A Hilbert Space Approach to Fractional Difference Equations

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Abstract We formulate fractional difference equations of Riemann–Liouville and Caputo type in a functional analytical framework. Main results are existence of solutions on Hilbert space-valued weighted sequence spaces and a condition for stability of linear fractional difference equations. Using a functional calculus, we relate the fractional sum to fractional powers of the operator $1 - \tau^{-1}$ with the right shift τ^{-1} on weighted sequence spaces. Causality of the solution operator plays a crucial role for the description of initial value problems.

Keywords Computational geometry · Graph theory · Hamilton cycles

1 Introduction

1.1 Notation

We write $\mathbb{R}_{>0} := \{x \in \mathbb{R};\, x > 0\}$ and for $\mu, \varrho \in \mathbb{R}$ we define for the comprehension $\mathbb{C}_{|\cdot|<\mu} := \{z \in \mathbb{C};\, |z| < \mu\}$ and $\mathbb{C}_{|\cdot|>\mu}$, $\mathbb{C}_{|\cdot|\leq\mu}$, $\mathbb{C}_{|\cdot|\geq\mu}$ and $\mathbb{C}_{\mu\geq|\cdot|\geq\varrho}$ are defined similarly. For $\varrho > 0$ we denote the complex ball with radius ϱ centered at 0 by

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$B(0,\varrho) := \{z \in \mathbb{C};\, |z| < \varrho\}$ and the circle with radius ϱ centered at 0 by $S_\varrho := \partial B(0,\varrho)$. We set $\mathbb{N} := \mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$. For sets X, Y we denote the set of functions from Y to X by $X^Y := \{f : Y \to X\}$ and for $f \in X^Y$ we write $\operatorname{ran} f := \{f(y) \in X;\, y \in Y\}$ for the range of f. In particular, for any $M \subseteq \mathbb{Z}$, X^M is the space of sequences in X on M and for $u \in X^M$, $n \in M$ we write $u_n := u(n)$. The identity mapping on a vector space V is denoted by 1. For a sequence $u \in V^{\mathbb{Z}}$ we denote $\operatorname{spt} u := \{n \in \mathbb{Z};\, u_n \neq 0\}$. If V is a normed vector space we denote with $\|\cdot\|_V$ the norm on V.

We recall the binomial coefficient and the binomial series including some of their properties. Proofs of the following propositions can be found in [11, 14].

Proposition 1 (Binomial coefficient [11, pp. 164–165], [14, p. 34]) *For $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 1}$ the binomial coefficient is defined by*

$$\binom{\alpha}{0} := 1, \qquad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

For $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$(-1)^n \binom{\alpha}{n} = \binom{-\alpha+n-1}{n} \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} = (-1)^n \binom{\alpha-1}{n}.$$

Proposition 2 (Binomial series [14, pp. 65, 73]) *Let $\alpha \in \mathbb{C}$. The binomial power series is defined by*

$$(1+z)^\alpha := \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

The series converges absolutely in $B(0,1)$. In particular, the mapping $\mathbb{C}_{|\cdot|>1} \to \mathbb{C}$, $z \mapsto (1-z^{-1})^\alpha$ is holomorphic. For each $\alpha, \beta \in \mathbb{C}$ we have $(1+z)^\alpha(1+z)^\beta = (1+z)^{\alpha+\beta}$.

Binomial coefficients can be expressed with the gamma function.

Lemma 3 (Falling factorial [11, p. 164]) *With the falling factorial*

$$(x)^{(n)} := \frac{\Gamma(x+1)}{\Gamma(x-n+1)}, \qquad x \in \mathbb{C} \setminus \mathbb{Z}, \qquad n \in \mathbb{N},$$

we have for each $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $n \in \mathbb{N}$

$$(-1)^n \binom{\alpha}{n} = \binom{-\alpha+n-1}{n} = \frac{1}{\Gamma(-\alpha)} (n-(1+\alpha))^{(-(1+\alpha))}. \tag{1}$$

Lemma 4 *Let $\alpha \in (0,1)$ and $\varrho > 1$. Then we have for each $z \in S_\varrho$*

$$(1-\varrho^{-1})^\alpha \leq \left|(1-z^{-1})^\alpha\right|.$$

Proof Let $z \in S_\varrho$. For every $n \in \mathbb{Z}_{\geq 1}$ we observe that $(-1)^n\binom{\alpha}{n} < 0$ and therefore $(-1)^n\binom{\alpha}{n}z^{-n} = -\left|\binom{\alpha}{n}\right| z^{-n}$. We show by induction that for every $n \in \mathbb{N}$

$$\left|\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}z^{-k}\right| \geq \left|\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\varrho^{-k}\right|$$

and when letting n tend to infinity the inequality follows. The induction basis is trivial. For the induction step for $n \in \mathbb{N}$ we use the lower triangle inequality to obtain

$$\begin{aligned}\left|\sum_{k=0}^{n+1}(-1)^k\binom{\alpha}{k}z^{-k}\right| &= \left|\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}z^{-k} + (-1)^{n+1}\binom{\alpha}{n+1}z^{-(n+1)}\right| \\ &\geq \left|\left|\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}z^{-k}\right| - \left|(-1)^{n+1}\binom{\alpha}{n+1}z^{-(n+1)}\right|\right| \\ &= \left|\left|\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}z^{-k}\right| + (-1)^{n+1}\binom{\alpha}{n+1}\varrho^{-(n+1)}\right| \\ &\geq \left|\sum_{k=0}^{n+1}(-1)^k\binom{\alpha}{k}\varrho^{-k}\right|.\end{aligned}$$

1.2 Fractional Difference Operators

Let V be a real or complex vector space.

