$.

One advantage of this algorithm is that in **Step 1** we only have to solve one strongly convex program on the constraint set C instead of two as in literature [25, 32] since H_k is a half-space. This may help to reduce the cost of computation, especially when C has got a complicated structure. However, to apply this algorithm one requires to know the Lipschitz-type constants of the bifunction f. In some cases, the Lipschitz-type constants of f are unknown or difficult to estimate so we cannot apply this algorithm directly.

In this paper, we modify Hieu's iteration process for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a quasi-nonexpansive mapping in a real Hilbert space in which the bifunction f is pseudomonotone on C with respect to $Sol(C, f)$. More precisely, we propose to use the subgradient extragradient algorithm [33,34] for solving the equilibrium problem combining with the Ishikawa's process for mapping T instead of the Halpern's iteration as in Algorithm 1. Moreover, our algorithm could be applied directly for the case the Lipschitz-type constants of the bifunction f are unknown.

The rest of paper is organized as follows. The next section contains some preliminaries on the metric projection, equilibrium problems and convex optimization. The Ishikawa subgradient extragradient algorithm and its convergence is presented in the third section. The last section is devoted to presentation of examples and numerical results.

2. Preliminaries

Let C be a nonempty closed convex subset of $\mathbb{H}$. We denote the metric projection onto C by P_C. Namely, for each $x \in \mathbb{H}, P_C(x)$ is the unique element in C such that

$$||x-P_C(x)|| \leq ||x-y||, \forall y \in C.$$

For example, if $C = H = \{y \in \mathbb{H} : \langle a, y\rangle + b \leq 0\}$, for some $a \in \mathbb{H}$ and $b \in \mathbb{R}$, is a half space. Then we have

$$P_H(x) = \begin{cases} x & \text{if } x \in H, \\ x - \dfrac{\langle a, x\rangle + b}{||a||^2} a & \text{if } x \notin H. \end{cases}$$

It is well-known that the metric projection P_C has the following properties:

Lemma 1. *Suppose that C is a nonempty closed convex subset in $\mathbb{H}$. Then we have*

(a) $P_C(x)$ is singleton and well defined for every x;
(b) $z = P_C(x)$ if and only if $\langle x-z, y-z\rangle \leq 0, \ \forall y \in C$;
(c) $||P_C(x)-P_C(y)||^2 \leq \langle P_C(x)-P_C(y), x-y\rangle, \forall x, y \in \mathbb{H}$;
(d) $||P_C(x)-P_C(y)||^2 \leq ||x-y||^2 - ||x-P_C(x)-y+P_C(y)||^2, \forall x, y \in \mathbb{H}$.

We recall the following definitions.

Definition 1. [1, 2, 35] Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction, and C be a nonempty, closed and convex subset of $\mathbb{H}, \emptyset \neq D \subset C$. Bifunction f is said to be:

(a) strongly monotone with constant $\tau > 0$ if

$$f(x, y) + f(y, x) \leq -\tau||x-y||^2, \forall x, y \in C;$$

(b) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(c) *pseudo monotone on C if*

$$\forall x, y \in C, f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

(d) *pseudo monotone on C with respect to D if*

$$\forall x^* \in D, \forall y \in C, f(x^*, y) \geq 0, \Rightarrow f(y, x^*) \leq 0;$$

(e) *Lipschitz-type continuous on C if there exist positive constants c_1 and c_2 such that*

$$f(x,y) + f(y,z) \geq f(x,z) - c_1||x-y||^2 - c_2||y-z||^2, \forall x, y, z \in C.$$

From Definition 1, we have the followings:

(i) $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \forall D \subset C$.
(ii) If $f(x,y) = \langle F(x), y-x \rangle$, for a mapping $F : \mathbb{H} \to \mathbb{H}$. Then the notions of monotonicity of bifunction f collapse to the notions of monotonicity of mapping F, respectively. Moreover, if mapping F is Lipschitz with constant L on C, i.e., $||F(x)-F(y)|| \leq L||x-y||, \forall x, y \in C$. Then, f is also Lipschitz-type continuous on C (see [35, 36]), for example, with constants $c_1 = \frac{L}{2\epsilon}, c_2 = \frac{L\epsilon}{2}$, for any $\epsilon > 0$.

Now, let $\varphi : C \to (-\infty; +\infty]$ be a proper, convex and lower semicontinuous function and $\rho > 0$, the proximal mapping of φ with ρ is defined by

$$prox_{\rho\varphi}(x) = \arg\min\{\rho\varphi(y) + \frac{1}{2}||y-x||^2 : y \in C\}, x \in \mathbb{H}.$$

The followings are well-known properties of the proximal mapping (see [37]).

Lemma 2. *For all $x \in \mathbb{H}, u \in C$, the following three claims are equivalent:*

(i) $u = prox_{\rho\varphi}(x)$.
(ii) $\frac{x-u}{\rho} \in \partial\varphi(u)$.
(iii) $\langle x-u, y-u \rangle \leq \rho(\varphi(y) - \varphi(u))$ *for any* $y \in C$.

The following lemmas will be used in the sequel.

Lemma 3. *([38, Lemma 2.5]) Let $\{s_k\}$ be a sequence of nonnegative real numbers such that:*

$$s_{k+1} \leq (1-\lambda_k)s_k + \lambda_k\delta_k + \eta_k, \forall k \geq 0,$$

where $\{\lambda_k\}, \{\delta_k\}, \{\eta_k\}$ satisfy the following conditions:

(i) $\{\lambda_k\} \subset [0,1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, *or* $\prod_{k=1}^{\infty}(1-\lambda_k) = 0$;
(ii) $\limsup_{k\to\infty} \delta_k \leq 0$;

(iii) $\eta_k \geq 0 \forall k \geq 0, \sum_{k=0}^{\infty} \eta_k < \infty.$

Then $\lim_{k\to\infty} s_k = 0.$

Lemma 4. *(see [39]) Let $\{r_k\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{r_{k_j}\}$ of $\{r_k\}$ such that $r_{k_j} < r_{k_j+1}, \forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} m_k = \infty$, and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$r_{m_k} \leq r_{m_k+1}, \text{ and } r_k \leq r_{m_k+1}.$$

In fact, m_k is the largest number m in the set $\{1, 2, ..., k\}$ such that $r_m < r_{m+1}$.

