Effects of Stiffness and the Variation of Center of Mass on Rocket Motion

Dam Viet Phuong[1], Vu Quoc Tru[1], Nguyen Anh Tuan[1]

Abstract – *This paper presents the effects of stiffness and the variation of center of mass on the motion of a rocket model. The equations of motion are based on Meshchersky's method and Lagrange's equations for flexible-variable systems. The rocket model used in this study is built based on the 9M22Y rocket. The obtained results show good agreement with actual flight data.*

Keywords: *Flexible rockets, Dynamics of elastic- variable mass systems.*

Nomenclature

ρ Mass density along the rocket length

ρ_{air} Mass density of the air

Δm Mass of the thrust

φ pitch angle in vertical plane

J_0 Moment of inertia

F_X, F_Y Resultant external force on the variable mass System

F_{thrust} Effective thrust force

$\mathfrak{M}(t)$ Static moment, $\mathfrak{M}(t) = \int\limits_{(rocket\ length)} \rho x dx$

$\mu = \dfrac{d\mathfrak{M}(t)}{dt}$ Derivation of Static moment

U Velocity of jet flow

m Mass of rocket

ζ Mass flow rate

$k_{thrust} = \dfrac{1}{\sqrt{1+\left(\dfrac{\partial v}{\partial x}(x_p)\right)^2}}$ Coefficient of effective thrust related to elasticity

$[J_{RG}], [R_{RG}], [H_{RG}]$ Additional inertial term related to the deformation of the rocket

$[G_{RG}]$ Additional stiffness term related to the deformation of the rocket

x_P Position of thrust

$\{d_{RG}\}$ Vector of nodal displacements

$[A]$ Transition matrix

I. Introduction

For rockets, it is necessary to reduce their mass and augment their slenderness in order to increase the effective range. Moreover, the reduction in the mass of rocket structures also may help install heavier payloads. On the other hand, it is relevant to note that more and more materials with advanced properties are being introduced to the aerospace industry. Due to these reasons, the structures of modern rockets may be more flexible. Previously, while studying rocket dynamics, a rocket was commonly regarded as a point mass [9], [12], [14] or a rigid body. This approach is acceptable for rockets with small deformation. For more flexible rockets, it is necessary to include the effect of flexibility to enhance the accuracy of the solution. Methods, which have been widely used to derive the equations of motions of rockets, are based on William Moore, Tsiolkovsky hay Meshchersky [dẫn chứng] or the Lagrange's equation [2], [13]. Using this approach, in several studies, researchers assume that thrust acts as an external force and the loss in mass generates only a force along the body axis [13]. Moreover, the force component normal to the body axis formed by the interaction between the loss in mass and the transverse vibration of the body, as well as the rotation of the rocket is ignored. Previous studies also did not consider the variation of the location of the mass center.

During flight, a rocket is affected by the aerodynamic force, the gravitational force and engine thrust. These forces make the rocket move, and at the same time, cause the vibrations of the rocket structure. These elastic vibrations affect the motions and the aerodynamics force, therefore, change the dynamic characteristics. In addition, while flying, the rocket loses mass, and thus, the location of the center of mass keeps varying. This variation also influence the dynamic characteristics of rocket flight. Up to the present moment, the research on this issue has not covered the entire phenomenon.

In this study, we analyze the effects of the bending deformation and the variation of the location of the mass center simultaneously. The force component normal to the body axis generated by the interaction between the loss in mass, the vibration and rotation of the body is added to the system of equations of motion. The rocket is modeled as a free-free Euler-Bernoulli beam.

II. Dynamics of rocket system

2.1. The system

Considering the system from the moment of time t_k to $t_k+\tau$, (in which τ is time variable), the rocket at the moment t_k transforms to a new system of exhaust and a rocket with different mass at the moment $t_k+\tau$. The mass of the rocket and exhaust system is conserved; hence, we can apply the Hamilton principle and the Lagrange equation in this case.

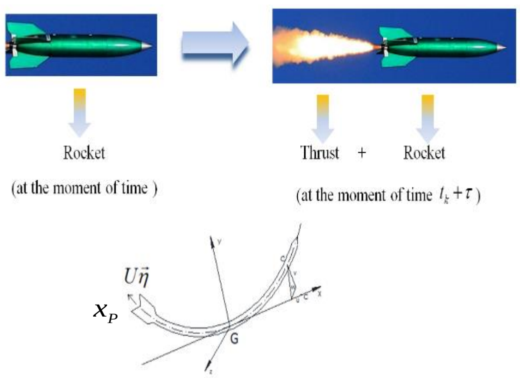


Fig.1 The variable mass system

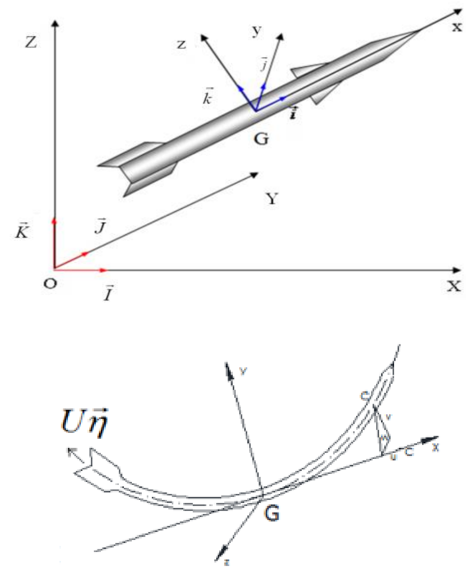


Fig.2 Ground-fixed coordinate system and local body-fixed coordinate system

The Lagrange equation can be written as follows:

$$\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}+\frac{\partial U_e}{\partial q_i}=Q_i \quad (1)$$

where:

$T=T_{rocket}+T_{thrust}$: Kinetic energy

U_e : Elastic potential energy

Q_i : Generalized force corresponding to the degree of freedom q_i

2.2. Kinetic Energy of Rocket System

The local velocity on the rocket body and the kinetic energy is:

$$T_{rocket}=\frac{1}{2}\int_{(V)}\rho\vec{V}_{C'}^2 dV$$

$$\vec{V}_{C'}=\frac{d\vec{r}_{C'}}{d\tau}=\frac{d(\vec{r}_G+\overrightarrow{GC}+\overrightarrow{CC'})}{d\tau}$$

$$=\begin{pmatrix}\left[\dot{X}_G\ \dot{Y}_G\ \dot{Z}_G\right]+\left[x\ y\ z\right]\left[\dot{A}\right]\\ +\left[\dot{u}\ \dot{v}\ \dot{w}\right]\left[A\right]+\left[u\ v\ w\right]\left[\dot{A}\right]\end{pmatrix}\begin{bmatrix}\vec{I}\\ \vec{J}\\ \vec{K}\end{bmatrix}$$

$$T_{rocket}=T_{rocket}^{(1)}+T_{rocket}^{(2)}+T_{rocket}^{(3)}$$

where:

$$T_{rocket}^{(1)} = \frac{1}{2}\left(\begin{array}{l} m(t)\left(\dot{X}_G^2 + \dot{Y}_G^2 + \dot{Z}_G^2\right) \\ + \int\limits_{(V)} \rho\left[x \;\; y \;\; z\right]\left[\dot{A}\right]\left[\dot{A}\right]^T\left[x \;\; y \;\; z\right]^T dV \\ +2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\dot{A}\right]^T \int\limits_{(V)} \rho\left[x \;\; y \;\; z\right]^T dV \end{array} \right)$$

$$T_{rocket}^{(2)} = \frac{1}{2}\left(\begin{array}{l} \int\limits_{(V)} \rho\left[u \;\; v \;\; w\right]\left[\dot{A}\right]\left[\dot{A}\right]^T\left[u \;\; v \;\; w\right]^T dV \\ +2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\dot{A}\right]^T \int\limits_{(V)} \rho\left[u \;\; v \;\; w\right]^T dV \\ +2\int\limits_{(V)} \rho\left[x \;\; y \;\; z\right]\left[\dot{A}\right]\left[\dot{A}\right]^T\left[u \;\; v \;\; w\right]^T dV \end{array} \right)$$

$$T_{rocket}^{(3)} = \frac{1}{2}\left(\begin{array}{l} \int\limits_{(V)} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV \\ +2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[A\right]^T \int\limits_{(V)} \rho\left[\dot{u} \;\; \dot{v} \;\; \dot{w}\right]^T dV \\ +2\int\limits_{(V)} \rho\left[x \;\; y \;\; z\right]\left[\dot{A}\right]\left[A\right]^T\left[\dot{u} \;\; \dot{v} \;\; \dot{w}\right]^T dV \\ +2\int\limits_{(V)} \rho\left[u \;\; v \;\; w\right]\left[\dot{A}\right]\left[A\right]^T\left[\dot{u} \;\; \dot{v} \;\; \dot{w}\right]^T dV \end{array} \right)$$

2.3. Kinetic Energy of Thrust

$$T_{thrust} = \frac{1}{2}\Delta m \vec{V}_{thrust}^2$$

The exhaust velocity can be derived as

$$\vec{V}_{thrust} = \vec{V}_P + U\vec{\eta}$$

$$\vec{V}_{thrust} = \left(\begin{array}{l} \left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right] + \left[x_P \; y_P \; z_P\right]\left[\dot{A}\right] \\ +\left[\dot{u}_P \; \dot{v}_P \; \dot{w}_P\right]\left[A\right] + \left[u_P \; v_P \; w_P\right]\left[\dot{A}\right] \\ -U\left[1 + \frac{\partial u}{\partial x}(x_P); \frac{\partial v}{\partial x}(x_P); \frac{\partial w}{\partial x}(x_P)\right]\left[\mathrm{A}\right] \end{array} \right)\left[\begin{array}{c} \vec{I} \\ \vec{J} \\ \vec{K} \end{array} \right]$$

$$T_{thrust} = \frac{1}{2}\Delta m\left(T_{thrust}^{(1)} + T_{thrust}^{(2)} + T_{thrust}^{(3)} + T_{thrust}^{(4)}\right)$$

where:

$$T_{thrust}^{(1)} = \left(\begin{array}{l} \left(\dot{X}_G^2 + \dot{Y}_G^2 + \dot{Z}_G^2\right) + 2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\dot{\mathrm{A}}\right]^{\mathrm{T}}\left[x_P \; y_P \; z_P\right]^T \\ +\left[x_P \; y_P \; z_P\right]\left[\dot{\mathrm{A}}\right]\left[\dot{\mathrm{A}}\right]^T\left[x_P \; y_P \; z_P\right]^T \end{array} \right)$$

$$T_{thrust}^{(2)} = \left(\begin{array}{l} -2U_{hd}\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\mathrm{A}\right]^T\left[1 + \frac{\partial u}{\partial x}(x_P) \;\; \frac{\partial v}{\partial x}(x_P) \;\; \frac{\partial w}{\partial x}(x_P)\right]^T \\ -2U_{hd}\left[x_P \; y_P \; z_P\right]\left[\dot{\mathrm{A}}\right]\left[\mathrm{A}\right]^T\left[1 + \frac{\partial u}{\partial x}(x_P) \;\; \frac{\partial v}{\partial x}(x_P) \;\; \frac{\partial w}{\partial x}(x_P)\right]^T \\ -2U_{hd}\left[u_P \; v_P \; w_P\right]\left[\dot{\mathrm{A}}\right]\left[\mathrm{A}\right]^T\left[1 + \frac{\partial u}{\partial x}(x_P) \;\; \frac{\partial v}{\partial x}(x_P) \;\; \frac{\partial w}{\partial x}(x_P)\right]^T \\ +U_{hd}^2\left(\left(1 + \frac{\partial u}{\partial x}(x_P)\right)^2 + \left(\frac{\partial v}{\partial x}(x_P)\right)^2 + \left(\frac{\partial w}{\partial x}(x_P)\right)^2\right) \end{array} \right)$$

