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A note on the combination of equilibrium problems

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Abstract

We show that the solution set of a strictly convex combination of equilibrium problems is not necessarily contained in the corresponding intersection of solution sets of equilibrium problems even if the bifunctions defining the equilibrium problems are continuous and monotone. As a consequence, we show that some results given in some recent papers are not always true. Therefore different numerical methods for computing common solutions of families of equilibrium problems proposed in the literature may not converge under the monotonicity assumption. Finally, we prove that if the bifunctions are also parapseudomonotone, then the solution set of any strictly convex combination of a family of equilibrium problems is equivalent to the solution set of the intersection of the same family of equilibrium problems.

Keywords Equilibria · Ky Fan inequality · Combination of equilibrium problems

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This paper is dedicated to Professor Le Dung Muu on the Occasion of His 70th Birthday.

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1 Introduction

Let C be a nonempty, closed and convex subset in a real Hilbert space $\mathbb{H}$ and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (shortly EP(C, f)), in the sense of Blum and Oettli (1994), Muu and Oettli (1992) (see also Fan 1972), consists of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C.$$

We denote the solution set of EP(C, f) by $Sol(C, f)$. Numerical methods for solving EP(C, f) can be found in Dinh and Kim (2016), Tran et al. (2008).

Let $f_i : C \times C \to \mathbb{R}, i = 1, 2, \ldots, N$, be bifunctions defined on C. Recently, many researchers were interested in finding a common solution of a finite family of equilibrium problems (Suwannaut and Kangtunyakarn 2013, 2014, 2016; Khan et al. 2018) (CSEP for short):

$$\text{Find } x^* \in C \text{ such that } f_i(x^*, y) \geq 0,\ \forall y \in C \text{ and } i = 1, 2, \ldots, N. \quad CSEP(C, f_i)$$

or, equivalently,

$$\text{find } x^* \in \Omega := \cap_{i=1}^N Sol(C, f_i).$$

Let $\alpha_i \in (0, 1), i = 1, \ldots, N$ such that $\sum_{i=1}^N \alpha_i = 1$ and set

$$f(x, y) = \sum_{i=1}^N \alpha_i f_i(x, y), \forall x, y \in C.$$

The combination of equilibrium problems (shortly, CEP$(C, \sum_{i=1}^N \alpha_i f_i)$) consists of finding $x^* \in C$ such that

$$f(x^*, y) = \sum_{i=1}^N \alpha_i f_i(x^*, y) \geq 0, \forall y \in C.$$

We denote by $Sol(C, \sum_{i=1}^N \alpha_i f_i)$ the solution set of the combination of equilibrium problems.

In Suwannaut and Kangtunyakarn (2013) said that under certain conditions one has

$$\Omega = \cap_{i=1}^N Sol(C, f_i) = Sol\left(C, \sum_{i=1}^N \alpha_i f_i\right).$$

Therefore, common solutions of a finite family of equilibrium problems can be computed by simply finding solutions of any strictly convex combination of the same family of equilibrium problems. Based on this relation, Suwannaut and Kangtunyakarn (2013, 2014, 2016), Khuangsatung and Kangtunyakarn (2014), Khan et al. (2018) gave algorithms for finding a common element of the fixed point sets of a family of mappings and the solution sets of equilibrium problems and/or the zero point sets of a family of mappings.

In this short paper, we show that, under the same conditions as the ones given in Suwannaut and Kangtunyakarn (2013), the relation

$$Sol\left(C, \sum_{i=1}^{N} \alpha_i f_i\right) \subset \cap_{i=1}^{N} Sol(C, f_i),$$

is not always true. Therefore, different results given in the recent papers Suwannaut and Kangtunyakarn (2013, 2014, 2016), Khuangsatung and Kangtunyakarn (2014), Khan et al. (2018) are not necessarily true because they rely on the wrong inclusion above. Moreover, we present a sufficient condition for which the above formula is correct not only when N is finite but also when $N = +\infty$.