The fractional sum can be motivated by the iterated sum formula and is also related to iterating the backward difference operator (see e.g. [15]). For $\alpha \in \mathbb{R}_{>0}$ the fractional sum $\nabla^{-\alpha}\colon V^{\mathbb{N}} \to V^{\mathbb{N}}$ is defined by (cf. [3, p. 3])

$$(\nabla^{-\alpha}u)_n = \sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}u_k = \sum_{k=0}^{n}(-1)^k\binom{-\alpha}{k}u_{n-k}. \tag{2}$$

There is also a definition motivated by iterating the forward difference operator which is studied at least since [15] and can be found in [3, p. 3] as well. Note that $(\nabla^{-\alpha}u)_n$ in general depends on $u_0, \ldots, u_n$.

The approach to defining the fractional differential operators in the Riemann–Liouville and Caputo sense (cf. [8]) was applied mutatis mutandis to difference operators (see e.g. [1] and the references therein). Recall that for $\varDelta : V^{\mathbb{N}} \to V^{\mathbb{N}}, u \mapsto (u_{n+1} - u_n)_{\mathbb{N}}$ we have $(\varDelta u)_n = (\nabla u)_{n+1}$ for $n \in \mathbb{N}$. For $\alpha \in (0, 1)$ the Riemann–Liouville forward fractional difference operator is defined by (cf. [16, p. 3813])

$$\varDelta^\alpha : V^{\mathbb{N}} \to V^{\mathbb{N}}, \qquad u \mapsto \varDelta\nabla^{-(1-\alpha)}u. \tag{3}$$

The Caputo forward fractional difference operator is defined by (cf. [16, p. 3813])

$$\Delta_C^\alpha : V^{\mathbb{N}} \to V^{\mathbb{N}}, \qquad u \mapsto \nabla^{-(1-\alpha)}\Delta u. \tag{4}$$

In this paper we study sequences in a Hilbert space $V = H$ on $\mathbb{Z}$ and define a fractional difference sum operator using the binomial series and a functional calculus which is not purely algebraic as in the case of $\nabla^{-\alpha}$. The connection between operators defined on $H^{\mathbb{Z}}$ with those defined on $H^{\mathbb{N}}$ will be causality and we analyze how the Riemann–Liouville and the Caputo operator fit into the calculus developed for sequences in $H^{\mathbb{Z}}$. An important step for the development of the discrete, functional analytic framework which is introduced in this paper has been done in the continuous case for fractional derivatives in [19]. Lastly we study the asymptotic stability of the zero solution of a linear fractional difference equation with the Riemann–Liouville and the Caputo forward difference operator. The interest in the study of linear problems in the context of stability analysis stems from Lyapunov's first method, which has been analyzed in [6] for fractional differential equations. The results regarding asymptotic stability will be in terms of the Matignon criterion (cf. [18]), however, for bounded operators on a Hilbert space H and will be compared to those in [1, 5]. A useful tool when analyzing the asymptotic stability of linear problems is the $\mathscr{Z}$ transform which is also used in [1, 5] but which is studied here for sequences in $H^{\mathbb{Z}}$. Asymptotic stability has also been studied using the Riemann–Liouville and the Caputo backward difference operators in [4, 16].

2 Exponentially Weighted ℓ_p Spaces

We denote by $(H, \|\cdot\|_H)$ a complex and separable Hilbert space. The scalar product $\langle\cdot,\cdot\rangle_H$ on H shall be conjugate linear in the first argument and linear in the second argument. We recall several of the concepts of weighted $\ell_{p,\varrho}(\mathbb{Z}; H)$ spaces and the $\mathscr{Z}$ transform (see also [13]).

Lemma 5 (Exponentially weighted ℓ_p spaces [13]) *Let* $1 \le p < \infty$, $\varrho > 0$. *Define*

$$\ell_{p,\varrho}(\mathbb{Z}; H) := \left\{ x \in H^{\mathbb{Z}};\ \sum_{k\in\mathbb{Z}} \|x_k\|_H^p \varrho^{-pk} < \infty \right\},$$

$$\ell_{\infty,\varrho}(\mathbb{Z}; H) := \left\{ x \in H^{\mathbb{Z}};\ \sup_{k\in\mathbb{Z}} \|x_k\|_H \varrho^{-k} < \infty \right\}.$$

Then $\ell_{p,\varrho}(\mathbb{Z}; H)$ *and* $\ell_{\infty,\varrho}(\mathbb{Z}; H)$ *are Banach spaces with norms*

$$\|x\|_{\ell_{p,\varrho}(\mathbb{Z};H)} := \left(\sum_{k\in\mathbb{Z}} \|x_k\|_H^p \varrho^{-pk} \right)^{\frac{1}{p}} \qquad (x \in \ell_{p,\varrho}(\mathbb{Z}; H))$$

and

$$\|x\|_{\ell_{\infty,\varrho}(\mathbb{Z};H)} := \sup_{k\in\mathbb{Z}} \|x_k\|_H\, \varrho^{-k} \qquad (x \in \ell_{\infty,\varrho}(\mathbb{Z};H)),$$

respectively. Moreover, $\ell_{2,\varrho}(\mathbb{Z};H)$ *is a Hilbert space with the inner product*

$$\langle x, y\rangle_{\ell_{2,\varrho}(\mathbb{Z};H)} := \sum_{k\in\mathbb{Z}} \langle x_k, y_k\rangle_H\, \varrho^{-2k} \qquad (x, y \in \ell_{2,\varrho}(\mathbb{Z};H)).$$

We write $\ell_p(\mathbb{Z};H) := \ell_{p,1}(\mathbb{Z};H)$ *for* $1 \le p \le \infty$.