$$T_{thrust}^{(3)} = \left(\begin{array}{l} 2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\dot{\mathrm{A}}\right]^{\mathrm{T}}\left[u_P \; v_P \; w_P\right]^T \\ +2\left[x_P \; y_P \; z_P\right]\left[\dot{\mathrm{A}}\right]\left[\dot{\mathrm{A}}\right]^T\left[u_P \; v_P \; w_P\right]^T \\ +\left[u_P \; v_P \; w_P\right]\left[\dot{\mathrm{A}}\right]\left[\dot{\mathrm{A}}\right]^T\left[u_P \; v_P \; w_P\right]^T \end{array} \right)$$

$$T_{thrust}^{(4)} = \left(\begin{array}{l} \left(\dot{u}_P^2 + \dot{v}_P^2 + \dot{w}_P^2\right) + 2\left[\dot{X}_G \; \dot{Y}_G \; \dot{Z}_G\right]\left[\mathrm{A}\right]^{\mathrm{T}}\left[\dot{u}_P \; \dot{v}_P \; \dot{w}_P\right]^T \\ +2\left[\dot{u}_P \; \dot{v}_P \; \dot{w}_P\right]\left[\mathrm{A}\right]\left[\dot{\mathrm{A}}\right]^T\left[u_P \; v_P \; w_P\right]^T \\ +2\left[x_P \; y_P \; z_P\right]\left[\dot{\mathrm{A}}\right]\left[\mathrm{A}\right]^T\left[\dot{u}_P \; \dot{v}_P \; \dot{w}_P\right]^T \\ -2U_{hd}\dot{u}_P\left(1 + \frac{\partial u}{\partial x}(x_P)\right) \\ -2U_{hd}\left(+\dot{v}_P\left(\frac{\partial v}{\partial x}(x_P)\right) + \dot{w}_P\left(\frac{\partial w}{\partial x}(x_P)\right)\right) \end{array} \right)$$

2.4. Elastic Potential Energy

In this study, we consider only bending deformation; hence, the elastic potential energy of the rocket, which is modeled as an Euler-Bernoulli beam, is determined by:

$$U_e = \frac{\mathrm{EJ}_z}{2}\int\limits_{(L)}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx + \frac{\mathrm{EJ}_y}{2}\int\limits_{(L)}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

2.5. Generalized forces

The generalized force used in the Lagrange equation is given as

$$\mathrm{Q}_{q_i} = \sum \vec{\mathrm{F}}\frac{\partial \vec{r}}{\partial q_i} = \sum\left[F_x \quad F_y \quad F_z\right]\frac{\partial}{\partial q_i}\left(\begin{array}{l} \left[X_G \; Y_G \; Z_G\right]^T \\ +\left[\mathrm{A}\right]^T\left[x \; y \; z\right]^T \\ +\left[\mathrm{A}\right]^T\left[u \quad v \quad w\right] \end{array} \right)$$

where F_x, F_y, F_z are external forces acting on the rocket system include the gravitational force and the aerodynamic force. It is noted that the thrust force is not included here.

III. Equation of motion of rocket in vertical plane by finite element method

Modelling the flexible rocket as an Euler-Bernoulli beam that consists of n-1 elements and each element includes 2 nodes (fig 3.1). Each node has two degrees of freedom which are $\left(v, \frac{\partial v}{\partial x}\right)$.

The rocket is assumed to fly only on the vertical plane (xy plane).

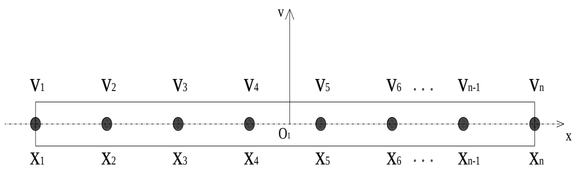


Fig.3 Nodes along rocket body axis

The displacements of each point in element i represented by:

$$v = \left[N_i(x)\right]\left\{d\right\}_i$$

where : $\left[N_i(x)\right]$ is the matrix of shape functions , $\left\{d\right\}_i$ is the nodal displacement vector of element i .

$$\left[N_i(x)\right] = \left[1 \quad x \quad x^2 \quad x^3\right]\begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 \\ 0 & 1 & 2x_i & 3x_i^2 \\ 1 & x_{i+1} & x_{i+1}^2 & x_{i+1}^3 \\ 0 & 1 & 2x_{i+1} & 3x_{i+1}^2 \end{bmatrix}^{-1}$$

$$\left\{d\right\}_i = \left[v_i \quad \frac{\partial v_i}{\partial x} \quad v_{i+1} \quad \frac{\partial v_{i+1}}{\partial x}\right]^T$$

Modelling the rocket as a free-free elastic beam, the boundary conditions of zero bending moment and shear force at the two ends must be satisfied.