The rest of paper is organized as follows. The next section contains first some preliminaries on equilibrium problems and then the inexact statements given in papers Suwannaut and Kangtunyakarn (2013, 2014, 2016), Khuangsatung and Kangtunyakarn (2014), Khan et al. (2018) related with combination of equilibrium problems. The main section is devoted to show that there exists a finite family of monotone equilibrium problems such that the set of their common solutions is strictly contained in the solution set of a combination of equilibrium problems. We also prove that under certain conditions these two sets are equal.

2 Preliminaries

In this section, we recall some definitions and statements presented in recent papers related to combination of equilibrium problems.

Let C be a nonempty, closed and convex subset of $\mathbb{H}$. We denote the metric projection onto C by P_C. Namely, for each $x \in \mathbb{H}$, $P_C(x)$ is the unique element in C such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

We also recall the following well-known definitions.

Definition 2.1 Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction defined on C. Bifunction φ is said to be:

(a) monotone on C if $\varphi(x, y) + \varphi(y, x) \leq 0, \forall x, y \in C$;
(b) pseudomonotone on C if

$$\forall x, y \in C: \ \varphi(x, y) \geq 0 \Rightarrow \varphi(y, x) \leq 0;$$

(c) paramonotone on C if φ is monotone on C and

$$\{x \in Sol(C,\varphi), y \in C, \varphi(x,y) = \varphi(y,x) = 0\} \Rightarrow y \in Sol(C,\varphi);$$

(d) parapseudomonotone on Cif φ is pseudomonotone on C and

$$\{x \in Sol(C,\varphi), y \in C, \varphi(x,y) = \varphi(y,x) = 0\} \Rightarrow y \in Sol(C,\varphi).$$

From the above definition, it can be seen that $a) \Rightarrow b)$ and $c) \Rightarrow d)$, $c) \Rightarrow a)$ and $d) \Rightarrow b)$.

In the sequel, we need the following blanket assumptions:

Assumptions $\mathcal{A}$.

($\mathcal{A}_1$) $\varphi(x,x) = 0$ for every $x \in C$;
($\mathcal{A}_2$) φ is monotone on C;
($\mathcal{A}_3$) φ is upper hemicontinuous, i.e., for each $x, y, z \in C$, we have

$$\limsup_{t \to 0^+} \varphi(tz + (1-t)x, y) \le \varphi(x,y);$$

($\mathcal{A}_4$) for each $x \in C$, $\varphi(x,\cdot)$ is lower semicontinuous and convex on C;
($\mathcal{A}_5$) for fixed $r > 0$ and $z \in C$, there exist a nonempty compact convex subset B of $\mathbb{H}$ and $x \in C \cap B$, such that

$$\varphi(y,x) + \frac{1}{r}\langle y - z, z - x\rangle < 0, \forall y \in C \backslash B.$$

The following five statements are displayed in Suwannaut and Kangtunyakarn (2013, 2014, 2016), Khuangsatung and Kangtunyakarn (2014), Khan et al. (2018), respectively.

Statement 2.1 (See Suwannaut and Kangtunyakarn 2013, Lemma 2.7) Let $f_i, i = 1, 2, \ldots, N$ be bifunctions satisfying $\mathcal{A}_1 - \mathcal{A}_4$ with $\cap_{i=1}^N Sol(C, f_i) \ne \emptyset$. Then

$$\cap_{i=1}^N Sol(C, f_i) = Sol(C, f),$$

where $f(x,y) = \sum_{i=1}^N \alpha_i f_i(x,y), \alpha_i > 0, \forall i = 1, 2, \ldots, N$ and $\sum_{i=1}^N \alpha_i = 1$. If Statement 2.1 holds true then it allows us to find common solutions of N equilibrium problems by solving a combination of equilibrium problems.

