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On Fractional *p*-Laplacian Equations at Resonance

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Abstract
This article shows the existence of weak solutions of a resonant problem for a fractional p-Laplacian equation in a bounded domain in $\mathbb{R}^N$. Our arguments are based on the Minimum principle, saddle point theorem and rely on a generalization of the Landesman–Lazer-type condition.

Keywords Fractional p-Laplacian equation · Minimum principle · saddle point theorem · Landesman–Lazer condition

Mathematics Subject Classification 35J20 · 35J60 · 58E05

1 Introduction and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$, $(N \geq 3)$ with smooth boundary $\partial\Omega$. In this article, we study the existence of weak solutions of the following Dirichlet problem at resonance for fractional p-Laplacian equation:

$$\begin{cases} (-\Delta)_p^s u = \lambda_1 |u|^{p-2}u + f(x,u) - k(x), & x \in \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

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where $p \geq 2$, $s \in (0; 1)$ [1–5].

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to +0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \mathrm{d}y, \quad x \in \mathbb{R}^N, \tag{1.2}$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, λ_1 denotes the first eigenvalue of the eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

The properties of eigenvalue problem will be specialited below.

Remark that the operation $(-\Delta)_p^s$ known as the fractional p-Laplacian leads naturally to the study of the quasilinear problem

$$\begin{cases} (-\Delta)_p^s u(x) = g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

One feature of the aforementioned operator is the nonlocality in the sense that the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on the whole Ω, but also on the whole $\mathbb{R}^N$, since $u(x)$ represents the expected value of a random variable tied to a process randomly jumping arbitrarily far from the points. The fractional p-Laplacian operator $(-\Delta)_p^s u(x)$, $p \geq 2$, and more generally pseudo differential operators, have been a classical topic in Hamonic analysis and partial differential equations. Nonlocal operator $(-\Delta)_p^s$ such as naturally arise in continuum mechanics, phase transition phenomena, population dynamics,...

In the literature, there are many works on the existence of solutions for fractional p-Laplacian equation, $p \geq 2$. The authors applied some different methods to study the existence, nonexistence or multiplicity results of weak solutions for nonlocal equations involving the fractional p-Laplacian in domain $\Omega \subset \mathbb{R}^N$. We refer the reader to some following paper. In [6], the authors investigated the fractional p-Laplacian equation (1.4) and established the existence and multiplicity results of weak solutions by using Morse Theory. In [7], the authors established the existence of multiple weak solutions for (1.4) with nonlinearity in form

$$\lambda f(x, u) + \mu g(x, u).$$

In [7–16], the authors applied some different methods (as Variational method via the Mountain Pass Theorem, fixed point method, etc.) to study the existence, nonexistence or multiplicity results of weak solutions for nonlocal equations involving the fractional p-Laplacian in domain $\Omega \subset \mathbb{R}^N$.

Our aim in this paper is to study the existence of weak solutions for a fractional p-Laplacian problem (1.1) by using the Minimum principle, the saddle point theorem together with a generalization of the Landesman–Lazer-type condition.

Now, let us introduce a variational setting for the problem (1.1).

We first recall some results related to the fractional Sobolev space and the fractional p-Laplacian, for more details see [6,17].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. For $p \in (1; +\infty)$, $s \in (0; 1)$, the fractional critical exponent is defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N \\ +\infty & \text{if } sp \geq N. \end{cases}$$

Define the Gagliardo seminorm by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

where $u : \mathbb{R}^N \to \mathbb{R}$ is a measurable function, and we define the fractional Sobolev space

$$W^{s,p}\left(\mathbb{R}^N\right) = \left\{ u \in L^p\left(\mathbb{R}^N\right) : u \text{ measurable}, [u]_{s,p} < +\infty \right\},$$

endowed with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}},$$

where $\|.\|_p$ denotes the norm of $L^p(\Omega)$.

Denote $X(\Omega)$ as the closed linear subspace

$$X(\Omega) = \left\{ u \in W^{s,p}\left(\mathbb{R}^N\right) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega \right\}.$$

which can be equivalently renormed by setting $\|.\| = [.]_{s,p}$ (see [6,17]).

Moreover $(X(\Omega), \|.\|)$ is a uniformly convex Banach space and that the embedding $X(\Omega)$ into $L^q(\Omega)$ is continuous for all $1 \leq q \leq p_s^*$ and compact for all $1 \leq q < p_s^*$ (see [6,17]).

We set the nonlinear operator $A : X(\Omega) \to X(\Omega)^*$ defined for all $u, v \in X(\Omega)$ by

$$\langle A(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

Remark that, if u is smooth enough, this definition coincides with that of (1.2).

Clearly for all $u \in X(\Omega)$, we have

$$\langle A(u), u \rangle = \|u\|^p, \qquad \|A(u)\|_* \leq \|u\|^{p-1}.$$

Since $X(\Omega)$ is uniformly convex Banach space, operator A satisfies the following compactness condition (see [6]).

Lemma 1.1 (S-property) *If $\{u_m\}$ is a sequence weakly converging to u in $X(\Omega)$ such that*

$$\langle A(u_m), u_m - u\rangle \to 0 \text{ as } m \to +\infty.$$

Then $\{u_m\}$ strongly converges to u in $X(\Omega)$.