Considering the boundary conditions, we can obtain the kinetic and elastic potential energy of the rocket, the kinetic energy of the jet flow, and generalized forces through the nodal displacements as

$$\begin{aligned} T_{rocket} &= \frac{1}{2}m(t)\left(\dot{X}_G^2 + \dot{Y}_G^2\right) + \frac{1}{2}J_0\dot{\varphi}^2 \\ &+ \frac{1}{2}\dot{\varphi}^2\left\{d_{RG}\right\}^T\left[N_3\right]^T\left[J_{\sum}\right]\left[N_3\right]\left\{d_{RG}\right\} \\ &- \mathfrak{M}(t)\dot{\varphi}\left(\dot{X}_G\sin\varphi - \dot{Y}_G\cos\varphi\right) + \dot{\varphi}\left[H_{\sum}\right]\left[N_3\right]\left\{\dot{d}_{RG}\right\} \\ &- \dot{\varphi}\left(\dot{X}_G\cos\varphi + \dot{Y}_G\sin\varphi\right)\left[R_{\sum}\right]\left[N_3\right]\left\{d_{RG}\right\} \\ &+ \frac{1}{2}\left\{\dot{d}_{RG}\right\}^T\left[N_3\right]^T\left[J_{\sum}\right]\left[N_3\right]\left\{\dot{d}_{RG}\right\} \\ &- \left(\dot{X}_G\sin\varphi - \dot{Y}_G\cos\varphi\right)\left[R_{\sum}\right]\left[N_3\right]\left\{\dot{d}_{RG}\right\} \end{aligned}$$

$$T_{thrust} = \frac{\Delta m}{2}\left(\begin{aligned} &\dot{X}_C^2 + \dot{Y}_C^2 + x_P^2\dot{\varphi}^2 + U^2 \\ &+\dot{\varphi}^2\left\{d_{RG}\right\}^T\left[N_3\right]^T\left[Z\right]^T\left[Z\right]\left[N_3\right]\left\{d_{RG}\right\} \\ &+\left\{d_{RG}\right\}^T\left[N_3\right]^T\left[W\right]^T\left[W\right]\left[N_3\right]\left\{d_{RG}\right\}U^2 \\ &-2\left(\dot{X}_C\sin\varphi - \dot{Y}_C\cos\varphi\right)x_P\dot{\varphi} \\ &+2\left(\dot{X}_C\sin\varphi - \dot{Y}_C\cos\varphi\right)U\left[W\right]\left[N_3\right]\left\{d_{RG}\right\} \\ &+2\left(\dot{X}_C\cos\varphi + \dot{Y}_C\sin\varphi\right)\left(-\left[Z\right]\left[N_3\right]\left\{d_{RG}\right\}\dot{\varphi} - U\right) \\ &+2\dot{\varphi}U\left(\left[Z\right]\left[N_3\right]\left\{d_{RG}\right\} - x_P\left[W\right]\left[N_3\right]\left\{d_{RG}\right\}\right) \end{aligned}\right)$$

$$U_e = \frac{1}{2}\left\{d_{RG}\right\}^T\left[G_{RG}\right]\left\{d_{RG}\right\}$$

$$Q_{q_i} = \sum \vec{F}\frac{\partial \vec{r}}{\partial q_i} = \sum\left[F_x \quad F_y \quad F_z\right]\frac{\partial}{\partial q_i}\left(\begin{aligned} &\left[X_G \ Y_G \ Z_G\right]^T \\ &+\left[A\right]^T\left[x \ y \ z\right]^T \\ &+\left[A\right]^T\left[N_i(x)\right]\left\{d_i\right\} \end{aligned}\right)$$

where:

$\left[N_3\right]$: Matrix used to simplify the vector of total nodal displacements.

$$\left[W\right] = \underbrace{\left[0 \quad 1 \quad 0 \quad ... \quad 0\right]}_{2n}; \ \left[Z\right] = \underbrace{\left[1 \quad 0 \quad ... \quad 0\right]}_{2n}$$

From (1):

$$\frac{d}{d\tau}\frac{\partial T_{rocket}}{\partial \dot{q}_i} + \frac{d}{d\tau}\frac{\partial T_{thrust}}{\partial \dot{q}_i} - \frac{\partial T_{rocket}}{\partial q_i} - \frac{\partial T_{thrust}}{\partial q_i} + \frac{\partial U_e}{\partial q_i} = Q_{q_i}$$

Then:

$$\lim_{\tau\to 0}\begin{pmatrix} \dfrac{d}{d\tau}\dfrac{\partial T_{rocket}\left(t_k+\tau\right)}{\partial \dot{q}_i}+\dfrac{d}{d\tau}\dfrac{\partial T_{thrust}\left(t_k+\tau\right)}{\partial \dot{q}_i} \\ -\dfrac{\partial T_{rocket}\left(t_k+\tau\right)}{\partial q_i}-\dfrac{\partial T_{thrust}\left(t_k+\tau\right)}{\partial q_i} \\ +\dfrac{\partial U_e\left(t_k+\tau\right)}{\partial q_i} \end{pmatrix} \quad (2)$$
$$=\lim_{\tau\to 0} Q_{q_i}\left(t_k+\tau\right)$$

Let Δm the mass lost during $t_k \to t_k+\tau$, ζ the mass flow rate, and U the jet flow speed.

Subtitute:
$$\lim_{\tau\to 0}(m(t_k+\tau))=m(t_k);\ \lim_{\tau\to 0}\Delta m=0;$$
$$-\lim_{\tau\to 0}\frac{dm(\mathrm{t}_k+\tau)}{d\tau}=\lim_{\tau\to 0}\frac{d\left(\Delta m\right)}{d\tau}=\zeta;$$