Statement 2.2 (See Suwannaut and Kangtunyakarn 2014, Theorem 3.1) Let F be a τ-contractive mapping on $\mathbb{H}$ and let A be a strongly positive linear bounded operator on $\mathbb{H}$ with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\tau}$. For every $i = 1, 2, \ldots, N$ let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $\mathcal{A}_1 - \mathcal{A}_4$ with $\Omega = \cap_{i=1}^N Sol(C, f_i) \ne \emptyset$. Let $\{x^k\}, \{y^k\}, \{z^k\}$ be sequences generated by $x^1 \in \mathbb{H}$ and

$$\begin{cases} \sum_{i=1}^{N} \alpha_i f_i(z^k, y) + \frac{1}{\rho_k}\langle y - z^k, z^k - x^k\rangle \geq 0, \forall y \in C, \\ y^k = \theta_k P_C(x^k) + (1-\theta_k)z^k, \\ x^{k+1} = \delta_k \gamma F(x^k) + (I - \delta_k A)y^k, \end{cases}$$

where $\{\delta_k\}, \{\theta_k\}, \{\rho_k\} \subset (0, 1), 0 < \alpha_i < 1, \forall i = 1, \ldots, N$. Suppose the following conditions $(i) - (v)$ hold.

(i) $\lim_{k\to\infty} \delta_k = 0$ and $\sum_{k=0}^{\infty} \delta_k = \infty$;
(ii) $0 < \underline{\theta} \leq \theta_k \leq \bar{\theta} < 1$, for some $\underline{\theta}, \bar{\theta} \in (0, 1)$;
(iii) $0 < \underline{\alpha} \leq \alpha_k \leq \bar{\alpha} < 1$, for some $\underline{\alpha}, \bar{\alpha} \in (0, 1)$;
(iv) $\sum_{i=1}^{N} \alpha_i = 1$;
(v) $\sum_{i=1}^{N} |\delta_{k+1} - \delta_k| < \infty, \sum_{i=1}^{\infty} |\theta_{k+1} - \delta_k| < \infty, \sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty$.

Then the sequences $\{x^k\}, \{y^k\}$, and $\{z^k\}$ converge to $q = P_\Omega(I - A + \gamma F)q$.

Statement 2.3 (See Khuangsatung and Kangtunyakarn 2014, Theorem 3.1) Let $f_i, i = 1, 2, \ldots, N$ satisfy assumptions $\mathcal{A}_1 - \mathcal{A}_4$. Assume that $\Omega = \cap_{i=1}^{N} Sol(C, f_i) \neq \emptyset$. Let the sequences $\{x^k\}$ and $\{y^k\}$ be generated by $u, x^1 \in \mathbb{H}$ and

$$\begin{cases} \sum_{i=1}^{N} \alpha_i f_i(y^k, y) + \frac{1}{\rho_k}\langle y - y^k, y^k - x^k\rangle \geq 0, \forall y \in C, \\ x^{k+1} = \lambda_k u + \mu_k x^k + \delta_k y^k \end{cases}$$

where $\{\lambda_k\}, \{\mu_k\}, \{\delta_k\} \subset (0, 1)$ and $\lambda_k + \mu_k + \delta_k = 1$; $\{\rho_k\} \subset (\underline{\rho}, \bar{\rho}) \subset (0, 1)$, $0 < \alpha_i < 1, \forall i = 1, \ldots, N$. Suppose the conditions $(i) - (iii)$ hold:

(i) $\lim_{k\to\infty} \lambda_k = 0$ and $\sum_{k=0}^{\infty} \lambda_k = \infty$;
(ii) $\sum_{i=1}^{N} \alpha_i = 1$;
(iii) $\sum_{i=1}^{N} |\delta_{k+1} - \delta_k| < \infty$.

Then the sequences $\{x^k\}, \{y^k\}$ converge to $q = P_\Omega(u)$.