Moreover A is the Gateaux derivative of the functional

$$u \to J(u) = \frac{\|u\|^p}{p} \text{ in } X(\Omega).$$

Now, we consider the nonlinear eigenvalue problem in $X(\Omega)$, namely

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u, & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.5}$$

Many properties of the eigenvalue problem (1.5) have been detected by several authors (can see [6,18,19]). Hence we can recall only the properties that using in our arguments below.

Let

$$\lambda_1 = \inf_{u\in X(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p} = \inf_{u\in X(\Omega)\setminus\{0\}} \frac{\langle A(u), u\rangle}{\|u\|_p^p}, \tag{1.6}$$

where $\|u\| = [u]_{s,p}, u \in X(\Omega)$. Then $\lambda_1 \in (0; +\infty)$ is the first eigenvalue of the eigenvalue problem (1.5). The number λ_1 plays an important role in arguments for our problem.

- $\lambda_1 = \min \sigma(s, p)$ is an isolated point of $\sigma(s, p)$, where $\sigma(s, p)$ is the spectrum of the operator $(-\Delta)_p^s$ in $X(\Omega)$. Moreover λ_1- eigenfunctions are proportionale.
- $\varphi_1(x)$ is a λ_1- eigenfunction, then either $\varphi_1(x) > 0$ a.e. in Ω or $\varphi_1(x) < 0$ a.e. in Ω. In below we always assume that $\varphi_1(x) > 0$ for a.e. $x \in \Omega$.

Definition 1.1 A function $u(x) \in X(\Omega)$ is said a weak solution of the problem (1.1) if only if

$$\langle A(u), v\rangle = \lambda_1 \int_\Omega |u|^{p-2}uv\mathrm{d}x + \int_\Omega f(x, u)v\mathrm{d}x - \int_\Omega k(x)v\mathrm{d}x \tag{1.7}$$

for all $v \in X(\Omega)$.

In order to establish our main theorem, we introduce the following hypotheses: (H_1) [1–5]

(i) $k(x) \neq 0$ a.e. $x \in \Omega$, $k(x) \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $f(x, 0) = 0$ and there exists a function $\tau(x) \in L^{p'}(\Omega)$ such that

$$|f(x, s)| \le \tau(x) \text{ for a.e.} x \in \Omega, \text{ all } s \in \mathbb{R}.$$

Denotes by

$$f^{+\infty}(x) = \lim_{s\to+\infty} f(x,s) \quad , \quad f_{-\infty}(x) = \lim_{s\to-\infty} f(x,s) \text{ a.e. } x \in \Omega. \tag{1.8}$$

$$F^{+\infty}(x) = \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau f(x, y\varphi_1)\mathrm{d}y, \qquad \text{a.e. } x \in \Omega. \tag{1.9}$$

$$F_{-\infty}(x) = \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau f(x, -y\varphi_1)\mathrm{d}y, \qquad \text{a.e. } x \in \Omega. \tag{1.10}$$

(H_{21})

(i) $$f_{-\infty}(x) < k(x) < f^{+\infty}(x), \text{ a.e. } x \in \Omega. \tag{1.11}$$

(ii) $$\int_\Omega F^{+\infty}(x)\varphi_1(x)\mathrm{d}x < \int_\Omega k(x)\varphi_1(x)\mathrm{d}x < \int_\Omega F_{-\infty}(x)\varphi_1(x)\mathrm{d}x. \tag{1.12}$$

(H_{22})

(i) $$f^{+\infty}(x) < k(x) < f_{-\infty}(x), \text{ a.e. } x \in \Omega. \tag{1.13}$$

(ii) $$\int_\Omega F_{-\infty}(x)\varphi_1(x)\mathrm{d}x < \int_\Omega k(x)\varphi_1(x)\mathrm{d}x < \int_\Omega F^{+\infty}(x)\varphi_1(x)\mathrm{d}x. \tag{1.14}$$

Our main result is given by the following theorem

Theorem 1.1 *Problem (1.1) admits a nonzero weak solution in $X(\Omega)$ if one of following two conditions*

(i) (H_1) *and* (H_{21}),

or

(ii) (H_1) *and* (H_{22}) *holds.*

Proof of the Theorem 1.1 is based on variational techniques via the Minimum principle and the saddle point theorem.

Theorem 1.2 (Minimum principle (see [20,21])) *Let $\mathfrak{F} \in C^1(Y)$, where Y is a Banach space.*

Assume that

(i) *$\mathfrak{F}$ is bounded from below, $c = \inf \mathfrak{F}$.*
(ii) *$\mathfrak{F}$ satisfies the Palais–Smale condition in Y.*

Then there exists $u_0 \in Y$ such that $\mathfrak{F}(u_0) = c$.

Theorem 1.3 (saddle point theorem-P.H.Rabinowitz (see [21,22])) *Let $X = E \oplus Y$ be a Banach space with Y closed in X and $\dim E < +\infty$. For $\rho > 0$ define*

$$M = \{u \in E : \|u\| < \rho\} \quad , \quad M_0 = \{u \in E : \|u\| = \rho\}.$$

Let $F \in C^1(X, R)$ be such that

$$b = \inf_{u \in Y} F(u) > a = \max_{u \in M_0} F(u).$$

If F satisfies the $(P-S)_c$ condition with

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} F(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} = 1\},$$

then c is a critical value of F.