$F_{thrust}=k_{thrust}\zeta U$ into Eq. (2) and let $Q_{\{d_{RG}\}}$ denote the generalized force vector of the whole structure corresponding to the generalized coordinate vector $\{d_{RG}\}$, we derive the equation system of motions of the flexible rocket in the vertical plane as follows:

$$\begin{cases}
\begin{aligned}
& m\ddot{X}_G-\left(\mu+\zeta x_P\right)\dot{\varphi}\sin\varphi-\mathfrak{M}(t)\left(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi\right) \\
& =F_X+F_{thrust}\cos\varphi+\left(\ddot{\varphi}\cos\varphi-\sin\varphi\dot{\varphi}^2\right)\left[R_{RG}\right]\left\{d_{RG}\right\} \\
& +\cos\varphi\dot{\varphi}\left(\left[\dot{R}_{RG}\right]+\zeta\left[Z_{RG}\right]\right)\left\{d_{RG}\right\} \\
& +2\cos\varphi\dot{\varphi}\left(\left[R_{RG}\right]\right)\left\{\dot{d}_{RG}\right\}+\sin\varphi\left[\dot{R}_{RG}\right]\left\{\dot{d}_{RG}\right\} \\
& +\sin\varphi\left[R_{RG}\right]\left\{\ddot{d}_{RG}\right\}-F_{thrust}\sin\varphi\left[\mathrm{W}_{RG}\right]\left\{d_{RG}\right\} \\[2ex]
& m\ddot{Y}_G+\left(\mu+\zeta x_P\right)\dot{\varphi}\cos\varphi+\mathfrak{M}(t)\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right) \\
& =F_Y+F_{thrust}\sin\varphi+\left(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi\right)\left[R_{RG}\right]\left\{d_{RG}\right\} \\
& +\dot{\varphi}\sin\varphi\left(\left[\dot{R}_{RG}\right]+\zeta\left[Z_{RG}\right]\right)\left\{d_{RG}\right\} \\
& +2\dot{\varphi}\sin\varphi\left[R_{RG}\right]\left\{\dot{d}_{RG}\right\}-\cos\varphi\left[\dot{R}_{RG}\right]\left\{\dot{d}_{RG}\right\} \\
& -\cos\varphi\left[R_{RG}\right]\left\{\ddot{d}_{RG}\right\}+F_{thrust}\cos\varphi\left[W_{RG}\right]\left\{d_{RG}\right\} \\[2ex]
& J_0\ddot{\varphi}+\left(\dot{J}_0+\zeta x_P^2\right)\dot{\varphi}-\left(\mu+\zeta x_P\right)\left(\dot{X}_G\sin\varphi-\dot{Y}_G\cos\varphi\right) \\
& -\mathfrak{M}\left(t\right)\left(\ddot{X}_G\sin\varphi-\ddot{Y}_G\cos\varphi\right)=Q_\varphi-\ddot{\varphi}\left\{d_{RG}\right\}^T\left[J_{RG}\right]\left\{d_{RG}\right\} \\
& -2\dot{\varphi}\left\{\dot{d}_{RG}\right\}^T\left[J_{RG}\right]\left\{d_{RG}\right\}-\left[\dot{H}_{RG}\right]\left\{\dot{d}_{RG}\right\} \\
& -\dot{\varphi}\left\{d_{RG}\right\}^T\left(\left[\dot{J}_{RG}\right]+\zeta\left[Z_{RG}\right]^T\left[Z_{RG}\right]\right)\left\{d_{RG}\right\} \\
& +\left(\ddot{X}_G\cos\varphi+\ddot{Y}_G\sin\varphi\right)\left[R_{RG}\right]\left\{d_{RG}\right\} \\
& -\left[H_{RG}\right]\left\{\ddot{d}_{RG}\right\}+F_{thrust}\left(x_P\left[W_{RG}\right]-\left[Z_{RG}\right]\right)\left\{d_{RG}\right\} \\
& +\left(\dot{X}_G\cos\varphi+\dot{Y}_G\sin\varphi\right)\left(\left[\dot{R}_{RG}\right]+\zeta\left[Z_{RG}\right]\right)\left\{d_{RG}\right\} \\[2ex]
& \left[J_{RG}\right]\left\{\ddot{d}_{RG}\right\}+\left[\dot{J}_{RG}\right]\left\{\dot{d}_{RG}\right\} \\
& +\left(\left[G_{RG}\right]-\dot{\varphi}^2\left[J_{RG}\right]\right)\left\{d_{RG}\right\} \\
& =Q_{\{d_{RG}\}}-\ddot{\varphi}\left[H_{RG}\right]^T+\left(\ddot{X}_G\sin\varphi-\ddot{Y}_G\cos\varphi\right)\left[R_{RG}\right]^T \\
& +\left(\dot{X}_G\sin\varphi-\dot{Y}_G\cos\varphi\right)\left[\dot{R}_{RG}\right]^T-\dot{\varphi}\left[\dot{H}_{RG}\right]^T
\end{aligned}
\end{cases} \quad (3)$$

IV. Numerical Simulation and Analysis

Performing a numerical simulation for a rocket model and launch at 20^0:

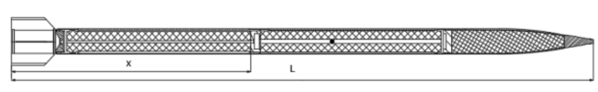


Fig.4 Model of rocket

Some basic parameters of the rocket are given as follows:

Mass of rocket:

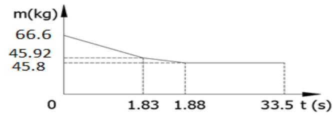


Fig.5 Mass of rocket depends on time

Thrust force:

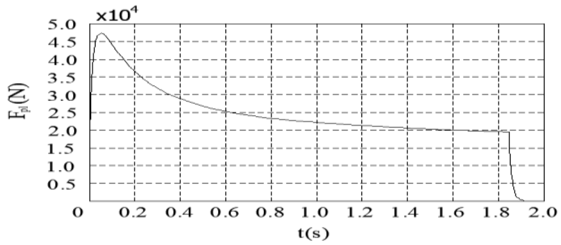


Fig.6 Thrust force depends on time

The distributions of mass and stiffness along the rocket body are given as:

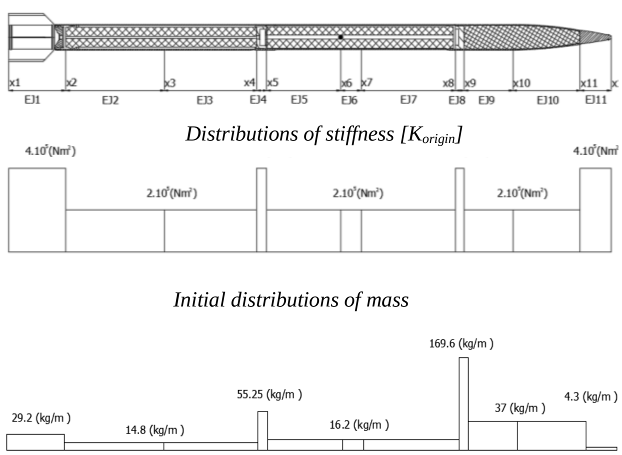


Fig.7 Distributions of mass and stiffness along the rocket

Aerodynamic forces: F_P

$$F_L = C_L(x,V,\alpha)\rho_{air}V^2 \quad ; \qquad F_D = C_D(x,V,\alpha)\rho_{air}V^2$$

C_L, C_D is aerodynamics coefficients, dependent on rocket's velocity, angle of attack, position on rocket- defined as follows:

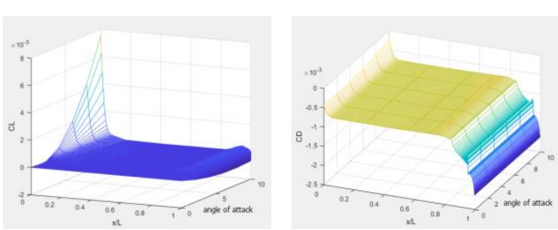


Fig.8 Coefficient of normal force C_L and drag force C_D at V=500m/s

4.1. Numerical Simulation Results

We perform the simulation in several cases, including:

- Rocket as variable mass - rigid body with the variation of center of mass.
- Rocket as variable mass- flexible body.
- Rocket as variable mass- flexible body with fixed center of mass.
- Transverse vibration of the rocket, which is modelled as a free-free Euler-Bernoulli beam.
  - ***Effects of stiffness***

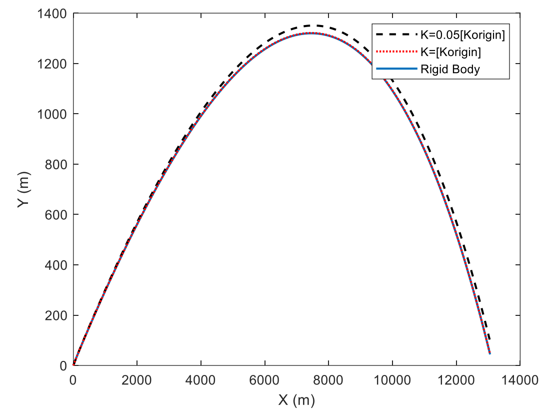


Fig.9 Trajectory of rocket

Tab. 1. Long range and altitude of rocket

Stiffness	0.05[K_{origin}]	[K_{origin}]	Rigid Body
X_{max}(m)	13239	13143	13140
Y_{max}(m)	1350.7	1321.1	1319.9

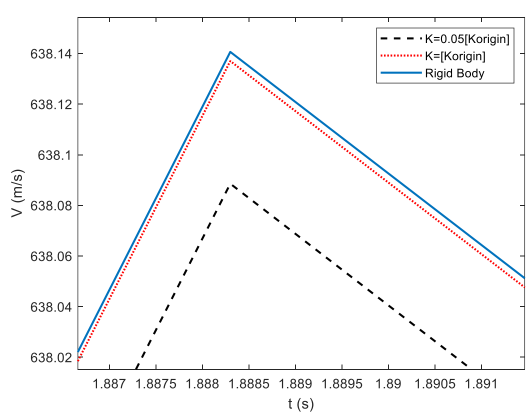


Fig.10 Velocity of center of mass

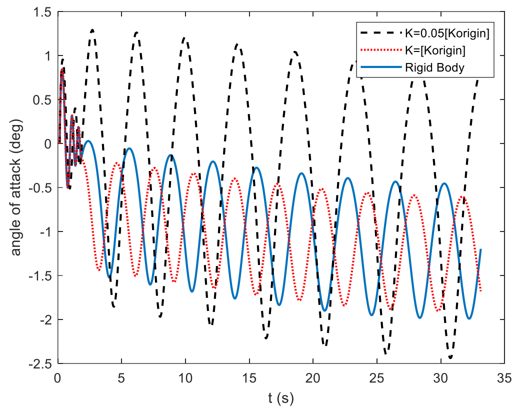


Fig.11 Variation of angle of attack

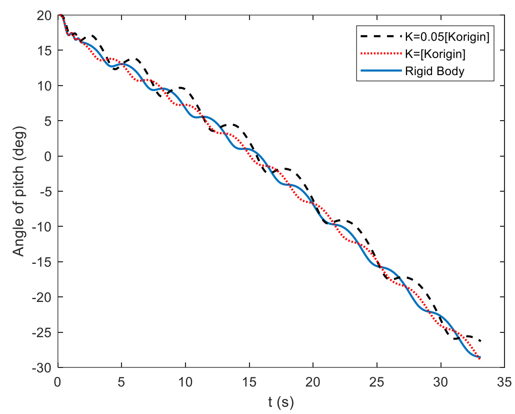
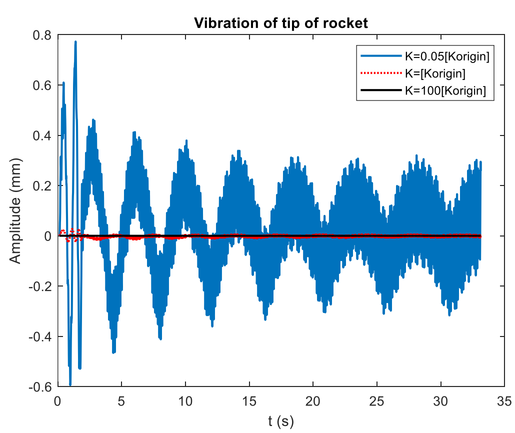