3. Main results

Now we are in a position to present an algorithm to find a common point of the solution set of an equilibrium problem and the fixed point of a mapping. To do so, we need the following blanket assumptions.

Asumption $\mathcal{A}$.

(A_1) f is weakly continuous on $C \times C$;
(A_2) $f(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C, for all $x \in C$;
(A_3) f is pseudomonotone on C with respect to $Sol(C, f)$;
(A_4) f is Lipschitz-type continuous on C.
(A_5) T is quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero.

It is well-known that if the bifunction f and the mapping T satisfy Assumptions $\mathcal{A}$, then the solution set $Sol(C, f)$ of EP(C, f) is closed and convex [4] and the set of fixed points of mapping T, $Fix(T)$ is closed and convex [37]. Therefore $\Omega = Sol(C,f) \cap Fix(T)$ is closed and convex. However, it is not given explicitly, we cannot find one point $x^* \in \Omega$ directly. The following algorithm give us a way to get $x^* \in \Omega$.

Algorithm 3.1

Initialization. Pick $x^0 = x^g \in C, \rho_0 > 0, \delta \in (0,1)$, and choose sequences $\{\mu_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ such that $\{\mu_k\} \subset [0,1]$, $\lim_{k\to\infty} \mu_k = 1$, $\{\alpha_k\} \subset [_\alpha, \bar{\alpha}] \subset (0,1), \{\beta_k\} \subset [_\beta, \bar{\beta}] \subset (0,1), \{\gamma_k\} \subset [_\gamma, \bar{\gamma}] \subset (0,1)$ and $\alpha_k + \beta_k + \gamma_k = 1, \forall k$.

Iteration k (k = 0, 1, 2, …). Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min\left\{f(x^k,y)+\frac{1}{2\rho_k}||y-x^k||^2 : y\in C\right\}CP(x^k)$$

to obtain its unique solution y^k.

Step 2. Take $w^k\in\partial_2 f(x^k,y^k)$,

$$H_k=\{x\in\mathbb{H} : \langle x^k-\rho_k w^k-y^k, x-y^k\rangle\le 0\},$$

and

$$z^k=\operatorname{argmin}\left\{f(y^k,y)+\frac{1}{2\rho_k}||y-x^k||^2 : y\in H_k\right\}.$$

Step 3. Compute

$$\begin{aligned} t^k &= \lambda_k x^g+(1-\lambda_k)z^k,\\ u^k &= \mu_k x^k+(1-\mu_k)Tx^k,\\ x^{k+1} &= \alpha_k u^k+\beta_k z^k+\gamma_k Tt^k. \end{aligned}$$

Set $\rho=f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k)$ and set

$$\rho_{k+1}=\begin{cases}\min\left\{\dfrac{\delta}{2\rho}(||x^k-y^k||^2+||z^k-y^k||^2),\rho_k\right\}, & \text{if } \rho>0\\ \rho_k, & \text{otherwise,}\end{cases}$$

and go to Iteration k with k replaced by $k+1$.

Remark 1. • *If $f(x,y)=\langle F(x),y-x\rangle$, where $F:C\to\mathbb{H}$ is a mapping. Then Step 1 and Step 2 becomes*

$$y^k=P_C(x^k-\rho_k F(x^k)),$$

and

$$z^k=P_{H_k}(x^k-\rho_k F(y^k)).$$

Since H_k is a half space, z^k can be computed explicitly.

• *From the definition of $\{\rho_k\}$ in Algorithm 2, it can be seen that $\{\rho_k\}$ is a decreasing sequence. In addition, if f satisfies the Lipschitz-type condition with constants L_1* and *L_2* on *C then we have.*

$$\begin{aligned}\rho&=f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k)\\ &\le L_1||x^k-y^k||^2+L_2||y^k-z^k||^2\\ &\le \max\{L_1,L_2\}(||x^k-y^k||^2+||y^k-z^k||^2).\end{aligned}$$

Therefore $\rho_k\ge\min\{\frac{\delta}{2\max\{L_1,L_2\}},\rho_0\},\forall k$.

The following theorem give us the convergence of Algorithm 3.1.

Theorem 1. *Suppose that the set* $\Omega = Sol(C,f) \cap Fix(T)$ *is nonempty, and the sequence* $\{\lambda_k\} \subset (0,1)$, satisfies $\sum_{k=0}^{\infty} \lambda_k = \infty$, $\lim_{k\to\infty} \lambda_k = 0$. *Then under Assumptions* $\mathcal{A}$ *the sequences* $\{x^k\}, \{y^k\}, \{z^k\}$ *generated by Algorithm 3.1 converge strongly to the solution* $x^* = P_\Omega(x^g)$.

Before presenting the proof of Theorem 1, we need the following lemma.

Lemma 5. *The sequences* $\{x^k\}, \{z^k\}, \{t^k\}$ and $\{u^k\}$ *are bounded.*

Proof. Let $x^* \in \Omega = Sol(C,f) \cap Fix(S)$. It follows from the definition of t^k that

$$\begin{aligned} ||t^k - x^*|| &= ||\lambda_k x^g + (1-\lambda_k) z^k - x^*|| \\ &= ||\lambda_k(x^g - x^*) + (1-\lambda_k)(z^k - x^*)|| \\ &\le \lambda_k ||x^g - x^*|| + (1-\lambda_k)||z^k - x^*||. \end{aligned}$$

From the definition of u^k we have

$$\begin{aligned} ||u^k - x^*|| &= ||\mu_k x^k + (1-\mu_k) T x^k - x^*|| \\ &= ||\mu_k(x^k - x^*) + (1-\mu_k)(T x^k - x^*)|| \\ &\le \mu_k ||x^k - x^*|| + (1-\mu_k)||x^k - x^*|| \\ &= ||x^k - x^*||. \end{aligned} \tag{3.1}$$

Since

$$z^k = \operatorname{argmin}\left\{ f(y^k, y) + \frac{1}{2\rho_k}||y - x^k||^2 : y \in H_k \right\},$$

by Lemma 2, we obtain

$$\rho_k(f(y^k, y) - f(y^k, z^k)) \ge \langle x^k - z^k, y - z^k \rangle, \forall y \in H_k.$$

Substituting $y = x^*$ in to the above inequality, we get

$$\rho_k(f(y^k, x^*) - f(y^k, z^k)) \ge \langle x^k - z^k, x^* - z^k \rangle. \tag{3.2}$$