2 Proof of Theorem 1.1(i) (Minimum principle and Existence of Weak Solutions)

We define the Euler–Lagrange functional associated with the problem (1.1) as

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x + \int_\Omega k(x)u\mathrm{d}x \\ &= J(u) + T(u), \quad u \in X(\Omega), \end{aligned} \tag{2.1}$$

where

$$J(u) = \frac{1}{p}\|u\|^p = \frac{1}{p}\langle A(u), u\rangle, \quad u \in X(\Omega). \tag{2.2}$$

$$T(u) = -\frac{\lambda_1}{p}\int_\Omega |u|^p \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x + \int_\Omega k(x)u\mathrm{d}x, \quad u \in X(\Omega)$$

$$F(x,u) = \int_0^u f(x,s)\mathrm{d}s.$$

We deduce that $I \in C^1(X(\Omega))$ (see [6]) and the derivative of I is defined by

$$\begin{aligned} \left\langle I'(u), v\right\rangle = \langle A(u), v\rangle - \lambda_1 \int_\Omega |u|^{p-2}uv\mathrm{d}x - \int_\Omega f(x,u)v\mathrm{d}x \\ + \int_\Omega k(x)v\mathrm{d}x, \forall u, v \in X(\Omega). \end{aligned} \tag{2.3}$$

Therefore the critical points of I are weak solutions of the problem (1.1).

Proposition 2.1 *Assuming the hypotheses (H_1) and (H_{21}) are fulfilled, then the functional $I : X(\Omega) \to \mathbb{R}$ given by (2.1) satisfies the (P–S) condition in $X(\Omega)$.*

Proof Let $\{u_m\}$ be a Palais–Smale sequence in $X(\Omega)$, i.e.,

$$|I(u_m)| \leq M \text{ with a positive constant } M. \tag{2.4}$$

$$I'(u_m) \to 0 \text{ in } X(\Omega)^* \text{ as } m \to +\infty. \tag{2.5}$$

First, we shall prove that the sequence $\{u_m\}$ is bounded in $X(\Omega)$. We suppose by contradiction that the sequence $\{u_m\}$ is not bounded in $X(\Omega)$. Without loss of generality, we assume that

$$\|u_m\| \to +\infty \text{ as } m \to +\infty.$$

Let $\widehat{u}_m = \frac{u_m}{\|u_m\|}$. Thus the sequence $\{\widehat{u}_m\}$ is bounded in $X(\Omega)$. Then there exists a subsequence $\{\widehat{u}_{m_k}\}$ which converges weakly to $\widehat{u}$ in $X(\Omega)$. Since the embedding $X(\Omega)$ into $L^p(\Omega)$ is compact, $\{\widehat{u}_{m_k}\}$ converges strongly to $\widehat{u}$ in $L^p(\Omega)$.

From (2.4), we have

$$\lim_{k\to+\infty} \sup \left\{ \frac{1}{p}\|\widehat{u}_{m_k}\|^p - \frac{\lambda_1}{p}\int_\Omega |\widehat{u}_{m_k}|^p \mathrm{d}x - \int_\Omega \frac{F\left(x, u_{m_k}\right)}{\|u_{m_k}\|^p}\mathrm{d}x + \int_\Omega \frac{k(x)\widehat{u}_{m_k}}{\|u_{m_k}\|^{p-1}}\mathrm{d}x \right\} \leq 0. \tag{2.6}$$

By hypotheses (H_1), we have:

$$\lim_{k\to+\infty} \sup \int_\Omega \frac{F\left(x, u_{m_k}\right)}{\|u_{m_k}\|^p}\mathrm{d}x = 0.$$

$$\lim_{k\to+\infty} \sup \int_\Omega \frac{k(x)\widehat{u}_{m_k}}{\|u_{m_k}\|^{p-1}}\mathrm{d}x = 0.$$

Moreover

$$\lim_{k\to+\infty} \int_\Omega |\widehat{u}_{m_k}|^p \mathrm{d}x = \int_\Omega |\widehat{u}|^p \mathrm{d}x = \|\widehat{u}\|_p^p.$$

Then, from (2.6) we obtain that

$$\lim_{k\to+\infty} \sup \|\widehat{u}_{m_k}\|^p \mathrm{d}x \leq \lambda_1 \|\widehat{u}\|_p^p. \tag{2.7}$$

From (2.7) and since the functional $J(u) = \frac{\|u\|^p}{p}$ is sequentially weakly lower semicontinous in $X(\Omega)$, we come to a conclusion that

$$\begin{aligned} \frac{\lambda_1}{p}\|\widehat{u}\|_p^p = \frac{\lambda_1}{p}\int_\Omega |\widehat{u}|^p \mathrm{d}x &\leq J\left(\widehat{u}\right) \leq \lim_{k\to+\infty} \inf J\left(\widehat{u}_{m_k}\right) \\ &\leq \lim_{k\to+\infty} \sup J\left(\widehat{u}_{m_k}\right) = \lim_{k\to+\infty} \sup \frac{1}{p}\left\|\widehat{u}_{m_k}\right\|^p \leq \frac{\lambda_1}{p}\|\widehat{u}\|_p^p. \end{aligned}$$

Hence,

$$J\left(\widehat{u}\right) = \frac{1}{p}\|\widehat{u}\|^p = \frac{\lambda_1}{p}\|\widehat{u}\|_p^p. \tag{2.8}$$

By definition of λ_1, from (2.8) we deduce that $\widehat{u} = \pm\varphi_1$, where $\varphi_1(x)$ is λ_1 eigenfunction of the eigenvalue problem (1.5).