Proposition 6 (One sided weighted sequence spaces [13]) *For* $1 \le p \le \infty$, $a \in \mathbb{Z}$ *and* $\varrho > 0$ *we define*

$$\ell_{p,\varrho}(\mathbb{Z}_{\ge a};H) := \left\{ x|_{\mathbb{Z}_{\ge a}};\, x \in \ell_{p,\varrho}(\mathbb{Z};H)\right\}.$$

And for $1 \le p \le \infty$, $\varrho > 0$, $a \in \mathbb{Z}$ *and for* $x \in H^{\mathbb{Z}_{\ge a}}$, *we define* $\iota x \in H^{\mathbb{Z}}$ *by*

$$(\iota x)_k := \begin{cases} 0 & \text{if } k < a, \\ x_k & \text{if } k \ge a. \end{cases}$$

Then $\ell_{p,\varrho}(\mathbb{Z}_{\ge a};H)$ *is a Banach space with norm* $\|\cdot\|_{\ell_{p,\varrho}(\mathbb{Z}_{\ge a};H)} := \|\iota\cdot\|_{\ell_{p,\varrho}(\mathbb{Z};H)}$, *and*

$$\iota\colon \ell_{p,\varrho}(\mathbb{Z}_{\ge a};H) \hookrightarrow \ell_{p,\varrho}(\mathbb{Z};H)$$

is an isometric embedding. Write $\ell_{p,\varrho}(\mathbb{Z}_{\ge a};H) \subseteq \ell_{p,\varrho}(\mathbb{Z};H)$.
For $1 \le p < q \le \infty$, $\varrho, \varepsilon > 0$, $a \in \mathbb{Z}$ *we have*

$$\begin{aligned} &(a) \quad \ell_{p,\varrho}(\mathbb{Z}_{\ge a};H) \subsetneq \ell_{q,\varrho}(\mathbb{Z}_{\ge a};H), \\ &(b) \quad \ell_{q,\varrho}(\mathbb{Z}_{\ge a};H) \subsetneq \ell_{p,\varrho+\varepsilon}(\mathbb{Z}_{\ge a};H). \end{aligned}$$

Definition 7 For $x \in H$ and $n \in \mathbb{Z}$ we define $\delta_n x \in H^{\mathbb{Z}}$ by

$$(\delta_n x)_m := \begin{cases} x, & \text{if } m = n, \\ 0, & \text{if } m \ne n, \end{cases}$$

and $\chi_{\mathbb{Z}_{\ge n}} x \in H^{\mathbb{Z}}$ by

$$(\chi_{\mathbb{Z}_{\ge n}} x)_m := \begin{cases} x, & \text{if } m \ge n, \\ 0, & \text{if } m < n. \end{cases}$$

Note that for $\varrho > 0$, $\delta_n x \in \ell_{p,\varrho}(\mathbb{Z};H)$ and for $\varrho > 1$, $\chi_{\mathbb{Z}_{\ge n}} x \in \ell_{p,\varrho}(\mathbb{Z};H)$.

Lemma 8 (Shift operator [13]) *Let* $1 \le p \le \infty$, $\varrho > 0$. *Then*

$$\begin{aligned}\tau\colon \ell_{p,\varrho}(\mathbb{Z}; H) &\to \ell_{p,\varrho}(\mathbb{Z}; H),\\ (x_k)_{k\in\mathbb{Z}} &\mapsto (x_{k+1})_{k\in\mathbb{Z}},\end{aligned}$$

is linear, bounded, invertible, and

$$\left\|\tau^n\right\|_{L(\ell_{p,\varrho}(\mathbb{Z};H))} = \varrho^n \qquad (n \in \mathbb{Z}).$$

3 $\mathscr{Z}$ Transform

Lemma 9 (L_2 space on circle and orthonormal basis [13]) *Let $\varrho > 0$. Define*

$$L_2(S_\varrho; H) := \left\{ f\colon S_\varrho \to H;\ \int_{S_\varrho} \|f(z)\|_H^2 \frac{\mathrm{d}z}{|z|} < \infty \right\}.$$

Then $L_2(S_\varrho; H)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L_2(S_\varrho;H)} := \frac{1}{2\pi}\int_{S_\varrho} \left\langle f(z), g(z)\right\rangle_H \frac{\mathrm{d}z}{|z|} \qquad (f, g \in L_2(S_\varrho; H)).$$

Moreover, let $(\psi_n)_{n\in\mathbb{Z}}$ be an orthonormal basis in H. Then $(p_{k,n})_{k,n\in\mathbb{Z}}$ with

$$p_{k,n}(z) := \varrho^k z^{-k}\psi_n \qquad (z \in S_\varrho)$$

is an orthonormal basis in $L_2(S_\varrho; H)$.

Theorem 10 ($\mathscr{Z}$ transform [13]) *Let $\varrho > 0$. The operator*

$$\begin{aligned}\mathscr{Z}_\varrho\colon \ell_{2,\varrho}(\mathbb{Z}; H) &\to L_2(S_\varrho; H),\\ x &\mapsto \Big(z \mapsto \sum_{k\in\mathbb{Z}} \langle\psi_n, \varrho^{-k}x_k\rangle_H\, p_{k,n}(z)\Big)\end{aligned}$$

is well-defined and unitary. For $x \in \ell_{1,\varrho}(\mathbb{Z}; H) \subseteq \ell_{2,\varrho}(\mathbb{Z}; H)$ we have

$$\mathscr{Z}_\varrho(x) = \left(z \mapsto \sum_{k\in\mathbb{Z}} x_k z^{-k}\right).$$

Remark 11 ($\mathscr{Z}$ transform of $x\in\ell_{2,\varrho}(\mathbb{Z}; H) \setminus \ell_{1,\varrho}(\mathbb{Z}; H)$) Let $\varrho > 0$, $x\in\ell_{2,\varrho}(\mathbb{Z}; H) \setminus \ell_{1,\varrho}(\mathbb{Z}; H)$. Then

$$\sum_{k\in\mathbb{Z}} x_k z^{-k}$$

does not necessarily converge for all $z \in S_\varrho$. For example if $H = \mathbb{C}$, $x \in \ell_{2,\varrho}(\mathbb{Z}; H) \setminus \ell_{1,\varrho}(\mathbb{Z}; H)$ with $x_k := \frac{\varrho^k}{k}$ and $z = \varrho$.