Statement 2.4 (Suwannaut and Kangtunyakarn 2016, Theorem 3.1) Let F be τ-contractive mapping on $\mathbb{H}$ and let $f_i, i = 1, 2, \ldots, N$ satisfy assumptions $\mathcal{A}_1 - \mathcal{A}_4$. Assume that $\Omega = \cap_{i=1}^{N} Sol(C, f_i) \neq \emptyset$. Let the sequence $\{x^k\}$ and $\{y^k\}$ be generated by $x^1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} \alpha_i f_i(y^k, y) + \frac{1}{\rho_k}\langle y - y^k, y^k - x^k\rangle \geq 0, \forall y \in C, \\ x^{k+1} = \lambda_k F(x^k) + \mu_k P_C(x^k) + \delta_k y^k \end{cases}$$

where $\{\lambda_k\}, \{\mu_k\}, \{\delta_k\} \subset (0, 1)$ such that $\lambda_k + \mu_k + \delta_k = 1\, \forall k$; $\{\rho_k\} \subset (\underline{\rho}, \bar{\rho}) \subset (0, 1)$, $0 < \alpha_i < 1, \forall i = 1, \ldots, N$. In addition, suppose the conditions $(i) - (iii)$ hold:

(i) $\lim_{k\to\infty} \lambda_k = 0$ and $\sum_{k=0}^{\infty} \lambda_k = \infty$;
(ii) $\sum_{i=1}^{N} \alpha_i = 1$;
(iii) $\sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty$.

Then the sequences $\{x^k\}, \{y^k\}$ converge to $q = P_\Omega(u)$.

Statement 2.5 (Khan et al. 2018, Theorem 4.2) Let $f_i, i = 1, 2, \ldots, N$ satisfy assumption $\mathcal{A}$. Assume that $\Omega = \cap_{i=1}^{N} Sol(C, f_i) \neq \emptyset$. For given $x^0, x^1 \in \mathbb{H}$, let the sequence $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ be generated by

$$\begin{cases} y^k = x^k + \theta_k(x^k - x^{k-1}) \\ \sum_{i=1}^{N} \alpha_i f_i(z^k, y) + \frac{1}{\rho_k}\langle y - z^k, z^k - y^k\rangle \geq 0, \forall y \in C, \\ x^{k+1} = \lambda_k x^k + \mu_k z^k \end{cases}$$

where $\{\theta_k\} \subset [0, \theta], \theta \in [0; 1]$, $\{\lambda_k\}, \{\mu_k\} \subset (0, 1)$ and $\lambda_k + \mu_k = 1$ for all k; $\{\rho_k\} \subset (\underline{\rho}, \bar{\rho}) \subset (0, 1)$, $0 < \alpha_i < 1, \forall i = 1, \ldots, N$. Suppose that the following conditions hold:

(i) $\theta_k \|x^k - x^{k-1}\| < \infty$;
(ii) $\sum_{i=1}^{\infty} \alpha_i < \infty$ and $\lim_{i\to\infty} \alpha_i = 0$;
(iii) $\sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty$, $\sum_{i=1}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty$.

Then the sequence $\{x^k\}$ converges to $q = P_\Omega(u)$.

Remark 2.1 • Each Statement 2.2–2.5 implies that the sequence $\{x^k\}$ obtained by the numerical precedure converges to a solution of the CSEP.
• In Corollary 3.1b–e it is shown that this is not necessarily true.

3 Main results

Let C be a nonempty, closed convex subset of $\mathbb{H}$ and let f_i $(i = 1, \ldots, N)$ be bifunctions defined on C such that

$$\cap_{i=1}^{N} Sol(C, f_i) \neq \emptyset.$$

For $\alpha_i \in (0, 1), i = 1, \ldots, N$ and $\sum_{i=1}^{N} \alpha_i = 1$ we consider the bifunction f defined by

$$f(x, y) = \sum_{i=1}^{N} \alpha_i f_i(x, y), \forall x, y \in C.$$

It is clear that if $x^* \in \cap_{i=1}^{N} Sol(C, f_i)$ then $f_i(x^*, y) \geq 0, \forall y \in C$, and $i = 1, 2, \ldots, N$. Therefore

$$f(x^*, y) = \sum_{i=1}^{N} \alpha_i f_i(x^*, y) \geq 0, \forall y \in C.$$

Hence $x^* \in Sol(C, f)$ and

$$\cap_{i=1}^{N} Sol(C, f_i) \subset Sol(C, f).$$

The following theorem shows that under assumptions $\mathcal{A}_1 - \mathcal{A}_4$, the inverse inclusion is not always true.