We shall consider following two cases.

First, we assume that $\widehat{u}_{m_k} \to \varphi_1$ in $L^p(\Omega)$ as $k \to +\infty$; hence, $u_{m_k}(x) \to +\infty$ a.e. $x \in \Omega$ and $\widehat{u}_{m_k}(x) \to \varphi_1(x)$ a.e. $x \in \Omega$.

From (2.4), we have

$$\begin{aligned} -pM \leq -\left\|u_{m_k}\right\|^p + \lambda_1 \int_\Omega \left|u_{m_k}\right|^p \mathrm{d}x + p \int_\Omega F\left(x, u_{m_k}\right) \mathrm{d}x \\ -p \int_\Omega k(x) u_{m_k}(x) \mathrm{d}x \leq pM. \end{aligned} \tag{2.9}$$

and from (2.5), there exists a sequence $\{\epsilon_k\}$, $\epsilon_k > 0$, $\epsilon_k \to 0$ as $k \to +\infty$ such as

$$\left|\left\langle I'\left(u_{m_k}\right), u_{m_k}\right\rangle\right| \leq \varepsilon_k \left\|u_{m_k}\right\|, \qquad (k = 1, 2, \ldots.)$$

that is

$$\begin{aligned} -\varepsilon_k \left\|u_{m_k}\right\| \leq \left\|u_{m_k}\right\|^p - \lambda_1 \int_\Omega \left|u_{m_k}\right|^p \mathrm{d}x - \int_\Omega f\left(x, u_{m_k}\right) u_{m_k} \mathrm{d}x \\ + \int_\Omega k(x) u_{m_k}(x) \mathrm{d}x \leq \varepsilon_k \left\|u_{m_k}\right\|. \end{aligned} \tag{2.10}$$

By summing (2.9) and (2.10), we have

$$\begin{aligned} -pM - \varepsilon_k \left\|u_{m_k}\right\| \leq p \int_\Omega F\left(x, u_{m_k}\right) \mathrm{d}x - \int_\Omega f\left(x, u_{m_k}\right) u_{m_k} \mathrm{d}x \\ + (1-p) \int_\Omega k(x) u_{m_k}(x) \mathrm{d}x \leq pM + \varepsilon_k \left\|u_{m_k}\right\|. \end{aligned} \tag{2.11}$$

After dividing (2.11) by $\|u_{m_k}\|$, remark that

$$\begin{aligned} \lim_{k\to+\infty} \int_\Omega f\left(x, u_{m_k}\right) \widehat{u}_{m_k}(x) \mathrm{d}x = \int_\Omega f^{+\infty}(x) \varphi_1(x) \mathrm{d}x, \\ \lim_{k\to+\infty} \int_\Omega k\left(x\right) \widehat{u}_{m_k}(x) \mathrm{d}x = \int_\Omega k(x) \varphi_1(x) \mathrm{d}x \end{aligned}$$

and due to the Lebesgue Theorem, we have

$$\lim_{k\to+\infty} \sup \int_\Omega \left(p \frac{F\left(x, u_{m_k}\right)}{\left\|u_{m_k}\right\|} - f^{+\infty}(x) \varphi_1(x) \right) \mathrm{d}x = (p-1) \int_\Omega k(x) \varphi_1(x) \mathrm{d}x. \tag{2.12}$$

Denote $l_k = \|u_{m_k}\| \to +\infty$ as $k \to +\infty$, by hypotheses (H_1), and we have

$$\left|\int_\Omega \frac{1}{l_k}\left(\int_0^{u_{m_k}} f(x,s)\,\mathrm{d}s - \int_0^{l_k\varphi_1} f(x,s)\,\mathrm{d}s\right)\right| \le \int_\Omega \frac{1}{l_k}\left|u_{m_k} - l_k\varphi_1\right| \tau(x)\,\mathrm{d}x$$
$$\le \|\tau\|_{p'} \left\|\widehat{u}_{m_k} - \varphi_1\right\|_p \to 0 \text{ as } k \to +\infty.$$

This implies

$$\lim_{k\to+\infty} \sup \int_\Omega \frac{F\left(x, u_{m_k}\right)}{\left\|u_{m_k}\right\|}\mathrm{d}x = \lim_{k\to+\infty} \int_\Omega \left(\frac{1}{l_k}\int_0^{l_k\varphi_1} f(x,s)\,\mathrm{d}s\right)\mathrm{d}x.$$

By changing $s = y\varphi_1$, $\mathrm{d}s = \varphi_1 \mathrm{d}y$, we deduce that

$$\lim_{k\to+\infty} \frac{1}{l_k}\int_0^{l_k\varphi_1} f(x,s)\,\mathrm{d}s = \lim_{k\to+\infty} \frac{1}{l_k}\int_0^{l_k} f(x, y\varphi_1)\,\varphi_1 \mathrm{d}y = F^{+\infty}(x)\varphi_1(x),$$

where $F^{+\infty}(x)$ is given by (1.9). Hence

$$\lim_{k\to+\infty} \sup \int_\Omega \frac{F\left(x, u_{m_k}\right)}{\left\|u_{m_k}\right\|}\mathrm{d}x = \int_\Omega F^{+\infty}(x)\varphi_1(x)\mathrm{d}x. \tag{2.13}$$