Since $x^* \in Sol(C,f)$ and $y^k \in C, f(x^*, y^k) \ge 0$. By the pseudo monotonicity of f we have $f(y^k, x^*) \le 0$. So, we get from (3.2) that

$$-\rho_k f(y^k, z^k) \ge \langle x^k - z^k, x^* - z^k \rangle. \tag{3.3}$$

Since $\omega^k \in \partial_2 f(x^k, y^k)$, we have

$$f(x^k, y) - f(x^k, y^k) \ge \langle \omega^k, y - y^k \rangle, \forall y \in C.$$

Therefore

$$f(x^k, z^k) - f(x^k, y^k) \ge \langle \omega^k, z^k - y^k \rangle.$$

So

$$\rho_k(f(x^k,z^k)-f(x^k,y^k)) \geq \rho_k\langle\omega^k,z^k-y^k\rangle. \tag{3.4}$$

Since $z^k \in H_k$, we have

$$\langle x^k-\rho_k\omega^k-y^k,z^k-y^k\rangle \leq 0.$$

So

$$\rho_k\langle\omega^k,z^k-y^k\rangle \geq \langle x^k-y^k,z^k-y^k\rangle. \tag{3.5}$$

From (3.3), (3.4), and (3.5) we deduce that

$$2\rho_k(f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k)) \geq 2(\langle x^k-z^k,x^*-z^k\rangle + \langle x^k-y^k,z^k-y^k\rangle)$$

or

$$\begin{aligned}2\rho_k(f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k)) \geq\ & ||z^k-x^*||^2-||x^k-x^*||^2\\ &+||x^k-y^k||^2+||z^k-y^k||^2.\end{aligned}$$

Thus

$$\begin{aligned}||z^k-x^*||^2 \leq\ & ||x^k-x^*||^2-||x^k-y^k||^2-||z^k-y^k||^2\\ &+2\rho_k(f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k)).\end{aligned} \tag{3.6}$$

By definition of ρ_k we have

$$\begin{aligned}||z^k-x^*||^2 \leq\ & ||x^k-x^*||^2-||x^k-y^k||^2-||z^k-y^k||^2\\ &+2\frac{\rho_k}{\rho_{k+1}}\rho_{k+1}(f(x^k,z^k)-f(y^k,z^k)-f(x^k,y^k))\\ \leq\ & ||x^k-x^*||^2-||x^k-y^k||^2-||z^k-y^k||^2\\ &+\frac{\rho_k}{\rho_{k+1}}\delta(||x^k-y^k||^2+||z^k-y^k||^2)\\ =\ & ||x^k-x^*||^2-\left(1-\frac{\rho_k}{\rho_{k+1}}\delta\right)(||x^k-y^k||^2+||z^k-y^k||^2).\end{aligned} \tag{3.7}$$

Since $\lim_{k\to\infty}\frac{\rho_k}{\rho_{k+1}}\delta = \delta \in (0;1)$, there exists $N \geq 0$ such that

$$||z^k-x^*|| \leq ||x^k-x^*||, \forall k \geq N. \tag{3.8}$$

It is clear that

$$\begin{aligned}||x^{k+1}-x^*|| &= ||\alpha_k u^k+\beta_k z^k+\gamma_k Tt^k-x^*||\\ &= ||\alpha_k(u^k-x^*)+\beta_k(z^k-x^*)+\gamma_k(Tt^k-x^*)||\\ &\leq \alpha_k||u^k-x^*||+\beta_k||z^k-x^*||+\gamma_k||Tt^k-x^*||.\end{aligned}$$

Therefore, for all $k \geq N$ we have

$$\begin{aligned}||x^{k+1}-x^*|| &\leq (\alpha_k+\beta_k)||x^k-x^*|| + \gamma_k||t^k-x^*||\\ &\leq (\alpha_k+\beta_k)||x^k-x^*|| + \gamma_k(\lambda_k||x^g-x^*|| + (1-\lambda_k)||x^k-x^*||)\\ &= [\alpha_k+\beta_k+\gamma_k(1-\lambda_k)]||x^k-x^*|| + \gamma_k\lambda_k||x^g-x^*||\\ &= (1-\gamma_k\lambda_k)||x^k-x^*|| + \gamma_k\lambda_k||x^g-x^*||\\ &\leq \max\{||x^g-x^*||, ||x^k-x^*||\} \leq \cdots \leq \max\{||x^g-x^*||, ||x^N-x^*||\}.\end{aligned}$$

Hence the sequence $\{x^k\}$ is bounded. Consequently, the sequences $\{z^k\}, \{y^k\}, \{t^k\}$, and $\{u^k\}$ are bounded. This completes the proof of Lemma 5. □

Now we prove Theorem 1.

Since Ω is a nonempty closed and convex subset in the real Hilbert space $\mathbb{H}$, there exists the unique element $x^* \in \Omega$ such that $x^* = P_\Omega(x^g)$. By Lemma 1, we have

$$\langle x^g-x^*, p-x^*\rangle \leq 0, \forall p \in \Omega. \tag{3.9}$$

From definition of x^{k+1}, we have

$$\begin{aligned}||x^{k+1}-x^*||^2 &= ||\alpha_k u^k + \beta_k z^k + \gamma_k Tt^k - x^*||^2\\ &= ||\alpha_k(u^k-x^*) + \beta_k(z^k-x^*) + \gamma_k(Tt^k-x^*)||^2\\ &= \alpha_k||u^k-x^*||^2 + \beta_k||z^k-x^*||^2 + \gamma_k||Tt^k-x^*||^2\\ &\quad -\alpha_k\beta_k||u^k-z^k||^2 - \alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2\\ &\leq \alpha_k||u^k-x^*||^2 + \beta_k||z^k-x^*||^2 + \gamma_k||Tt^k-x^*||^2\\ &\quad -\alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2\\ &\leq \alpha_k||u^k-x^*||^2 + \beta_k||z^k-x^*||^2 + \gamma_k||t^k-x^*||^2\\ &\quad -\alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2.\end{aligned}$$