Lemma 12 (Shift is unitarily equivalent to multiplication [13]) *Let* $\varrho > 0$. *Then*

$$\mathscr{Z}_\varrho \tau \mathscr{Z}_\varrho^* = \mathrm{m},$$

where m *is the multiplication-by-the-argument operator acting in* $L_2(S_\varrho; H)$*, i.e.,*

$$\begin{aligned}\mathrm{m}\colon L_2(S_\varrho; H) &\to L_2(S_\varrho; H),\\ f &\mapsto (z \mapsto z f(z)).\end{aligned}$$

Next, we present a Paley–Wiener type result for the $\mathscr{Z}$ transform.

Lemma 13 (Characterization of positive support [13]) *Let* $\varrho > 0$, $x \in \ell_{2,\varrho}(\mathbb{Z}; H)$. *Then the following statements are equivalent:*

(i) $\operatorname{spt} x \subseteq \mathbb{N}$,

(ii) $z \mapsto \sum_{k\in\mathbb{Z}} x_k z^{-k}$ *is analytic on* $\mathbb{C}_{|\cdot|>\varrho}$ *and*

$$\sup_{\mu>\varrho} \int_{S_\mu} \left\| \sum_{k\in\mathbb{Z}} x_k z^{-k} \right\|_H^2 \frac{\mathrm{d}z}{|z|} < \infty. \tag{5}$$

Definition 14 (*Causal linear operator*) We call a linear operator $B\colon \ell_{2,\varrho}(\mathbb{Z}; H) \to \ell_{2,\varrho}(\mathbb{Z}; H)$ *causal*, if for all $a \in \mathbb{Z}$, $f \in \ell_{2,\varrho}(\mathbb{Z}; H)$, we have

$$\operatorname{spt} f \subseteq \mathbb{Z}_{\geq a} \quad \Rightarrow \quad \operatorname{spt} Bf \subseteq \mathbb{Z}_{\geq a}.$$

Recall [12, VIII.3.6, p. 222] that for $A \in L(H)$ with spectrum $\sigma(A)$, the spectral radius

$$r(A) := \sup\{|z|;\, z \in \sigma(A)\}$$

of A satisfies

$$r(A) = \lim_{n\to\infty} \|A^n\|_{L(H)}^{1/n}.$$

Let $A \in L(H)$ and $\varrho > 0$. We denote the operators $\ell_{2,\varrho}(\mathbb{Z}, H) \to \ell_{2,\varrho}(\mathbb{Z}, H)$, $x \mapsto (Ax_k)$, and $L_2(S_\varrho, H) \to L_2(S_\varrho, H)$, $f \mapsto (z \mapsto Af(z))$, which have the same operator norm as A, again by A.

Proposition 15 (Convolution) *Let* $c \in \ell_{1,\varrho}(\mathbb{Z}; \mathbb{C})$ *and* $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$. *Then*

$$c * u := \left(\sum_{k=-\infty}^{\infty} c_k u_{n-k} \right)_{n\in\mathbb{Z}} \in \ell_{2,\varrho}(\mathbb{Z}; H).$$

We have Young's inequality

$$\|c * u\|_{\ell_{2,\varrho}(\mathbb{Z};H)} \leq \|c\|_{\ell_{1,\varrho}(\mathbb{Z};\mathbb{C})} \|u\|_{\ell_{2,\varrho}(\mathbb{Z};H)} .$$

Moreover,

$$\mathscr{Z}_\varrho(c * u) = \mathscr{Z}_\varrho c \mathscr{Z}_\varrho u.$$

Proof Let $n \in \mathbb{Z}$. With the Cauchy–Schwarz inequality we compute

$$\begin{aligned}\left(\sum_{k=-\infty}^{\infty} \|c_k u_{n-k}\|_H\right)^2 \varrho^{-2n} &= \left(\sum_{k=-\infty}^{\infty} |c_k|^{1/2}\varrho^{-k/2}|c_k|^{1/2}\varrho^{-k/2} \|u_{n-k}\|_H \varrho^{-(n-k)}\right)^2 \\ &\leq \|c\|_{\ell_{1,\varrho}(\mathbb{Z};\mathbb{C})} \left(\sum_{k=-\infty}^{\infty} |c_k|\varrho^{-k} \|u_{n-k}\|_H^2 \varrho^{-2(n-k)}\right).\end{aligned}$$

Therefore using Fubini's theorem

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \|(c * u)_n\|_H^2 \varrho^{-2n} &\leq \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \|c_k u_{n-k}\|_H\right)^2 \varrho^{-2n} \\ &\leq \|c\|_{\ell_{1,\varrho}(\mathbb{Z};\mathbb{C})} \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} |c_k|\varrho^{-k} \|u_{n-k}\|_H^2 \varrho^{-2(n-k)}\right) \\ &= \|c\|_{\ell_{1,\varrho}(\mathbb{Z};\mathbb{C})}^2 \|u\|_{\ell_{2,\varrho}(\mathbb{Z};H)}^2 .\end{aligned}$$

This shows Young's inequality. If additionally $u \in \ell_{1,\varrho}(\mathbb{Z}; H)$ then