Theorem 3.1 *For any integer $N \geq 2$, there exist a nonempty, closed convex set C and bifunctions $f_1, f_2, \ldots, f_N$ defined on C satisfying assumptions $\mathcal{A}_1 - \mathcal{A}_4$ and $\alpha_i \in (0, 1), i = 1, 2, \ldots, N, \sum_{i=1}^{N} \alpha_i = 1$ such that*

$$Sol\left(C, \sum_{i=1}^{N} \alpha_i f_i\right) \not\subset \cap_{i=1}^{N} Sol(C, f_i).$$

Proof It is clear that we only need to prove the case when $\mathbb{H} = \mathbb{R}^2$ and $N = 2$. In that purpose, for $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$ we consider the set C and bifunctions f_1 and f_2 given as follows

$$\begin{aligned} &C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}. \\ &f_1(x, y) = x_2 y_1 - x_1 y_2. \\ &f_2(x, y) = x_1 y_2 - x_2 y_1. \end{aligned}$$

Then we have: $f_1(x, x) = 0, \forall x \in C$ and for all $x, y \in C$, we obtain

$$f_1(x, y) + f_1(y, x) = x_2 y_1 - x_1 y_2 + y_2 x_1 - y_1 x_2 = 0.$$

Hence, f_1 is monotone on C. For each $x \in C$ we have also $f_1(x, y)$ is linear in y, and thus $f_1(x, \cdot)$ is convex. Furthermore, it is obvious that f_1 is continuous on $C \times C$.

Therefore the bifunction f_1 satisfies assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and $\mathcal{A}_4$.

Similarly, f_2 satisfies assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and $\mathcal{A}_4$. In addition, it can be seen that

$$\begin{aligned} Sol(C, f_1) &= \{0\} \times [0, +\infty). \\ Sol(C, f_2) &= [0, +\infty) \times \{0\}. \end{aligned}$$

So,

$$Sol(C, f_1) \cap Sol(C, f_2) = \{(0, 0)\}.$$

Now, we consider a combination of f_1, f_2 given as follows

$$\begin{aligned} f(x, y) &= \frac{1}{2} f_1(x, y) + \frac{1}{2} f_2(x, y) \\ &= \frac{1}{2}\big[f_1(x, y) + f_2(x, y)\big] \\ &= 0, \forall x, y \in C. \end{aligned}$$

It is obvious that f satisfies assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$. Moreover

$$Sol(C, f) = C = [0, +\infty) \times [0, +\infty).$$

Therefore

$$Sol(C, f) \not\subset Sol(C, f_1) \cap Sol(C, f_2).$$

□

From this theorem, we have the following corollary

Corollary 3.1 *Statements 2.1–2.5 are not always true.*

Proof Let $N = 2$ and take the set C and the bifunctions f_1 and f_2 defined as in Theorem 3.1. Consider the combination of f_1 and f_2 given by

$$f(x, y) = \frac{1}{2} f_1(x, y) + \frac{1}{2} f_2(x, y) = 0, \forall x, y \in C.$$

Hence,

$$\Omega = Sol(C, f_1) \cap Sol(C, f_2) = \{(0, 0)\},$$

$$Sol(C, f) = C = [0, +\infty) \times [0, +\infty).$$

Then we have the following results:

(a) Statement 2.1 is false because $Sol(C, f) \not\subset Sol(C, f_1) \cap Sol(C, f_2)$.

(b) Let us consider the numerical procedure given in Statement 2.2. Take $x^1 \in C$ such that $x^1 \neq (0, 0)$ and set $F(x) = x^1$, $Ax = x$, $\forall x \in \mathbb{R}^2$. Choose $\gamma = 1$, we have

$$\begin{cases} \frac{1}{\rho_k}\langle y - z^k, z^k - x^k\rangle \geq 0, \forall y \in C, \\ y^k = \theta_k P_C(x^k) + (1 - \theta_k) z^k, \\ x^{k+1} = \delta_k x^1 + (1 - \delta_k) y^k. \end{cases}$$

Hence

$$\begin{cases} z^k = P_C(x^k) \\ y^k = P_C(x^k) \\ x^{k+1} = \delta_k x^1 + (1 - \delta_k) P_C(x^k). \end{cases}$$

Because $x^1 \in C$ we can conlude that $x^k = x^1, \forall k$. Since $x^1 \notin \Omega$ this means that Statement 2.2 is false.