Fig.12 Variation of angle of pitch



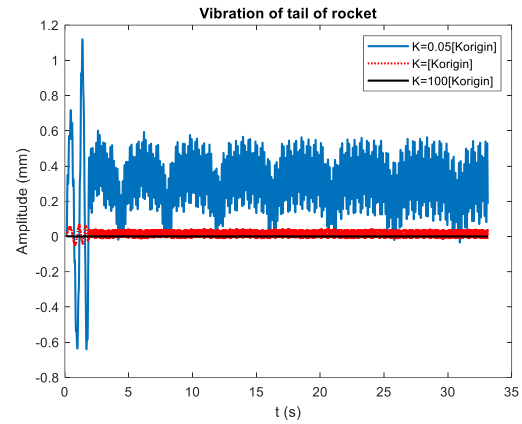
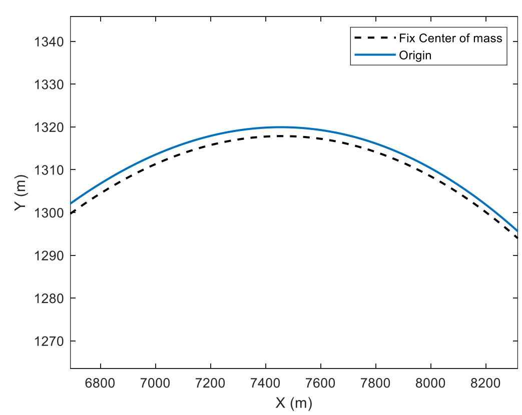


Fig.13 Vibrations of Rocket's Tail and Tip when $K=0.05[K_{origin}]$, $K=[K_{origin}]$, $K=100[K_{origin}]$

Tab. 2. Amplitude of bending vibration

Stiffness	$0.05[K_{origin}]$	$[K_{origin}]$	$100[K_{origin}]$
Amplitude-Tip (mm)	0.77	0.028	2.65×10^{-4}
Amplitude-Tail(mm)	1.12	0.07	6.85×10^{-4}

- ***Effects of the variation of center of mass***



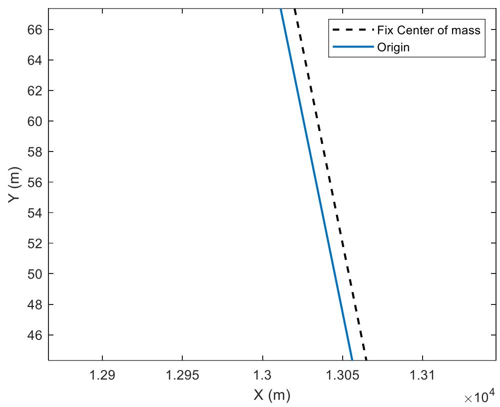


Fig. 14 Effect of the variation of center of mass on trajectory

Tab. 3. Long range and altitude of rocket

	Fix Center of mass	Origin
X_{max}(m)	13149	13143
Y_{max}(m)	1317.9	1321.1

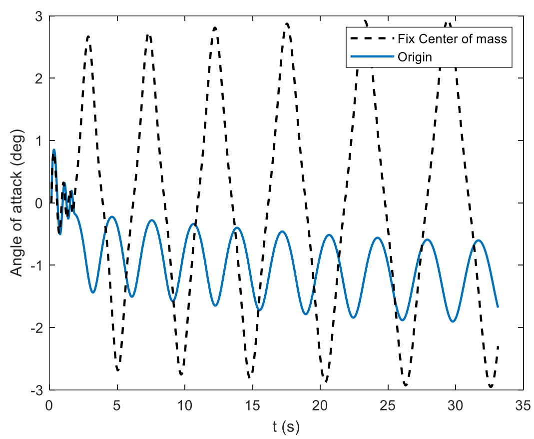


Fig.15 Effect of the variation of center of mass on the angle of attack

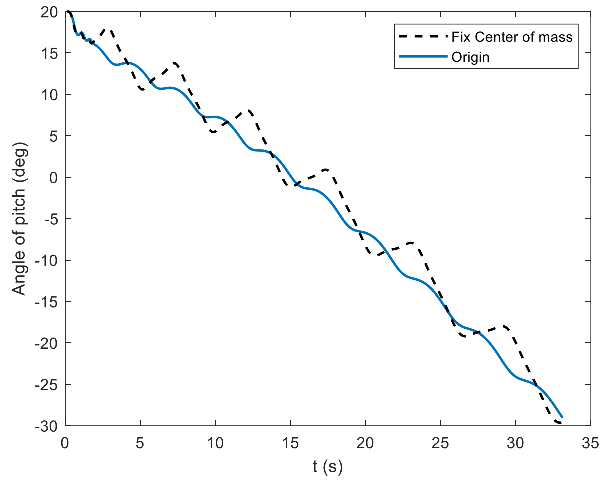


Fig.16 Effect of the variation of center of mass on the angle of pitch

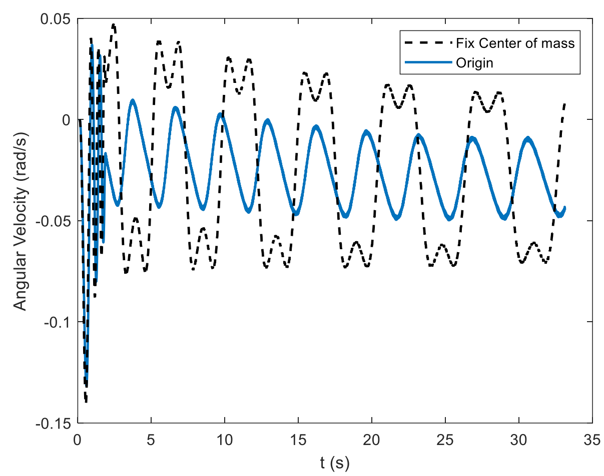


Fig.17 Effect of the variation of center of mass on the angular velocity (derivative of angle of pitch)