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \|(c * u)_n\|_H \varrho^{-n} &\leq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \|c_k u_{n-k}\|_H \varrho^{-n} \\ &= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |c_k|\varrho^{-k} \|u_{n-k}\|_H \varrho^{-(n-k)} \\ &= \|c\|_{\ell_{1,\varrho}(\mathbb{Z};\mathbb{C})} \|u\|_{\ell_{1,\varrho}(\mathbb{Z};H)} ,\end{aligned}$$

i.e., $c * u \in \ell_{1,\varrho}(\mathbb{Z}; H) \cap \ell_{2,\varrho}(\mathbb{Z}; H)$ which simplifies the $\mathscr{Z}$ transform of $c * u$. Using Fubini's theorem, we compute for $u \in \ell_{1,\varrho}(\mathbb{Z}; H) \cap \ell_{2,\varrho}(\mathbb{Z}; H)$ and $z \in S_\varrho$

$$\begin{aligned}\mathscr{Z}_\varrho(c * u)(z) &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_k u_{n-k}\right) z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_k z^{-k} u_{n-k} z^{-(n-k)}\right) \\ &= \sum_{k=-\infty}^{\infty} c_k z^{-k} \left(\sum_{n=-\infty}^{\infty} u_{n-k} z^{-(n-k)}\right) = \mathscr{Z}_\varrho(c)\mathscr{Z}_\varrho(u).\end{aligned}$$

For $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ the formula follows by density of $\ell_{1,\varrho}(\mathbb{Z}; H) \cap \ell_{2,\varrho}(\mathbb{Z}; H) \subseteq \ell_{2,\varrho}(\mathbb{Z}; H)$.

Example 16 (The operator $(1-\tau^{-1})^{\alpha}$) Let $\varrho > 1$ and $\alpha \in \mathbb{C}$. For the operator $1 - \tau^{-1} : \ell_{2,\varrho}(\mathbb{Z}; H) \to \ell_{2,\varrho}(\mathbb{Z}; H)$, we compute

$$(1-\tau^{-1}) = \mathscr{Z}_{\varrho}^{*}(1-z^{-1})\mathscr{Z}_{\varrho}.$$

We have $|z^{-1}| < 1$ for all $z \in S_{\varrho}$ and therefore

$$(1-\tau^{-1})^{\alpha} := \mathscr{Z}_{\varrho}^{*}(1-z^{-1})^{\alpha}\mathscr{Z}_{\varrho} : \ell_{2,\varrho}(\mathbb{Z}; H) \to \ell_{2,\varrho}(\mathbb{Z}; H)$$

is well-defined. This is an application of the holomorphic functional calculus (cf. [10, pp. 13–18], [9, p. 601]).

We define $c \in \ell_{1,\varrho}(\mathbb{Z}; \mathbb{C})$ by

$$c_k := \begin{cases} (-1)^k \binom{\alpha}{k} & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Then

$$\mathscr{Z}_{\varrho}c = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} z^{-k} = (1-z^{-1})^{\alpha}.$$

Thus we compute for $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$

$$\mathscr{Z}_{\varrho}(c * u) = \mathscr{Z}_{\varrho}c\,\mathscr{Z}_{\varrho}u = (1-z^{-1})^{\alpha}\mathscr{Z}_{\varrho}u.$$

Thus for $\alpha \in \mathbb{C}$ and $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ we obtain

$$(1-\tau^{-1})^{\alpha}u = c * u = \left(\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}u_{n-k}\right)_{n\in\mathbb{Z}} = \left(\sum_{k=-\infty}^{n}(-1)^{n-k}\binom{\alpha}{n-k}u_k\right)_{n\in\mathbb{Z}},$$

i.e., $(1-\tau^{-1})^{\alpha}$ is a convolution operator and by Young's Theorem $(1-\tau^{-1})^{\alpha}$ is bounded and $\left\|(1-\tau^{-1})^{\alpha}\right\|_{L(\ell_{2,\varrho}(\mathbb{Z};H))} = \|c\|_{\ell_{1,\varrho}(\mathbb{Z};H)}$.

If $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ with $\operatorname{spt} u \subseteq \mathbb{N}$, we have

$$(1-\tau^{-1})^{\alpha}u = \left(\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}u_{n-k}\right)_{n\in\mathbb{Z}}.$$

Since τ commutes with $(1-\tau^{-1})^{\alpha}$, we deduce that $(1-\tau^{-1})^{\alpha}$ is causal.

On $\ell_{2,\varrho}(\mathbb{Z}; H)$ we compute for $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned}(1-\tau^{-1})^{\alpha}(1-\tau^{-1})^{\beta} &= \mathscr{Z}_{\varrho}^{*}(1-z^{-1})^{\alpha}\mathscr{Z}_{\varrho}\mathscr{Z}_{\varrho}^{*}(1-z^{-1})^{\beta}\mathscr{Z}_{\varrho} \\ &= \mathscr{Z}_{\varrho}^{*}(1-z^{-1})^{\alpha+\beta}\mathscr{Z}_{\varrho} = (1-\tau^{-1})^{\alpha+\beta}.\end{aligned}$$

In particular, for $\alpha \in \mathbb{C}$, $(1-\tau^{-1})^{\alpha}$ is invertible with inverse $(1-\tau^{-1})^{-\alpha}$.

4 Fractional Difference Equations on $\ell_{2,\varrho}(\mathbb{Z}; H)$

Fractional Operators

Let $\varrho > 1$ and $\alpha \in (0,1)$. We consider the operators (2), (3) and (4) defined on $V = H$. For comparing operators defined on spaces of sequences on $\mathbb{Z}$ with those defined for sequences on $\mathbb{N}$, we recall the embedding of $\ell_{2,\varrho}(\mathbb{N}; H)$ into $\ell_{2,\varrho}(\mathbb{Z}; H)$ by ι in Proposition 6. Moreover, we extend the operator Δ on $\mathbb{N}$ to $\mathbb{Z}$ by

$$\Delta : \ell_{2,\varrho}(\mathbb{Z}; H) \to \ell_{2,\varrho}(\mathbb{Z}; H), \qquad u \mapsto \chi_{\mathbb{N}}(\tau - 1)u = \chi_{\mathbb{N}}\tau(1-\tau^{-1})u.$$

Note that the left shift on $\mathbb{N}$ cuts of the first value of a sequence and embedded sequences have positive support. This is the reason for multiplying with $\chi_{\mathbb{N}}$ in the definition of Δ on $\ell_{2,\varrho}(\mathbb{Z}; H)$.