So

$$\begin{aligned}||x^{k+1}-x^*||^2 &\leq \alpha_k||u^k-x^*||^2 + \beta_k||z^k-x^*||^2 + \gamma_k||\lambda_k x^g + (1-\lambda_k)z^k - x^*||^2\\ &\quad -\alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2\\ &\leq \alpha_k||u^k-x^*||^2 + \beta_k||z^k-x^*||^2 + 2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle\\ &\quad + (1-\lambda_k)\gamma_k||z^k-x^*||^2 - \alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2\\ &= \alpha_k||u^k-x^*||^2 + [\beta_k + (1-\lambda_k)\gamma_k]||z^k-x^*||^2\\ &\quad + 2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle - \alpha_k\gamma_k||Tt^k-u^k||^2 - \beta_k\gamma_k||Tt^k-z^k||^2.\end{aligned}$$

Combining with (3.1) and (3.7), for all $k \geq N$, we have

$$
\begin{aligned}
||x^{k+1}-x^*||^2 \leq & \alpha_k||x^k-x^*||^2 \\
& + [\beta_k + (1-\lambda_k)\gamma_k](||x^k-x^*||^2-||x^k-y^k||^2-||z^k-y^k||^2) \\
& + [\beta_k + (1-\lambda_k)\gamma_k)]\frac{\rho_k}{\rho_{k+1}}\delta(||x^k-y^k||^2+||z^k-y^k||^2) \\
& + 2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle-\alpha_k\gamma_k||Tt^k-u^k||^2-\beta_k\gamma_k||Tt^k-z^k||^2 \\
= & (1-\lambda_k\gamma_k)||x^k-x^*||^2-\left(1-\frac{\rho_k}{\rho_{k+1}}\delta\right)\tau_k(||x^k-y^k||^2+||z^k-y^k||^2) \\
& + 2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle-\alpha_k\gamma_k||Tt^k-u^k||^2-\beta_k\gamma_k||Tt^k-z^k||^2,
\end{aligned}
\tag{3.10}
$$

where $\tau_k = 1-\alpha_k-\lambda_k\gamma_k$.

Therefore,

$$
\begin{aligned}
||x^{k+1}-x^*||^2 \leq & ||x^k-x^*||^2-\left(1-\frac{\rho_k}{\rho_{k+1}}\delta\right)\tau_k(||x^k-y^k||^2+||z^k-y^k||^2) \\
& + 2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle-\alpha_k\gamma_k||Tt^k-u^k||^2-\beta_k\gamma_k||Tt^k-z^k||^2.
\end{aligned}
\tag{3.11}
$$

We now consider two distinct cases.

Case 1. There exists $M \geq N$ such that

$$||x^{k+1}-x^*|| \leq ||x^k-x^*||, \forall k \geq M.$$

In this case, the limit of $\{||x^k-x^*||\}$ exists, say $\lim_{k\to\infty} \|x^k - x^*\| = a \geq 0$.

From (3.11), we have

$$
\begin{aligned}
\left(1-\frac{\rho_k}{\rho_{k+1}}\delta\right)\tau_k(||x^k-y^k||^2+||z^k-y^k||^2) \leq & ||x^k-x^*||^2-||x^{k+1}-x^*||^2 \\
& +2\lambda_k\gamma_k\langle x^g-x^*, t^k-x^*\rangle.
\end{aligned}
$$

Since $\alpha_k \in [\underline{\alpha}, \bar{\alpha}] \subset (0,1)$ and $\lim_{k\to\infty} \lambda_k = 0$, we have that $\liminf_{k\to\infty} \tau_k \geq 1-\bar{\alpha}>0$. In addition, $\lim_{k\to\infty}\left(1-\frac{\rho_k}{\rho_{k+1}}\delta\right) = 1-\delta>0, \{t^k\}$ is bounded, we get in the limit of the above inequality that

$$\lim_{k\to\infty} ||x^k-y^k|| = 0 \text{ and } \lim_{k\to\infty} ||z^k-y^k|| = 0. \tag{3.12}$$

Therefore, $\lim_{k\to\infty} ||x^k-z^k|| = 0$.

Because $||u^k-x^k|| = (1 - \mu_k)||Tx^k-x^k||$ and $\lim_{k\to\infty} \mu_k = 1$, we get

$$\lim_{k\to\infty} ||u^k-x^k|| = 0. \tag{3.13}$$

So, $\lim_{k\to\infty} ||z^k-u^k|| = 0$.

Similarly, we have

$$\lim_{k\to\infty} ||t^k - z^k|| = 0, \text{ and } \lim_{k\to\infty} \|t^k - x^k\| = 0. \tag{3.14}$$

Since the sequence $\{x^k\}$ is bounded, there exists a subsequence $\{x^{n_k}\}$ that converges weakly to some $p^0 \in \mathbb{H}$, such that

$$\limsup_{k\to\infty} \langle x^g - x^*, x^k - x^* \rangle = \lim_{n\to\infty} \langle x^g - x^*, x^{n_k} - x^* \rangle = \langle x^g - x^*, p^0 - x^* \rangle. \tag{3.15}$$

Combining with (3.12) and (3.14) we get the sequences $\{y^{n_k}\}, \{z^{n_k}\}, \{t^{n_k}\}, \{u^{n_k}\}$ converge weakly to p^0 and $p^0 \in C$.

Since $y^{n_k} = prox_{\rho_{n_k} f(x^{n_k},.)}(x^{n_k})$, by Lemma 2, we have

$$\rho_{n_k}\Big(f(x^{n_k}, y) - f(x^{n_k}, y^{n_k})\Big) \geq \langle x^{n_k} - y^{n_k}, y - y^{n_k} \rangle, \forall y \in C.$$

Let $k \to \infty$, using the continuity of f and $\lim_{n\to\infty} \rho_{n_k} = \rho > 0$, we obtain

$$f(p^0, y) - f(p^0, p^0) \geq 0.$$

So

$$f(p^0, y) \geq 0, \forall y \in C.$$

This means that p^0 is a solution of $EP(C,f)$.