Therefore from (2.12), (2.13), we obtain that

$$\int_\Omega \left(pF^{+\infty}(x) - f^{+\infty}(x)\right)\varphi_1(x)\mathrm{d}x = (p-1)\int_\Omega k(x)\varphi_1(x)\mathrm{d}x. \tag{2.14}$$

On the other hand, from the hypotheses (1.11) we have

$$f^{+\infty}(x) - k(x) \ge 0 \text{ a.e. } x \in \Omega.$$

Hence (2.14) implies that

$$\int_\Omega pF^{+\infty}(x)\varphi_1(x)\mathrm{d}x = p\int_\Omega k(x)\varphi_1(x)\mathrm{d}x + \int_\Omega \left(f^{+\infty}(x) - k(x)\right)\varphi_1(x)\mathrm{d}x$$
$$\ge p\int_\Omega k(x)\varphi_1(x)\mathrm{d}x$$

which contradicts (H_{21}). □

In the case when $\widehat{u}_{m_k} \to -\varphi_1(x)$ as $k \to +\infty$, by similar arguments we also have

$$\int\limits_{\Omega} (pF_{-\infty}(x) - f_{-\infty}(x))\,\varphi_1(x)\mathrm{d}x = (p-1)\int\limits_{\Omega} k(x)\varphi_1(x)\mathrm{d}x. \tag{2.15}$$

By hypotheses, (1.11), (2.15) imply that

$$\begin{aligned}\int\limits_{\Omega} pF_{-\infty}(x)\varphi_1(x)\mathrm{d}x &= p\int\limits_{\Omega} k(x)\varphi_1(x)\mathrm{d}x + \int\limits_{\Omega} (f_{-\infty}(x) - k(x))\varphi_1(x)\mathrm{d}x\\ &\le p\int\limits_{\Omega} k(x)\varphi_1(x)\mathrm{d}x,\end{aligned}$$

which contradicts (H_{21}).

This implies that the (P–S) sequence $\{u_m\}$ is bounded in $X(\Omega)$. Then there exists a subsequence $\{u_{m_k}\}$ which converges weakly to u_0 in $X(\Omega)$. We will prove that the subsequence converges strongly to u_0 in $X(\Omega)$.

Indeed, since $u_{m_k} \rightharpoonup u_0$ in $X(\Omega)$ and the embedding $X(\Omega)$ into $L^p(\Omega)$ is compact, $\{u_{m_k}\}$ converges strongly to u_0 tin $L^p(\Omega)$.

Firstly we remark that by (H_1)

$$\begin{aligned}\left|\left\langle T'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle\right| &\le \lambda_1 \left\|u_{m_k}\right\|_p^{p-1} \left\|u_{m_k} - u_0\right\|_p\\ &\quad + \|\tau\|_{p'}\left\|u_{m_k} - u_0\right\|_p + \|k\|_{p'}\left\|u_{m_k} - u_0\right\|_p\\ &\le \left(\lambda_1 \left\|u_{m_k}\right\|_p^{p-1} + \|\tau\|_{p'} + \|k\|_{p'}\right)\left\|u_{m_k} - u_0\right\|_p.\end{aligned}$$

Since $\{u_{m_k}\}$ is bounded in $L^p(\Omega)$, $\left\|u_{m_k} - u_0\right\|_p \to 0$ as $k \to +\infty$, we obtain that

$$\lim_{k\to+\infty}\left\langle T'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle = 0. \tag{2.16}$$

Combining (2.16) and the fact

$$\lim_{k\to+\infty}\left\langle I'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle = 0$$

we get

$$\begin{aligned}\lim_{k\to+\infty}\left\langle J'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle &= \lim_{k\to+\infty}\left\langle I'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle\\ &\quad - \lim_{k\to+\infty}\left\langle T'\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle = 0.\end{aligned}$$

That is

$$\lim_{k\to+\infty}\left\langle A\left(u_{m_k}\right), u_{m_k} - u_0\right\rangle = 0. \tag{2.17}$$

From (2.17), by the (S)-property of the operator A (see Lemma 1.1), we deduce that the subsequence $\{u_{m_k}\}$ converges strongly to u_0 in $X(\Omega)$. Therefore the functional I satisfies the Palais–Smale condition in $X(\Omega)$. Proposition 2.1 is proved.

Proposition 2.2 *The functional I given by* (2.1) *is coercive on $X(\Omega)$ provided* (H_{21}) *holds.*

Proof Firstly we noted that, in the proof of the Proposition 2.1, we have proved that if $\{I(u_m)\}$ is a sequence bounded from above with a sequence $\{u_m\}$ in $X(\Omega)$ such that $\|u_m\| \to +\infty$ as $m \to +\infty$, then (up to a subsequence),

$$\widehat{u}_m = \frac{u_m}{\|u_m\|} \to \pm\varphi_1(x) \text{ in } X(\Omega) \text{ as } m \to +\infty.$$

Using this fact, we will prove that the functional I is coercive in $X(\Omega)$ if (H_{21}) satisfied.