Let $v \in \ell_{2,\varrho}(\mathbb{N}; H)$ and set $u := \iota v \in \ell_{2,\varrho}(\mathbb{Z}; H)$. We compare the operator $(1-\tau^{-1})^{-\alpha}$ defined on $\ell_{2,\varrho}(\mathbb{Z}; H)$ and the fractional sum (2). We have $\operatorname{spt}\big((1-\tau^{-1})^{-\alpha}u\big) \subseteq \mathbb{N}$ and obtain

$$\iota\nabla^{-\alpha}v = (1-\tau^{-1})^{-\alpha}u.$$

Using definitions (3) and (4) of the Riemann–Liouville and Caputo difference operators, and the fact that $\Delta u = (\tau-1)(u - \chi_{\mathbb{N}}u_0) = \tau(1-\tau^{-1})(u-\chi_{\mathbb{N}}u_0)$, we compute

$$\begin{aligned}
&\Delta(1-\tau^{-1})^{-(1-\alpha)}u = \chi_{\mathbb{N}}\tau(1-\tau^{-1})^{\alpha}u = \tau(1-\tau^{-1})^{\alpha}u - \delta_{-1}u_0,\\
&(1-\tau^{-1})^{\alpha-1}\Delta u = (1-\tau^{-1})^{\alpha-1}\chi_{\mathbb{N}}\tau(1-\tau^{-1})u = \tau(1-\tau^{-1})^{\alpha}(u-\chi_{\mathbb{N}}u_0).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\iota\Delta^{\alpha}v &= \chi_{\mathbb{N}}\tau(1-\tau^{-1})^{\alpha}u,\\
\iota\Delta^{\alpha}_{C}v &= \tau(1-\tau^{-1})^{\alpha}(u-\chi_{\mathbb{N}}u_0).
\end{aligned}$$

In view of $\tau(1-\tau^{-1})^{\alpha}$, the Caputo and the Riemann–Liouville operators are equal whereby the Caputo operator regularizes u first. In particular, for $n \in \mathbb{N}$ by Proposition 1, we have $((1-\tau^{-1})^{\alpha}\chi_{\mathbb{N}}u_0)_n = \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}u_0 = \binom{-\alpha+n}{n}u_0$ and so

$$(\Delta^{\alpha}v)_n = (\Delta^{\alpha}_{C}v)_n + \binom{-\alpha+n+1}{n+1}u_0.$$

It is notable that the operator $(1-\tau^{-1})^{\alpha}$ when $H=\mathbb{C}$ maps real valued sequences to real valued sequences. We could have started with a real Hilbert space H and analyze $(1-\tau^{-1})^{\alpha}$ spectral-wise by the complexification $H\oplus H$.

Proposition 17 (Equivalence of difference equation and sequence equation) *Let* $\varrho>1$ *and* $\alpha\in(0,1)$. *Let* $x\in H$, $F:\ell_{2,\varrho}(\mathbb{Z};H)\to\ell_{2,\varrho}(\mathbb{Z};H)$ *and* $u\in\ell_{2,\varrho}(\mathbb{Z};H)$. *Let* $\operatorname{spt} u\subseteq\mathbb{N}$ *and* $\operatorname{spt} F(u)\subseteq\mathbb{N}$. *In view of the Riemann–Liouville operator, the following are equivalent:*

(i) $\tau(1-\tau^{-1})^{\alpha}u=F(u)+\delta_{-1}x$,

(ii) $u_0=x,\ ((1-\tau^{-1})^{\alpha}u)_{n+1}=F(u)_n$ *for* $n\in\mathbb{N}$,

(iii) $u_0=x,\ u_{n+1}=(-1)^{n+1}\binom{-\alpha}{n+1}u_0+\sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}F(u)_k$ *for* $n\in\mathbb{N}$.

In view of the Caputo operator, the following are equivalent:

(iv) $\tau(1-\tau^{-1})^{\alpha}u=F(u)+(1-\tau^{-1})^{\alpha}\chi_{\mathbb{Z}_{\geq -1}}x$,

(v) $u_0=x,\ ((1-\tau^{-1})^{\alpha}u)_{n+1}=F(u)_n+(-1)^{n+1}\binom{\alpha-1}{n+1}u_0$ *for* $n\in\mathbb{N}$,

(vi) $u_0=x,\ u_{n+1}=u_0+\sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}F(u)_k$ *for* $n\in\mathbb{N}$.

Proof We only proof the equivalence of (i), (ii) and (iii).
$(i)\Leftrightarrow(ii)$: If we evaluate (i) at $n\in\mathbb{Z}$ we obtain

$$(\tau(1-\tau^{-1})^{\alpha}u)_n=((1-\tau^{-1})^{\alpha}u)_{n+1}=F(u)_n+(\delta_{-1}x)_n.$$

Since $((1-\tau^{-1})^{\alpha}u)_n$ and $F(u)_n=0$ for $n\in\mathbb{Z}_{<0}$, and since $(\delta_{-1}x)_n=x$ if and only if $n=-1$ and $((1-\tau^{-1})^{\alpha}u)_0=u_0$, it follows that (i) and (ii) are equivalent.
$(i)\Leftrightarrow(iii)$: If we apply $(1-\tau^{-1})^{-\alpha}$ to (i) we see that (i) is equivalent to

$$\tau u=(1-\tau^{-1})^{-\alpha}F(u)+(1-\tau^{-1})^{-\alpha}\delta_{-1}u.$$

This equation is equivalent to (iii), since

$$(1-\tau^{-1})^{-\alpha}\delta_{-1}x=\begin{cases}0, & \text{if } n<-1,\\ (-1)^{n+1}\binom{-\alpha}{n+1}x, & \text{if } n\geq -1,\end{cases}$$

and since $\operatorname{spt} F(u)\subseteq\mathbb{N}$,

$$(1-\tau^{-1})^{-\alpha}F(u)=\sum_{k=0}^{n}(-1)^{n-k}\binom{-\alpha}{n-k}F(u)_k.$$