Next, we need to show that $p^0 \in Fix(T)$. Indeed, from (3.10), we have

$$\begin{aligned} ||x^{k+1} - x^*||^2 &\leq (1 - \lambda_k\gamma_k)||x^k - x^*||^2 + 2\lambda_k\gamma_k\langle x^g - x^*, t^k - x^* \rangle \\ &\quad - \alpha_k\gamma_k\|Tt^k - u^k\|^2 - \beta_k\gamma_k\|Tt^k - z^k\|^2 \\ &\leq (1 - \lambda_k\gamma_k)||x^k - x^*||^2 + 2\lambda_k\gamma_k\langle x^g - x^*, t^k - x^* \rangle \\ &\quad - \alpha_k\gamma_k||Tt^k - u^k||^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} (||x^{k+1} - x^*||^2 - ||x^k - x^*||^2) \\ &\leq \liminf_{k\to\infty} (-\lambda_k\gamma_k||x^k - x^*||^2 + 2\lambda_k\gamma_k\langle x^g - x^*, t^k - x^* \rangle - \alpha_k\gamma_k||Tt^k - u^k||^2) \\ &= -\limsup_{k\to\infty} \alpha_k\gamma_k||Tt^k - u^k||^2 \leq -_{\alpha\gamma} \limsup_{k\to\infty} ||Tt^k - u^k||^2. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} ||Tt^k - u^k|| = 0.$$

It is clear that

$$||Tt^k - t^k|| \leq ||Tt^k - u^k|| + ||u^k - x^k|| + ||x^k - t^k||.$$

Combining this with (3.13) and (3.14) we get

$$\lim_{k\to\infty} ||Tt^k - t^k|| = 0.$$

Since $\{t^{n_k}\}$ converges weakly to p^0, $\lim_{k\to\infty} ||Tt^{n_k} - t^{n_k}|| = 0$ and $I - T$ is demiclosed at zero, we can conclude that $p^0 \in Fix(T)$. Hence

$$p^0 \in \Omega = Sol(C, f) \cap Fix(T).$$

From (3.9), (3.15), we obtain

$$\limsup_{k\to\infty} \langle x^g - x^*, x^k - x^* \rangle = \langle x^g - x^*, p^0 - x^* \rangle \le 0.$$

So

$$\limsup_{k\to\infty} \langle x^g - x^*, t^k - x^* \rangle = \limsup_{k\to\infty} \langle x^g - x^*, t^k - x^k \rangle + \limsup_{k\to\infty} \langle x^g - x^*, x^k - x^* \rangle \le 0.$$

From (3.10), $\forall k \ge M$, we get

$$||x^{k+1} - x^*||^2 \le (1 - \lambda_k \gamma_k) ||x^k - x^*||^2 + 2\lambda_k \gamma_k \langle x^g - x^*, t^k - x^* \rangle.$$

By Lemma 3, we have

$$\lim_{k\to\infty} ||x^k - x^*||^2 = 0.$$

Hence x^k converges strongly to x^*.

Case 2. There exists a subsequence $\{||x^{n_k} - x^*||\}$ of $\{||x^k - x^*||\}$ such that $||x^{n_k} - x^*|| < ||x^{n_{k+1}} - x^*||, \forall k \in \mathbb{N}$. From Lemma 4, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$, and the following inequalities are satisfied by all $k \in \mathbb{N}$

$$||x^{m_k} - x^*|| \le ||x^{m_k+1} - x^*||, ||x^k - x^*|| \le ||x^{m_k+1} - x^*||. \tag{3.16}$$

From (3.11), we have

$$\left(1 - \frac{\rho_{m_k}}{\rho_{m_k+1}} \delta\right) \tau_{m_k} (||x^{m_k} - y^{m_k}||^2 + ||z^{m_k} - y^{m_k}||^2) \le ||x^{m_k} - x^*||^2 - ||x^{m_k+1} - x^*||^2 + 2\lambda_{m_k} \alpha_{m_k} \langle x^g - x^*, t^{m_k} - x^* \rangle.$$

Since $\liminf_{k\to\infty} \tau_{m_k} \ge 1 - \bar{\alpha} > 0$ and $\lim_{k\to\infty} \left(1 - \frac{\rho_{m_k}}{\rho_{m_k+1}} \delta\right) = 1 - \delta > 0$, we get from the above inequality that

$$\lim_{n\to\infty} ||x^{m_k} - y^{m_k}|| = 0, \lim_{n\to\infty} ||z^{m_k} - y^{m_k}|| = 0.$$

Using the same argument as in the proof of the Case 1, we obtain

$$\limsup_{k\to\infty}\langle x^g - x^*, t^{m_k} - x^*\rangle \le 0.$$

For all $m_k \ge N$, we have

$$||x^{m_k+1} - x^*||^2 \le (1-\lambda_{m_k}\gamma_{m_k})||x^{m_k} - x^*||^2 + 2\lambda_{m_k}\gamma_{m_k}\langle x^g - x^*, t^{m_k} - x^*\rangle.$$

From (3.16) we have

$$||x^{m_k+1} - x^*||^2 \le (1-\lambda_{m_k}\gamma_{m_k})||x^{m_k+1} - x^*||^2 + 2\lambda_{m_k}\gamma_{m_k}\langle x^g - x^*, t^{m_k} - x^*\rangle.$$

Therefore,

$$||x^{m_k+1} - x^*||^2 \le 2\langle x^g - x^*, t^{m_k} - x^*\rangle, \forall m_k \ge N.$$

Since

$$\limsup_{k\to\infty}\langle x^g - x^*, t^{m_k} - x^*\rangle \le 0,$$

we get in the limit that

$$\lim_{k\to\infty}||x^{m_k+1} - x^*|| = 0 \text{and} \lim_{k\to\infty}||x^{m_k} - x^*|| = 0.$$

From (3.16), we have $||x^k - x^*|| \le ||x^{m_k+1} - x^*||$.

Therefore, $\lim_{k\to\infty}||x^k - x^*|| = 0$. □

When $T \equiv I$- the identity mapping of $\mathbb{H}$, we get the following algorithm for solving the EP(C,f) in which the Lipschitz-type constants of the bifunction f are not required to be known.

Algorithm 3.2

Initialization. Pick $x^0 = x^g \in C, \rho_0 > 0, \delta \in (0,1)$, and choose sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ such that $\{\alpha_k\} \subset [_\alpha, \bar{\alpha}] \subset (0,1), \{\beta_k\} \subset [_\beta, \bar{\beta}] \subset (0,1), \{\gamma_k\} \subset [_\gamma, \bar{\gamma}] \subset (0,1)$ and $\alpha_k + \beta_k + \gamma_k = 1, \forall k$.

Iteration k (k = 0, 1, 2, …). Having x^k do the following steps:

Step 1. Solve the following strongly convex program

$$\min\left\{f(x^k, y) + \frac{1}{2\rho_k}||y - x^k||^2 : y \in C\right\} CP(x^k)$$

to obtain its unique solution y^k.