Indeed, suppose by contradiction that I is not coercive, it is possible to choose a sequence $\{u_m\}$ in $X(\Omega)$ such that $\|u_m\| \to +\infty$ as $m \to +\infty$, $I(u_m) \leq$ const and

$$\widehat{u}_m = \frac{u_m}{\|u_m\|} \to \pm\varphi_1(x) \text{ in } X(\Omega) \text{ as } m \to +\infty.$$

Remark that by (1.6) we deduce that

$$-\int_\Omega F(x, u_m)\mathrm{d}x + \int_\Omega k(x)u_m(x)\mathrm{d}x \leq I(u_m) \leq \text{ const}, \quad m = 1, 2, \ldots \tag{2.18}$$

We now consider following two cases

Case 1: Assume that $\widehat{u}_m \to \varphi_1$ as $m \to +\infty$.

Dividing (2.18) by $\|u_m\|$, we get

$$\begin{aligned} -\int_\Omega F^{+\infty}(x)\varphi_1(x)\mathrm{d}x + \int_\Omega k(x)\varphi_1(x)\mathrm{d}x \\ = \lim_{k\to+\infty} \sup\left(-\int_\Omega \frac{F(x, u_m)}{\|u_m\|}\mathrm{d}x + \int_\Omega k(x)\widehat{u}_m(x)\mathrm{d}x\right) \leq \lim_{m\to+\infty} \sup \frac{\text{const}}{\|u_m\|} = 0 \end{aligned}$$

which gives

$$\int_\Omega k(x)\varphi_1(x)\mathrm{d}x \leq \int_\Omega F^{+\infty}(x)\varphi_1(x)\mathrm{d}x.$$

which contradicts (H_{21}).

Case 2: Assume that $\widehat{u}_m \to -\varphi_1$ as $m \to +\infty$.

By similar computation above, we get

$$\int_\Omega F_{-\infty}(x)\varphi_1(x)\mathrm{d}x - \int_\Omega k(x)\varphi_1(x)\mathrm{d}x \le 0,$$

that is

$$\int_\Omega F_{-\infty}(x)\varphi_1(x)\mathrm{d}x \le \int_\Omega k(x)\varphi_1(x)\mathrm{d}x$$

which contradicts (H_{21}).

It implies that I is coercive in $X(\Omega)$. The Proposition 2.2 is proved. □

Proof of Theorem 1.1(i): The coerciveness (see Proposition 2.2) and the Palais–Smale condition (see Proposition 2.1) are enough to prove that the functional I attains its proper infimum at some u_0 in $X(\Omega)$ (see Theorem 1.3) so that the problem (1.1) has at least a weak solution $u_0 \in X(\Omega)$. It is clear that u_0 is a nontrivial solution of the problem (1.1). □

3 Proof of the Theorem 1.1(ii): (saddle point theorem and Existence of Weak Solutions)

First, we remark that by similar arguments as in the proof of proposition 2.1, with hypotheses (H_{22}), we can prove that the functional I given by (2.1) satisfies the (P–S) condition in $X(\Omega)$.

Splitting $X(\Omega)$ as the sum: $X(\Omega) = E \oplus Y$, where

$$\begin{aligned} E &= \{t\varphi_1, t \in \mathbb{R}\}. \\ Y &= \{v \in X(\Omega) : \int_\Omega \varphi_1^{p-1} v \mathrm{d}x = 0\}, \end{aligned} \tag{3.1}$$

where φ_1 is normalized eigenfunction associated with the eigenvalue λ_1 of the problem (1.5), $\varphi_1 > 0$, $x \in \Omega$, $\|\varphi_1\| = 1$.

For $u = t\varphi_1 + v$, $v \in Y$; then, we have

$$\int_\Omega u\varphi_1^{p-1}\mathrm{d}x = t\int_\Omega |\varphi_1|^p \mathrm{d}x + \int_\Omega v\varphi_1^{p-1}\mathrm{d}x.$$

Since $v \in Y$, $\int_\Omega v\varphi_1^{p-1}\mathrm{d}x = 0$ and by definition of λ_1,

$$\int_\Omega \varphi_1^p \mathrm{d}x = \frac{1}{\lambda_1}\|\varphi_1\|^p = \frac{1}{\lambda_1}.$$

Hence $t = \lambda_1 \int_\Omega u\varphi_1^{p-1}\mathrm{d}x$.

On the other hand, for any $u \in X(\Omega)$, take $t = \lambda_1 \int_\Omega u\varphi_1^{p-1}\mathrm{d}x$, $v = u - t\varphi_1$. It is clear that $v \in Y$. Thus $u = t\varphi_1 + v$, $v \in Y$.

Lemma 3.1 *There exists* $\overline{\lambda} > \lambda_1$ *such that*

$$(Av, v) = \|v\|^p > \overline{\lambda} \int_\Omega |v|^p \mathrm{d}x, \text{ for all } v \in Y.$$

Proof Let

$$\lambda = \inf\{(Av, v) : \int_\Omega |v|^p \mathrm{d}x = 1, v \in Y\}.$$

We shall prove that value λ is attained in Y.

Let $\{v_m\}$ in Y be a minimizing sequence, i.e.,

$$\int_\Omega |v_m|^p \mathrm{d}x = 1, m = 1, 2, \ldots$$

$$\lim_{m\to+\infty} (Av_m, v_m) = \lim_{m\to+\infty} \|v_m\|^p = \lambda.$$

This implies that the sequence $\{v_m\}$ is bounded in $X(\Omega)$. Hence there exists a subsequence $\{v_{m_k}\}$ such as

$$v_{m_k} \rightharpoonup v_0 \text{ in } X(\Omega),$$
$$v_{m_k} \to v_0 \text{ in } L^p(\Omega)$$

provided the embedding $X(\Omega)$ into $L^p(\Omega)$ is compact.