(c) Take $u = x^1 \in C$ such that $x^1 \neq (0, 0)$. The sequence $\{x^k\}$ generated by Statement 2.3 becomes

$$\begin{cases} \frac{1}{\rho_k}\langle y - y^k, y^k - x^k\rangle \geq 0, \forall y \in C, \\ x^{k+1} = \lambda_k u + \mu_k x^k + \delta_k y^k. \end{cases}$$

Hence

$$\begin{cases} y^k = P_C(x^k) \\ x^{k+1} = \lambda_k u + \mu_k x^k + \delta_k P_C(x^k). \end{cases}$$

Since $x^1 = u \in C$, $y^k \in C$ and $\lambda_k + \mu_k + \delta_k = 1$, the sequence $\{x^k\} \subset C$ and thus

$$x^k = u, \forall k.$$

This leads to $x^k \to u \notin \Omega$ and thus Statement 2.3 is false.

(d) Take $x^1 \in C$ such that $x^1 \neq (0, 0)$ and set $F(x) = x^1$. Then the sequences $\{x^k\}$ and $\{y^k\}$ generated by Statement 2.4 take the form

$$\begin{cases} \frac{1}{\rho_k}\langle y - y^k, y^k - x^k\rangle \geq 0, \forall y \in C, \\ x^{k+1} = \lambda_k F(x^k) + \mu_k P_C(x^k) + \delta_k y^k. \end{cases}$$

So

$$x^{k+1} = \lambda_k x^1 + \mu_k x^k + \delta_k P_C(x^k).$$

Since $x^1 \in C$ and $\lambda_k + \mu_k + \delta_k = 1$ we have $x^k = x^1, \forall k$. Because $x^1 \notin \Omega$ the Statement 2.4 is false.

(e) Firstly, we show that bifunction f_1 satisfies assumption $\mathcal{A}_5$. In that purpose, fixed $r > 0$ and $z \in C$, we have

$$\begin{aligned} f_1(y, x) + \frac{1}{r}\langle y - z, z - x\rangle &= x_1 y_2 - x_2 y_1 \\ &\quad + \frac{1}{r}\big((y_1 - z_1)(z_1 - x_1) + (y_2 - z_2)(z_2 - x_2)\big) \\ &= y_1\big(\frac{1}{r}(z_1 - x_1) - x_2\big) + y_2\big(x_1 + \frac{1}{r}(z_2 - x_2)\big) \\ &\quad - \frac{1}{r}\big(z_1(z_1 - x_1) + z_2(z_2 - x_2)\big). \end{aligned}$$

By choosing $x = (z_1, z_2 + r(z_1 + 1))$. Then, we have that $x \in C$ and

$$f_1(y, x) + \frac{1}{r}\langle y - z, z - x\rangle = -y_1\big(z_2 + r(z_1 + 1)\big) - y_2 + z_2(z_1 + 1\big). \quad (3.1)$$

By setting $B = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1| \le z_1+1, |y_2| \le z_2+r(z_1+1)+z_2(z_1+1)\}$, we have that B is a nonempty compact, convex subset of $\mathbb{R}^2$ and $x = (z_1, z_2 + r(z_1 + 1)) \in C \cap B$. From (3.1) we have that

$$f_1(y, x) + \frac{1}{r}\langle y - z, z - x\rangle < 0, \ \forall y \in C\backslash B.$$

Similarly, f_2 satisfies assumption $\mathcal{A}_5$.
Now, take $x^1 = x^0 \in C$ such that $x^0 \neq (0, 0)$. Then $y^k = x^k$ and $z^k = x^k$ for all k and the sequence $\{x^k\}$ generated by Statement 2.5 becomes

$$x^k = x^1, \forall k.$$

So, Statement 2.5 is false.