Step 2. Take $w^k \in \partial_2 f(x^k, y^k)$,

$$H_k = \{x \in \mathbb{H} : \langle x^k - \rho_k w^k - y^k, x - y^k\rangle \le 0\}.$$

Compute

$$z^k = \arg\min\left\{f(y^k, y) + \frac{1}{2\rho_k}||y - x^k||^2 : y \in H_k\right\}.$$

Step 3. Compute

$$t^k = \lambda_k x^g + (1-\lambda_k)z^k,$$
$$x^{k+1} = \alpha_k x^k + \beta_k z^k + \gamma_k t^k.$$

Set $\rho = f(x^k, z^k) - f(y^k, z^k) - f(x^k, y^k)$ and set

$$\rho_{k+1} = \begin{cases} \min\{\frac{\delta}{2\rho}(||x^k - y^k||^2 + ||z^k - y^k||^2), \rho_k\}, & \text{if } \rho > 0 \\ \rho_k, & \text{otherwise,} \end{cases}$$

and go to Iteration k with k replaced by $k+1$.

The following corollary can be deduced immediately from Theorem 1.

Corollary 1. *Suppose that the solution set Sol(C,f) of EP(C,f) is nonempty and* $\{\lambda_k\} \subset (0,1)$ *is a sequence such that* $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\lim_{k\to\infty} \lambda_k = 0$. *Then under assumptions* (A_1), (A_2), (A_3), *and* (A_4), *the sequences* $\{x^k\}, \{y^k\}$, *and* $\{z^k\}$ *generated by Algorithm 3.2 converge strongly to the solution* $x^* = P_{Sol(C,f)}(x^g)$.

When $f(x,y) = \langle F(x), y-x\rangle$ for every $x, y \in C$, where $F : C \to \mathbb{H}$ is a mapping, the equilibrium problem (1.1) reduces to the following variational inequality problem (VIP):

$$\text{find } x^* \in C \text{ such that } \langle F(x), y-x\rangle \geq 0 \forall y \in C.$$

In this case, we get the following algorithm for finding a common element of the set of solutions of (VIP) and the set of fixed points of a quasi-nonexpansive mapping in a real Hilbert space.

Algorithm 3.3

Initialization. Pick $x^0 = x^g \in C, \rho_0 > 0, \delta \in (0,1)$, and choose sequences $\{\mu_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ such that $\{\mu_k\} \subset [0,1]$, $\lim_{k\to\infty} \mu_k = 1, \{\alpha_k\} \subset [\underline{\alpha}, \bar{\alpha}] \subset (0,1), \{\beta_k\} \subset [\underline{\beta}, \bar{\beta}] \subset (0,1), \{\gamma_k\} \subset [\underline{\gamma}, \bar{\gamma}] \subset (0,1)$ and $\alpha_k + \beta_k + \gamma_k = 1, \forall k$.

Iteration k (k = 0, 1, 2, …). Having x^k do the following steps:

Step 1. Compute

$$y^k = P_C(x^k - \rho_k F(x^k)).$$

Step 2. Take $w^k = x^k$,

$$H_k = \{x \in \mathbb{H} : \langle x^k - \rho_k w^k - y^k, x - y^k\rangle \leq 0\}.$$

Compute

$$z^k = P_{H_k}(x^k - \rho_k F(y^k)).$$

Step 3. Compute

$$\begin{aligned} t^k &= \lambda_k x^g + (1-\lambda_k) z^k, \\ u^k &= \mu_k x^k + (1-\mu_k) T x^k, \\ x^{k+1} &= \alpha_k u^k + \beta_k z^k + \gamma_k T t^k. \end{aligned}$$

Set $\rho = \langle F(x^k), z^k - x^k \rangle - \langle F(y^k), z^k - y^k \rangle - \langle F(x^k), y^k - x^k \rangle$ and set

$$\rho_{k+1} = \begin{cases} \min\{\frac{\delta}{2\rho}(||x^k - y^k||^2 + ||z^k - y^k||^2), \rho_k\}, & \text{if } \rho > 0 \\ \rho_k, & \text{otherwise,} \end{cases}$$

and go to Iteration k with k replaced by $k+1$.

From Theorem 1, we get the following corollary.

Corollary 2. Suppose that $\Omega = Sol(C, F) \cap Fix(T)$ is nonempty, and the sequence $\{\lambda_k\} \subset (0, 1)$, satisfies $\sum_{k=0}^{\infty} \lambda_k = \infty$, $\lim_{k\to\infty} \lambda_k = 0$. Then under Assumptions $\mathcal{A}$ the sequences $\{x^k\}, \{y^k\}, \{z^k\}$ generated by *Algorithm 3.3* converge strongly to the solution $x^* = P_\Omega(x^g)$.

4. Examples and numerical results

In this section, we consider four examples to illustrate the convergence of Algorithm 3.1 with the aim to compare its numerical behavior with an existing strongly convergent algorithm, namely the Halpern subgradient extragradient method (HSEM) introduced in [30] as follows.

Algorithm 4.4 (see [30, Algorithm 4.4]).

Initialization. Choose $x_0 \in \mathbb{H}$ and parameters $\lambda, \{\alpha_k\}, \{\beta_k\}$ satisfy the following conditions:

i.$0 < \lambda < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$.

ii. $\alpha_k \in (0, 1)$, $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = +\infty$.

ii. $0 < a < \beta_k < \frac{1-\kappa}{2}$.

Iteration k ($k = 0, 1, ...$). Having x^k do the following steps:

Step 1. Solve two strongly convex optimization problems

$$\begin{cases} y^k = & \operatorname{argmin}\{\lambda f(x^k, y) + \frac{1}{2}||x^k - y||^2 : y \in C\} \\ z^k = & \operatorname{argmin}\{\lambda f(y^k, y) + \frac{1}{2}||x^k - y||^2 : y \in H_k\}, \end{cases}$$

where $H_k = \{v \in \mathbb{H} : \langle x^k - \lambda\omega^k - y^k, v - y^k \rangle \le 0\}, \omega^k \in \partial_2 f(x^k, y^k)$.

Step 2. Compute $x^{k+1} = (1-\beta_k) t^k + \beta_k T(t^k)$, where $t^k = \alpha_k x_0 + (1-\alpha_k) z^k$. Set $k := k+1$ and go back Step 1.

All the programs are written in Mathlab R2014 and performed on a Laptop with Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz, 1700 Mhz, 2

Table 1. The number of Iterations and CPU times (in second) computed by Algorithm 3.1 and 4.4 in Example 1.