Observe that

$$\left| \int_\Omega \varphi_1^{p-1} \left(v_{m_k} - v_0\right) \mathrm{d}x \right| \le \|\varphi_1\|_p^{p-1} \left\|v_{m_k} - v_o\right\|_p \to 0 \text{ as } m \to +\infty.$$

Hence

$$0 = \lim_{k\to+\infty} \int_\Omega v_{m_k}(x)\varphi_1^{p-1}(x)\mathrm{d}x = \int_\Omega v_0(x)\varphi_1^{p-1}(x)\mathrm{d}x$$

this implies that $v_0 \in Y$.

Besides

$$1 = \lim_{k\to+\infty} \int_\Omega \left|v_{m_k}(x)\right|^p \mathrm{d}x = \int_\Omega |v_0(x)|^p \mathrm{d}x,$$

so $v_0 \neq 0$.

By the lower weak semicontinous of the functional $v \mapsto \|v\|^p$, $v \in X(\Omega)$, we have

$$\lambda \le (Av_0, v_0) = \|v_0\|^p \le \lim_{k\to+\infty} \inf \left\|v_{m_k}\right\|^p \le \lim_{k\to+\infty} \left\|v_{m_k}\right\|^p = \lambda.$$

hence $\lambda = \|v_0\|^p$. It means that the value λ is attained at v_0.

By the variational characterization of the eigenvalue λ_1, it is clear that $\lambda \ge \lambda_1$.

If $\lambda = \lambda_1$, by simplicity of λ_1, there exists $t \in \mathbb{R}$ such that $v_0 = t\varphi_1$.

But since $v_0 \in Y$, we have

$$0 = \int_\Omega \varphi_1^{p-1} v_0 \mathrm{d}x = t \int_\Omega \varphi_1^p \mathrm{d}x = t \|\varphi_1\|_p^p,$$

hence $t = 0$ and then $v_0 = 0$ which a contradiction due to $v_0 \neq 0$.

This implies that $\overline{\lambda} = \lambda > \lambda_1$ and the proof of the Lemma 2.1 is complete. □

Proposition 3.1 *The function I given by (2.1) is coercive on Y provided hypotheses (H_1) and (H_{22}) hold.*

Proof Observe that by Holder inequality, Lemma 3.1 and hypotheses (H_1) we have for any $v \in Y$:

$$\begin{aligned} |I(v)| &\geq \frac{1}{p}\|v\|^p - \frac{\lambda_1}{p}\int_\Omega |v|^p \mathrm{d}x - \int_\Omega |F(x, v)|\,\mathrm{d}x - \int_\Omega k(x)\,|v|\,\mathrm{d}x \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right)\|v\|^p - \left(\|\tau\|_{p'} + \|k\|_{p'}\right)\|v\|_p \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right)\|v\|^p - M\left(\|\tau\|_{p'} + \|k\|_{p'}\right)\|v\|, \end{aligned} \tag{3.2}$$

with M is positive.

From (3.2), since $p \geq 2$, $1 - \frac{\lambda_1}{\overline{\lambda}} > 0$ it follows $|I(v)| \to +\infty$ as $\|v\| \to +\infty$. So that the functional I is coercive on Y and Proposition 3.1 is proved. □

From Proposition 3.1, it implies that

$$B_Y = \min_{v \in Y} I(v) > -\infty.$$

Remark that for every $t \in R$, we have

$$\frac{1}{p}\|t\varphi_1\|^p - \frac{\lambda_1}{p}\int_\Omega |t\varphi_1|^p \mathrm{d}x = 0,$$

as follows from the definition of λ_1 and φ_1. Thus

$$\begin{aligned} I(t\varphi_1) &= t\int_\Omega k(x)\varphi_1(x)\mathrm{d}x - \int_\Omega F(x, t\varphi_1)\,\mathrm{d}x \\ &= t\int_\Omega \left(k(x)\varphi_1(x) - \frac{F(x, t\varphi_1)}{t}\right)\mathrm{d}x, \end{aligned} \tag{3.3}$$

where

$$\frac{F(x, t\varphi_1)}{t} = \frac{1}{t}\int_0^{t\varphi_1} f(x, s)\,\mathrm{d}s.$$

Note that

$$\begin{aligned}\lim_{t\to+\infty}\frac{F(x,t\varphi_1)}{t}&=\lim_{t\to+\infty}\frac{1}{t}\int_0^{t\varphi_1}f(x,s)\,\mathrm{d}s\\&=\lim_{t\to+\infty}\frac{1}{t}\left(\int_0^{t}f(x,y\varphi_1)\,\mathrm{d}y\right)\varphi_1=F^{+\infty}(x)\varphi_1\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\lim_{t\to-\infty}\frac{F(x,t\varphi_1)}{t}&=\lim_{t\to-\infty}-\frac{1}{|t|}\int_0^{t\varphi_1}f(x,s)\,\mathrm{d}s\\&=\lim_{t\to-\infty}\frac{1}{|t|}\left(\int_0^{|t|}f(x,-y\varphi_1)\,\mathrm{d}y\right)\varphi_1=F_{-\infty}(x)\varphi_1.\end{aligned}\tag{3.5}$$