□

From Theorem 3.1 we can see that under assumptions $\mathcal{A}_1 - \mathcal{A}_4$ the statement

$$\cap_{i=1}^{N} Sol(C, f_i) = Sol(C, f),$$

is not always true. So a natural question is to find under which conditions this formula is correct. The following theorem gives us an answer under the assumption:

($\mathcal{A}'_2$) φ is parapseudomonotone on C.

Theorem 3.2 *Let $f_i, i = 1, 2, \dots$ be bifunctions satisfying $\mathcal{A}_1, \mathcal{A}'_2, \mathcal{A}_3$ and $\mathcal{A}_4$ such that $\cap_{i=1}^{\infty} Sol(C, f_i) \neq \emptyset$ and bifunction $f(x, y) = \sum_{i=1}^{\infty} \alpha_i f_i(x, y)$, where $\alpha_i > 0, \forall i = 1, 2, \dots$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, is well-defined on C, i.e., $f(x, y) < \infty \ \forall x, y \in C$. Then*

$$\cap_{i=1}^{\infty} Sol(C, f_i) = Sol(C, f). \tag{3.2}$$

Proof By the assumption and the observation above, we have that

$$\emptyset \neq \cap_{i=1}^{\infty} Sol(C, f_i) \subset Sol(C, f).$$

Therefore, we only have to show the inverse inclusion. In that purpose, we take $x^* \in Sol(C, f)$. Then we have

$$f(x^*, y) = \sum_{i=1}^{\infty} \alpha_i f_i(x^*, y) \ge 0, \forall y \in C. \tag{3.3}$$

Taking $\bar{x} \in \cap_{i=1}^{\infty} Sol(C, f_i)$ we can write

$$f_i(\bar{x}, y) \ge 0, \forall y \in C \text{ and } \forall i = 1, 2, \dots$$

and, in particular,

$$f_i(\bar{x}, x^*) \geq 0, \forall i. \tag{3.4}$$

Since the bifunctions f_i are pseudomonotone on C, we have

$$f_i(x^*, \bar{x}) \leq 0, \ \forall i. \tag{3.5}$$

Fixing $j \geq 1$ and replacing y by $\bar{x}$ in (3.3), we get

$$0 \leq \alpha_j f_j(x^*, \bar{x}) + \sum_{j \neq i=1}^{\infty} \alpha_i f_i(x^*, \bar{x})$$

and by (3.5) that

$$f_j(x^*, \bar{x}) \geq 0. \tag{3.6}$$

From (3.5) and (3.6) for all j we deduce that

$$f_j(x^*, \bar{x}) = 0, \ \text{for all } j. \tag{3.7}$$

By the pseudomonotonicity of the bifunctions f_j, we get $f_j(\bar{x}, x^*) \leq 0$, for all j. Combining this with (3.4) we get

$$f_j(\bar{x}, x^*) = 0 \text{ for all } j. \tag{3.8}$$

Using (3.7) and (3.8) with the parapseudomonotonicity of each f_j, we obtain directly that $x^* \in Sol(C, f_j), \forall j$. So

$$x^* \in \cap_{i=1}^{\infty} Sol(C, f_i).$$

The proof is completed. □

Remark 3.1 • From the proof above, we can see that Theorem 3.2 is still valid when $\mathbb{H}$ is a real Banach space.