	Alg 3.1	Alg 4.4
Iter.	1139	1888
Cpu(s)	45.74	74.02

Core(s), 4 Logical Processor(s), Ram 4.00 GB. To terminate the Algorithms, we use the stopping criteria $Err = \left\|x^{k+1} - x^k\right\| \le \varepsilon$ for some $\varepsilon > 0$.

Example 1. In this test, we consider our problem in the space $\mathbb{H} = \mathbb{R}^5$ and the bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ which comes from the Nash-Cournot equilibrium model [36, 40] defined as follows.

$$f(x, y) = \langle Px + Qy + q, y - x \rangle.$$

where, $P, Q \in \mathbb{R}^{5\times5}$ are two matrices of order 5 such that Q is symmetric, positive semidefinite and $Q-P$ is negative semidefinite. From the results in [36], the bifunction f satisfies conditions (A_1) - (A_4) with the Lipschitz-type constants $L_1 = L_2 = \frac{||P-Q||}{2}$. We take the data in [36], in detail,

$$P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

and $r = (1, -2, -1, 2, -1)^T$, $C = \{x \in \mathbb{R}^5 : \sum_{i=1}^5 x_i \ge -1, -5 \le x_i \le 5, i = \overline{1;5}\}$.

We consider the mapping $T : \mathbb{R}^5 \to \mathbb{R}^5$ is given by the following formula:

$$T(x) = \frac{1}{5}\sum_{i=1}^{5} T_i(x),$$

where

$$T_i(x) = \begin{cases} (x_1, \ldots, x_i, \ldots, x_5)^T & \text{if } x_i \le a_i \\ (x_1, \ldots, a_i, \ldots, x_5)^T & \text{if } x_i > a_i, \end{cases}$$

with $i = \overline{1, 5}$ and $a = (1, 1, 1, 1, 1)^T$. From [28] we can see that T is a quasi-nonexpansive mapping and $Fix(T) = \cap_{i=1}^5 Fix(T_i)$. We illustrate the convergence of Algorithm 3.1 and compare it with Algorithm 4.4 in [30]. In Algorithm 4.4, we take $\lambda = \frac{||P-Q||}{4}, \alpha_k = \frac{1}{k+1}, \beta_k = \frac{1}{2}$. While, in Algorithm 3.1, we choose $\rho_0 = 1000, \delta = 0.9, \lambda_k = \frac{1}{k+1}, \mu_k = 1 - \frac{1}{k+1}, \alpha_k = 0.2, \beta_k = 0.4, \gamma_k = 0.4$. The starting point is $x^0 = (1, 3, 1, 1, 2)^T$. The stopping criteria is $Err = ||x^{k+1} - x^k|| \le \varepsilon$ with $\varepsilon = 10^{-6}$. In this case, the approximate solution computed by Algorithm 3.1 is

$$(-0.724815, 0.803666, 0.720101, -0.866164, 0.200635)^T$$

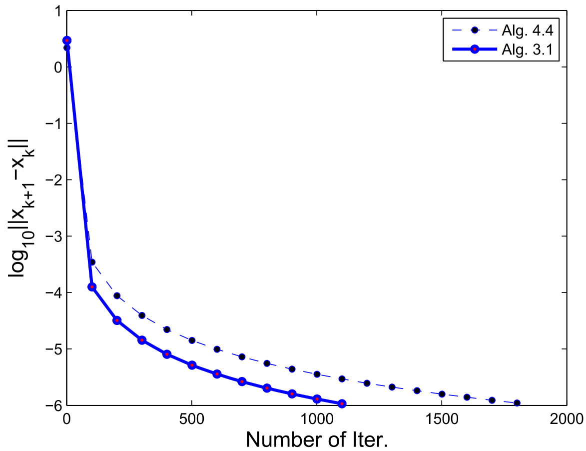

Figure 1. The number of iterations in Algorithms 3.1 and 4.4 - Example 1.

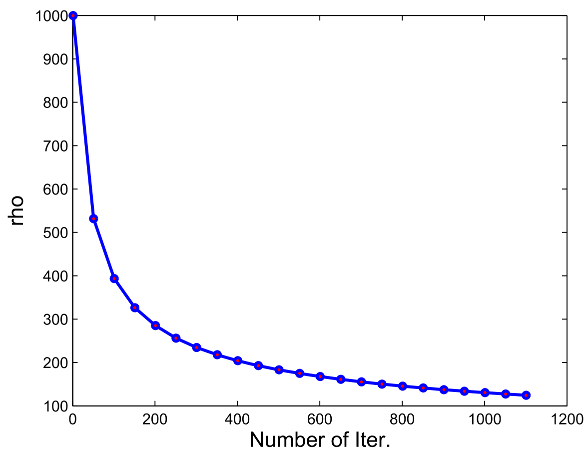

Figure 2. The change of ρ_k in Example 1.

while the approximate solution computed by Algorithm 4.4 is

$$(-0.724586, 0.80402, 0.719916, -0.865651, 0.201025)^T.$$

The detailed result is reported in Table 1 and in Figure 1. Moreover, the number of changes of parameter ρ_k is also reported in Figure 2.

Example 2. In this example, two matrices $P, Q \in \mathbb{R}^{n\times n}$ are generated randomly and vector q is chosen randomly with its elements in $[-n, n]$. The feasible set C is a polyhedral convex set and is defined as follows

$$C = \{x \in \mathbb{R}^n : Ax \leq b, lb \leq x \leq ub\}$$

where A is a matrix of order $m \times n$ ($n = 5, 10, 15, 20$ and $m = 50, 100$) with its entries generated randomly in $[-2, 2]$ and $b \in \mathbb{R}^n$ is a vector with its elements generated randomly in $[1, 3]$, lb is the zero vector, ub is a vector

Table 2. Experiment for Example 2.