Hence by the hypotheses (H_{22}), from (3.3), (3.4), (3.5), we have

$$\begin{aligned}\lim_{t\to+\infty}I(t\varphi_1)&=\lim_{t\to+\infty}t\int_\Omega\left(k(x)\varphi_1(x)-\frac{F(x,t\varphi_1)}{t}\right)\mathrm{d}x\\&=\lim_{t\to+\infty}t\int_\Omega\big(k(x)\varphi_1(x)-F^{+\infty}(x)\varphi_1\big)\mathrm{d}x=-\infty\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\lim_{t\to-\infty}I(t\varphi_1)&=\lim_{t\to-\infty}t\int_\Omega\left(k(x)\varphi_1(x)-\frac{F(x,t\varphi_1)}{t}\right)\mathrm{d}x\\&=\lim_{t\to-\infty}t\int_\Omega(k(x)\varphi_1(x)-F_{-\infty}(x)\varphi_1)\mathrm{d}x=-\infty.\end{aligned}\tag{3.7}$$

Thus there exists $R>0$ such that for any $t:|t|=R$ we have

$$I(t\varphi_1)<B_Y\le I(v)\text{ for all }v\in Y.$$

From this, we can finish the proof of Theorem 1.1 (ii).

Proof of Theorem 1.1(ii): By Proposition 3.1, applying the saddle point theorem (see Theorem 1.4) we deduce that the functional I attains its proper infimum at some $u_0\in X(\Omega)$, so that the problem (1.1) has at least a weak solution $u_0\in X(\Omega)$ and it is clear that $u_0\neq 0$.

The Theorem 1.2 is proved. □

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References

1. Fiscella, A., Pucci, P.: p-fractional Kirchhoff equations involving critical nonlinearities. Nonlinear Anal. RWA **35**, 350–378 (2017)
2. Areoya, D., Orsina, L.: Landesman–Lazer condition and quasilinear elliptic equations. Nonlinear Anal. **28**, 1623–1632 (1997)
3. Hung, B.Q., Toan, H.Q.: On existence of weak solutions for a p-Laplacian system at resonance. RACSAM **110**, 33–47 (2016)
4. Iannizzotto, A., Squassina, M.: Qeyl-type lows for fractional p-eigenvalue problems. Asymptot. Anal. **88**, 233–245 (2014)
5. Ngo, Q.A., Toan, H.Q.: Some remarks on a class of nonuniformly elliptic equations of p-Laplacian type. Acta Appl. Math. **106**, 229–239 (2009)
6. Iannizzotto, A., Liu, S., Perera, K., Squassina, M.: Existence results for fractional p-Laplacian problems via Morse theory. Adv. Calc. Var. **9**, 101–125 (2016)
7. Ferrara, M., Gurrini, L., Zang, B.: Multiple solutions for perturbed non-local fractional Laplacian equations. Electron. J. Differ. Equ. **260**, 1–10 (2013)
8. Dong, W., Xu, J., Wei, Z.: Infinity many weak solutions for a fractional Schrodinger equation. Bound. Value Probl. **2014**, 159 (2014)
9. Xiang, M.Q., Zhang, B.L., Qiu, H.: Existence of solutions for a critical fractional Kirchhoff type problem in $\mathbb{R}^N$. Sci. China Math. **60**, 1647–1660 (2017)
10. Xiang, M.Q., Pucci, P., Squassina, M., Zhang, B.L.: Nonlocal Schrödinger–Kirchhoff equations with external magnetic field. Discrete Contin. Dyn. Syst. **37**, 1631–1649 (2017)
11. Jiafa, X., Oregan, D., Dong, W.: Existence of weak solutions for a fractional p-Laplacian equation in $\mathbb{R}^N$. RACSAM **111**, 515–529 (2017)
12. Souza, M.: On a class of nonhomogenerous fractional quasilinear equations in $\mathbb{R}^N$ with exponential growth. Nonlinear Differ. Equ. Appl. **22**, 499–511 (2015)
13. Xiang, M., Zhang, B., Radulescu, V.D.: Existence of solutions for perturbed fractional p-Laplacian equations. J. Differ. Equ. **260**, 1392–1413 (2015)
14. Xu, J., Wei, Z., Dong, W.: Existence of weak solutions for a fractional schrodinger equation. Commun. Nonlinear. Sci. Numer. Simul. **22**, 1215–1222 (2015)
15. Mingqi, X., Radulescu, V.D., Zhang, B.: Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions. Nonlinearity **31**, 3228–3250 (2018)
16. Mingqi, X., Radulescu, V.D., Zhang, B.: Combined effects for fractional Schrödinger–Kirchhoff systems with critical nonlinearities. ESAIM: COCV **24**, 1249–1273 (2018)
17. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev space. Bull. Sci. Math. **136**, 521–573 (2012)
18. Franzina, G., Palatucci, G.: Fractional p-eigenvalue. Riv. Mat. Univ. Parma **8**(5), 315–328 (2014)
19. Lindgren, E., Lindqvist, P.: Fractional eigenvalue. Cal. Var. Partial Differ. Equ. **49**, 795–826 (2014)
20. Ngo, Q.A., Toan, H.Q.: Existence of solutions for a resonant problem under Landesman–Lazer condition. Electron. J. Differ. Equ. **98**, 1–10 (2008)
21. Struwe, M.: Variational Methods, 2nd edn. Springer, Berlin (2008)
22. Drabek, P., Milota, J.: Methods of Nonlinear Analysis Application to Differential Equations. Birkhausen, Basel (2007)