Conclusions:

This paper presents the effects the flexibility of the rocket body and the variation of the location of the mass center on the motions of a rocket model. The result has shown that the coupling between the bending vibration and the mass loss creates an equivalent force acting on the rocket in a direction normal to its body axis. When including the effect of body deformation, the effective thrust decreases, the range and the maximum altitude may increase despite the reduction of the initial velocity of the center of mass. When the variation of the location of the mass center is ignored, the maximum altitude becomes greater while the range declines. In this case, the amplitudes and the periods of the vibrations of the pitch angle and the angle of attack become larger; however, the solution is more linear

References

[1] L. Meirovitch (1970), "General Motion of a Variable Mass Flexible Rocket with Internal Flow", *Journal of Spacecraft and Rockets, Vol.7, No.2, pp.186-195.*

[2] A. Nanjangud, F.O. Eke (2012), "Lagrange's Equations for Rocket- Type Variable Mass Systems", *International Review of Aerospace Engineering (IREASE), Vol 5, No. 5, p256-260.*

[3] Shui-Lin Weng, Donald T. Greenwood (1992), "General Dynamical Equations of Motion for Elastic Body Systems", *Journal of Guidance, control, and dynamics, Vol. 15, No6, pp.1434-1441.*

[4] Meshchersky (1949), *Works on the Mechanics of Variable-Mass Bodies*, Moscow, 2nd edition, Moscow.

[5] Lei Wu, ChangchuanXie, ChaoYang (2012), "Aeroelastic Stability of a Slender Missile with Constant Thrust", *Procedia Engineering, Elselvier, pp. 128-135.*

[6] Tianfu Xu, Jili Rong, Dalin Xiang, Chenglong Pan, Xinzhe Yin (2016), "Dynamic Modeling and Stability Analysis of a Flexible Spinning Missile under Thrust", *International Journal of Mechanical Sciences, Vol 119, pp. 144-154.*

[7] Ryan C. Kitson, Carlos E. S. Cesnik, (2016) "Aeroelastic Modeling and Simulation of High-Speed Flexible Vehicles", *15th Dynamics Specialists Conference, San Diego, California, USA.*

[8] Weihua Su, Cecilia K. King, Scott R. Clark, Edwin D. Griffin, Jeffrey D. Suhey, Michael G. Wolf (2017), "Dynamic Beam Solutions for Real-Time Simulation and Control Development of Flexible Rockets", *Journal of Spacecraft and Rockets, Vol. 54, No. 2, pp. 403-416.*

[9] Chernet Tuge, Natinael Gezahegn (2017), "Stability Analysis of Dynamics of Variable Mass System", *Asian Journal of Applied Science and Technology, pp. 305-313.*

[10] Puneet Sharma, Ashish Gupta, Anuj Kumar Jain, Vikas Rastogi (2017), "Dynamic Modelling and Validation of Continuous Beam with free-free boundary conditions", *International Advanced Research Journalin Science, Engineering and Technology, Vol 4, Issue 3, pp. 47-53.*

[11] F. Mingireanu, L. Georgescu, G. Murariu, I. Mocanu (2014), "Trajectory modeling of GRAD rocket with low-cost terminal guidance upgrade coupled to range increase through step-like thrust-curves", *Romanian journal of physics, Vol. 59, pp.369-381, Bucharest.*

[12] M. V. Dileep, Kamath Surekha, Nair Vishnu (2016), "Ascent Phase Trajectory Optimization of Launch Vehicle Using Theta-Particle Swarm Optimization with Different Thrust Scenarios", *International Review of Aerospace Engineering (IREASE), Vol.9 (6), pp.200-207*

[13] Minjiao Li, Xiaoting Rui, Laith K. Abbas (2015), "Elastic Dynamic Effects on the Trajectory of a Flexible Launch Vehicle", *Journal of Spacecraft and Rockets,* pp. Vol 52, No. 6, pp. 1-17.

[14] M. V. Dileepa, Surekha Kamatha, Vishnu G. Nairb (2015), "Particle Swarm Optimization Applied to Ascent Phase Launch Vehicle Trajectory Optimization Problem", *Procedia Computer Science 54, pp.516 – 522*

,

Authors' information

[1]Faculty of Aerospace, Le Quy Don University, Ha Noi, Viet Nam

Dam Viet Phuong has a MSc in Applied Mechanics from Le Quy Don University. He is currently a doctoral student in Le Quy Don University, Ha Noi, Viet Nam. His research focus is on the dynamics of flexible- variable mass systems.

Vu Quoc Tru holds a Ph.D. in Applied Mechanics from Le Quy Don University, Ha Noi, Viet Nam. He is currently Associate Professor of Applied Mechanics at Le Quy Don University. His research interests are in the general areas of dynamics and control of aerospace vehicles.

Nguyen Anh Tuan holds a Ph.D. in Aerospace Engineering from KAIST, Daejeon, Korea. His research focus is aerodynamics, multi-body dynamics and bio-inspired micro air vehicles.