		$m = 50$				$m = 100$			
		Alg 3.1		Alg 4.4		Alg 3.1		Alg 4.4	
n	ϵ	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*
5	10^{-5}	20.82	430	23.99	594	23.96	440	26.09	622
10	–	41.15	805	49.89	1136	40.60	800	56.22	1134
15	–	43.91	1139	67.91	1888	53.83	923	71.93	1306
20	–	137.37	1133	172.51	1602	160.82	1160	215.40	1640

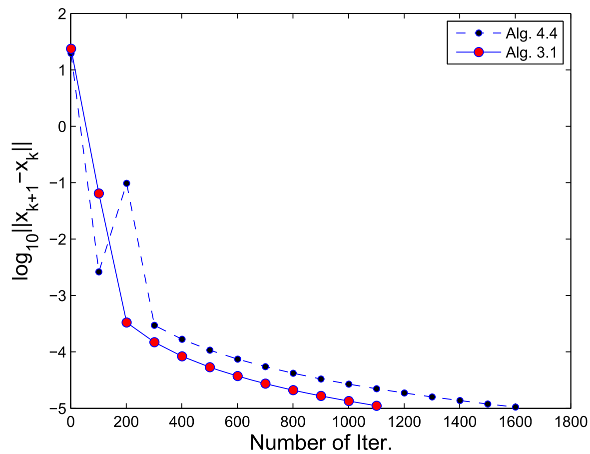

Figure 3. Number of iterations in Algorithm 3.1 and 4.4 - Example 2.

with its elements generated randomly in $[0, n]$. The mapping T is defined by

$$T(x) = \begin{cases} x & \text{if } ||x|| \leq 2 \\ \dfrac{2x}{||x||} & \text{if } ||x|| > 2. \end{cases}$$

The parameters in both algorithms are chosen as in Example 1. The stopping criteria is $Err = ||x^{k+1} - x^k|| \leq \varepsilon = 10^{-5}$. The results computed by Algorithm 3.1 and 4.4 are showed in Table 2 and in Figure 3 (with $m = 100, n = 20$).

Example 3. In the next example, two matrices $P, Q \in \mathbb{R}^{n \times n}$, the feasible set C, and the parameters in Algorithms 3.1 and 4.4 are chosen as in Example 2, q is the zero vector. While, the mapping T is given by

$$T(x) = \frac{1}{3} \sum_{i=1}^{3} T_i(x),$$

where

Table 3. Experiment for Example 3.

		$m=50$				$m=100$			
		Algorithm 3.1		Algorithm 4.4		Algorithm 3.1		Algorithm 4.4	
n	ϵ	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*	*CPU(s)*	*Iter.*
5	10^{-5}	20.14	429	20.14	429	18.65	382	24.18	540
10	–	42.49	747	54.49	1056	43.22	688	50.10	1134
15	–	47.65	848	50.63	1047	60.10	941	67.26	1333
20	–	146.50	1192	191.64	1685	149.02	1111	190.30	1479

$$T_i(x) = \begin{cases} x & \text{if } ||x-a_i|| \le 2 \\ \dfrac{2(x-a_i)}{||x-a_i||} & \text{if } ||x-a_i||>2, \end{cases}$$

with $i=1,2$ and

$$T_3(x) = \begin{cases} x & \text{if } \langle c,x\rangle \le 1 \\ x-\dfrac{\langle c,x\rangle c}{||c||^2} & \text{if } \langle c,x\rangle>1, \end{cases}$$
$$a_1=(-2,0,\dots,0)^T, a_2=(2,0,\dots,0)^T, c=(1,1,\dots,1)^T.$$

As before, we choose the stopping criteria is $Err=||x^{k+1}-x^k|| \le \varepsilon = 10^{-5}$. The results are showed in Table 3. From these results, we can see that, under the same tolerance, Algorithm 3.1 has a competitive advantage over Algorithm 4.4, especially in the number of iterative steps (Iter.) and the time of execution in second (CPU(s)).

Example 4. In the last example, we suppose that $\mathbb{H}=L^2([0,1])$ with the inner product

$$\langle x,y\rangle := \int_0^1 x(t)y(t)dt$$

and the induced norm

$$||x|| := \sqrt{\int_0^1 |x(t)|^2 dt}, \forall x \in \mathbb{H}.$$

We consider the feasible set $C=\{x\in\mathbb{H} : ||x|| \le 2\}$ and the following bifunction $f:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$ given by (see [41]).

$$f(x,y) = \left\langle \left(\frac{3}{2}-||x||\right)x, y-x \right\rangle.$$

From the results of [41], the bifunction f satisfies Condition (A4) with the Lipschitz-type constants $L_1=L_2=\frac{7}{4}$. The mapping T is given by

Table 4. Experiment for Example 4.

	Iter.	Cpu(s)
Alg. 3.3	705	1.28

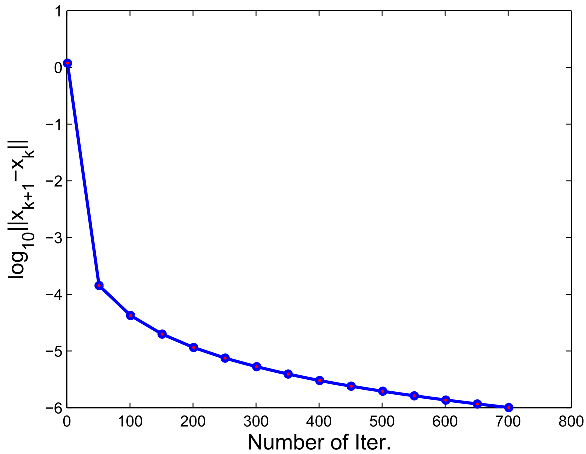

Figure 4. Numerical behavior of Algorithm 3.3 in Example 4.

$$T(x) = \begin{cases} x & \text{if } ||x|| \leq 1 \\ \dfrac{x}{||x||} & \text{if } ||x|| > 1. \end{cases}$$

We choose $\rho_0 = 6, \delta = 0.9, \lambda_k = \frac{1}{k+1}, \mu_k = 1 - \frac{1}{k+1}, \alpha_k = 0.2, \beta_k = 0.4, \gamma_k = 0.4$. The starting point is $x^0 = \frac{1}{200}(\sin(-3t) + \cos(-10t))$. We choose the stopping criteria is $Err = ||x^{k+1} - x^k|| \leq \varepsilon = 10^{-6}$. The results are showed in Table 4 and in Figure 4.

5. Conclusion

This paper has proposed a novel algorithm for finding a common point of the solution set of a pseudomonotone equilibrium problem and the set of fixed points of a quasi-nonexpansive mapping in a real Hilbert space. We have described how to incorporate the subgradient extragradient method with the Ishikawa iteration. The strong convergence of the presented algorithm is obtained when the Lipschitz-type constants of the bifunction is unknown. We have also considered some numerical examples to illustrate the convergence and also the advantage of the new algorithm over some existing methods in this field.

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