Remark 18 Note that the right hand side F in Proposition 17 (i), (iv) maps sequences instead of values of H. If we have a function $f : H \to H$ such that for $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ we have $(f(u_n))_{n\in\mathbb{Z}} \in \ell_{2,\varrho}(\mathbb{Z}; H)$, we may set $F(u) := (f(u_n))_{n\in\mathbb{Z}}$ in Proposition 17.

Remark 19 (Grünwald–Letnikov difference operator) The Grünwald–Letnikov difference operator is defined for $h > 0$ and $\alpha \in (0, 1)$ by (c.f. [17, p. 708]):

$$\tilde{\Delta}_h^\alpha : V^{h\mathbb{N}} \to V^{h\mathbb{N}}, \qquad u \mapsto \left(t \mapsto \frac{1}{h^\alpha} \sum_{k=0}^{t/h} (-1)^k \binom{\alpha}{k} u_{t-kh} \right), \tag{6}$$

where $h\mathbb{N} = \{hn; n \in \mathbb{N}\}$. It can be shown (cf. [17, p. 708], [20, p. 43]) that for $V = \mathbb{R}$ the Grünwald–Letnikov operator can be used to approximate the Riemann–Liouville integral of sufficiently smooth functions.

Let $\alpha \in (0, 1)$. For $v \in \ell_{2,\varrho}(\mathbb{N}; H)$ and $u := \iota v$ we calculate for the Grünwald–Letnikov operator (6), $(1 - \tau^{-1})^\alpha u = \tilde{\Delta}_1^\alpha v$. Let $h > 0$, $x \in H$ and $F : H \to H$. A Grünwald–Letnikov difference equation has the form

$$(\tilde{\Delta}_h^\alpha v)(t + h) = F(v(t)), \quad v(0) = x \qquad (t \in h\mathbb{N}).$$

For $h = 1$ the Grünwald–Letnikov equation resembles the Riemann–Liouville equation of Proposition 17 and for $h \in \mathbb{R}_{>0}$, we may treat a Grünwald–Letnikov problem by considering the problem

$$\tau(1 - \tau^{-1})^\alpha u = h^\alpha F(u) + \delta_{-1} x.$$

Linear Equations on Sequence Spaces

Remark 20 Let $A \in L(H)$ and $x \in H$. In view of the Riemann–Liouville difference operator we ask whether the linear equation

$$\tau(1 - \tau^{-1})^\alpha u = Au + \delta_{-1} x \tag{7}$$

of Proposition 17 has a unique so-called causal solution that is supported in $\mathbb{N}$. In the spaces $\ell_{2,\varrho}(\mathbb{Z}; H)$, we have a unique solution of (7) for every initial value if $\tau(1 - \tau^{-1})^\alpha - A$ is invertible in $\ell_{2,\varrho}(\mathbb{Z}; H)$. In view of Proposition 17, the solution $(\tau(1 - \tau^{-1})^\alpha - A)^{-1}\delta_{-1}x$ should be causal. For the corresponding Caputo equation

$$\tau(1 - \tau^{-1})^\alpha u = Au + (1 - \tau^{-1})^\alpha \chi_{\mathbb{Z}_{\geq -1}} x, \tag{8}$$

the treatment is similar since $\chi_{\mathbb{Z}_{\geq -1}} x = \chi_{\mathbb{N}} x + \delta_{-1} x$.

Lemma 21 *Let $\alpha \in (0, 1)$ and $A \in L(H)$. We define $f : \mathbb{C}_{|\cdot|>1} \to \mathbb{C}, z \mapsto z(1 - z^{-1})^\alpha$ and set $f_\varrho := f|_{S_\varrho}$ for $\varrho > 1$. For $\varrho > 1$, the operator $\tau(1 - \tau^{-1})^\alpha - A$ is invertible in $\ell_{2,\varrho}(\mathbb{Z}; H)$ if and only if* ran $f_\varrho \cap \sigma(A) = \emptyset$. *Moreover, there is $\varrho > 1$*

such that for all $\mu > \varrho$, $\operatorname{ran} f_\mu \cap \sigma(A) = \emptyset$*, that is* $\left\{z(1-z^{-1})^\alpha;\, |z| > \varrho\right\}$ *is in the resolvent set of* A*.*

Proof Recall the multiplication operator m of Lemma 12. Using the $\mathscr{Z}$ transform, the operator $\tau(1-\tau^{-1})^\alpha - A$ is invertible in $\ell_{2,\varrho}(\mathbb{Z}; H)$ if and only if $\mathrm{m}(1-\mathrm{m}^{-1})^\alpha - A$ is invertible in $L_2(S_\varrho, H)$, since $\mathscr{Z}_\varrho$ is unitary. This is the case, however, if and only if $\operatorname{ran} f_\varrho \cap \sigma(A) = \emptyset$. Using Lemma 4, there is $\varrho > 1$ such that for all $\mu > \varrho$ and $z \in S_\mu$, $r(A) < \mu(1-\mu^{-1})^\alpha \le \left|z(1-z^{-1})^\alpha\right|$. That is, for all $\mu > \varrho$, $\operatorname{ran} f_\mu \cap \sigma(A) = \emptyset$.