- Under Assumptions $\mathcal{A}_1, \mathcal{A'}_2, \mathcal{A}_3, \mathcal{A}_4$ and $\cap_{i=1}^{\infty} Sol(C, f_i) \neq \emptyset$ we may not get that f is well-defined on C. Indeed, let us consider the following example:

$$f_i(x, y) = 4^i x(y - x), \ \forall x, y \in C = [0, +\infty) \text{ and } i = 1, 2, \ldots$$

Then it can be seen that f_i satisfy Assumptions $\mathcal{A}_1, \mathcal{A'}_2, \mathcal{A}_3, \mathcal{A}_4$ for all $i \geq 1$ and $\cap_{i=1}^{\infty} Sol(C, f_i) = \{0\}$. However, with $\alpha_i = 2^{-i}$, the combination bifunction $f(x, y) = \sum_{i=1}^{\infty} \alpha_i f_i(x, y) = \sum_{i=1}^{\infty} 2^i x(y - x)$ is not well defined on C. For instance: $f(1, 2) = \sum_{i=1}^{\infty} 2^i = \infty$.

- Taking $f_i(x, y) = 0$ for all $i > N$, the formula (3.2) becomes

$$\cap_{i=1}^{N} Sol(C, f_i) = Sol(C, f).$$

  Therefore, Statement 2.1 is correct when assumption $\mathcal{A}_2$ is replaced by $\mathcal{A'}_2$.
- For the reader's convenience we report here the wrong implication made in the proof of Lemma 2.7 presented in Suwannaut and Kangtunyakarn (2013). The authors correctly proved in Suwannaut and Kangtunyakarn (2013) that under the assumptions $\mathcal{A}_1 - \mathcal{A}_4$ one gets $f_i(\bar{x}, x^*) = 0$ for all $i = 1, 2, \ldots, N$, where $\bar{x} \in Sol(C, \sum_{i=1}^{N} \alpha_i f_i)$ and $x^* \in \Omega$. But this fact with assumption $\mathcal{A}_1$ do not necessarily imply that $\bar{x} = x^*$. In fact, by recalling again our example of Theorem 3.1, we can set

$$Sol\left(C, \sum_{i=1}^{N} \alpha_i f_i\right) = [0, \infty) \times [0, \infty) \ni (1, 1) = \bar{x} \neq x^* = (0, 0) = \Omega.$$

- Statements 2.2–2.5 are correct if $\mathcal{A}_2$ is replaced by assumption $\mathcal{A}_{2bis}$:
  ($\mathcal{A}_{2bis}$) φ is paramonotone on C.

4 Conclusion

We have proved that there exists a finite family of monotone equilibrium problems such that the common solution set of them does not contain the solution set of a combination of those equilibrium problems. Based on this fact, we can say that some results given in some recent papers are not always true. We have also shown that under certain conditions the solution sets of any strictly convex combination and of the intersection of a family of equilibrium problems coincide not only in the finite case ($f_i, i = 1, 2, \ldots, N$) but also in the infinite case ($f_i, i = 1, 2, \ldots$).

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References

Blum E, Oettli W (1994) From optimization and variational inequalities to equilibrium problems. Math Stud 63:127–149

Dinh BV, Kim DS (2016) Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space. J Comput Appl Math 302:106–117

Fan K (1972) A minimax inequality and applications. In: Shisha O (ed) Inequalities III. Academic Press, New York, pp 103–113

Khan SA, Cholamjiak W, Kazmi KR (2018) An inertial forward-backward splitting method for solving combination of equilibrium problems and inclusion problems. Comput Appl Math 37(5):6283–6307

Khuangsatung W, Kangtunyakarn A (2014) Algorithm of a new variational inclusion problem and strictly pseudononspreading mapping with application. Fixed Point Theory Appl 2014:209

Muu LD, Oettli W (1992) Convergence of an adaptive penalty scheme for finding constrained equilibria. Nonlinear Anal TMA 18:1159–1166

Suwannaut S, Kangtunyakarn A (2013) The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem. Fixed Point Theory Appl 291:26

Suwannaut S, Kangtunyakarn A (2014) Convergence analysis for the equilibrium problems with numerical results. Fixed Point Theory Appl 2014(2014):167

Suwannaut S, Kangtunyakarn A (2016) Convergence theorem for solving the combination of equilibrium problems and fixed point problems in Hilbert spaces. Thai J Maths 14:67–87

Tran DQ, Dung ML, Nguyen VH (2008) Extragradient algorithms extended to equilibrium problems. Optimization 57:749–776