Proposition 22 (Causality of $(\tau(1-\tau^{-1})^\alpha - A)^{-1}$) *Let* $\varrho > 1$, $\alpha \in (0,1)$ *and* $A \in L(H)$*. Let* f_ϱ *be defined as in Lemma 21. The following are equivalent:*

$$
\begin{aligned}
&(i)\quad (\tau(1-\tau^{-1})^\alpha - A)^{-1} \in L\big(\ell_{2,\varrho}(\mathbb{Z}; H)\big) \text{ is causal},\\
&(ii)\quad (\tau(1-\tau^{-1})^\alpha - A)^{-1} \in L\big(\ell_{2,\varrho}(\mathbb{Z}; H)\big)\\
&\qquad\text{and}\quad \forall x \in H\colon\ \operatorname{spt}(\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_{-1}x \subseteq \mathbb{N},\\
&(iii)\quad \forall \mu \ge \varrho\colon\ \operatorname{ran} f_\mu \cap \sigma(A) = \emptyset.
\end{aligned}
$$

Proof $(i) \Rightarrow (ii)$: Let $x \in H$ and $u := (\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_{-1}x$. Using causality assumed in (i), we obtain $\operatorname{spt} u \subseteq \mathbb{Z}_{\ge -1}$. Moreover, $u_{-1} = ((1-\tau^{-1})^{-\alpha}Au)_{-2} + ((1-\tau^{-1})^{-\alpha}\delta_{-1}x)_{-2} = 0$ so that $\operatorname{spt} u \subseteq \mathbb{N}$.
$(ii) \Rightarrow (iii)$: Suppose by contradiction that there is $\varrho' > \varrho$ with $\operatorname{ran} f_{\varrho'} \cap \sigma(A) \ne \emptyset$. The set $\left\{z \in \mathbb{C}_{|\cdot|\ge\varrho'};\, z(1-z^{-1})^\alpha \in \sigma(A)\right\}$ is closed, since $\sigma(A)$ is closed and since f is continuous, and the set is bounded, since by Lemma 21 there is a $\tilde{\varrho} > \varrho'$ such that $f(\mathbb{C}_{|\cdot|\ge\tilde{\varrho}})$ is in the resolvent set. Thus, there is $z' \in \left\{z \in \mathbb{C}_{|\cdot|\ge\varrho'};\, z(1-z^{-1})^\alpha \in \sigma(A)\right\}$ with maximal absolute value. Therefore there is a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $|z_n| > \left|z'\right|$, that is, $z_n(1-z_n^{-1})^\alpha$ is in the resolvent set of A $(n \in \mathbb{N})$ and $\lim_{n\to\infty} z_n = z'$. Using the resolvent estimate (cf. [21, p. 378]), we have $\lim_{n\to\infty}\left\|(z_n(1-z_n^{-1})^\alpha - A)^{-1}\right\|_{L(H)} = \infty$. By applying the Banach–Steinhaus theorem (cf. [21, p. 141]), there is $x \in H$ with $\lim_{n\to\infty}\left\|(z_n(1-z_n^{-1})^\alpha - A)^{-1}x\right\|_H = \infty$. By assumption, $(\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_{-1}x \in \ell_{2,\varrho}(\mathbb{Z}; H)$ and $\operatorname{spt}(\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_{-1}x \subseteq \mathbb{N}$. Hence for $v := (\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_0 x \in \ell_{2,\varrho}(\mathbb{Z}; H)$, we have $v \in \ell_{2,\varrho}(\mathbb{Z}; H)$ and $\operatorname{spt} v \subseteq \mathbb{N}$. Applying Lemma 13, it follows that $F : \mathbb{C}_{|\cdot|>\varrho} \to H,\, z \mapsto \sum_{k=-\infty}^{\infty} v_k z^{-k}$ is analytic. Since $v \in \ell_{2,\mu}(\mathbb{Z}; H)$ for $\mu > \left|z'\right|$, it follows that for $G : \mathbb{C}_{|\cdot|>|z'|} \to H,\, z \mapsto (z(1-z^{-1})^\alpha - A)^{-1}x$, we have $G = F|_{\mathbb{C}_{|\cdot|>|z'|}}$. This means that $\lim_{n\to\infty}\|F(z_n)\|_H = \lim_{n\to\infty}\|G(z_n)\|_H = \infty$. Since F is continuous, this is a contradiction in that $\lim_{n\to\infty}\|F(z_n)\|_H \ne \infty$.
$(iii) \Rightarrow (i)$: We have $(\tau(1-\tau^{-1})^\alpha - A)^{-1} \in L(\ell_{2,\mu}(\mathbb{Z}; H))$ for $\mu > \varrho$ by Lemma 21. Since the resolvent of A is analytic, the mapping $z \mapsto (z(1-z^{-1})^\alpha - A)^{-1}$ is analytic on $\mathbb{C}_{|\cdot|>\varrho}$. Moreover the mapping $z \mapsto \left\|(z(1-z^{-1})^\alpha - A)^{-1}\right\|_{L(H)}$ is continuous and hence bounded on compact sets $\mathbb{C}_{\mu\ge|\cdot|\ge\varrho}$ where $\mu \ge \varrho$, i.e. the mapping attains its maximum on $\mathbb{C}_{\mu\ge|\cdot|\ge\varrho}$. By Lemma 4 and since A is bounded, $\sup_{z\in S_\mu}\left\|(z(1-z^{-1})^\alpha - A)^{-1}\right\|_{L(H)}$ decays to zero when μ tends to infinity. It follows that $\mu \mapsto \sup_{z\in S_\mu}\left\|(z(1-z^{-1})^\alpha - A)^{-1}\right\|_{L(H)}$ is bounded on $[\varrho, \infty)$ and there-

fore the conditions of Lemma 13(ii) are satisfied for $(\tau(1-\tau^{-1})^{\alpha}-A)^{-1}u$, where $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$, spt $u \subseteq \mathbb{N}$. It follows that $(\tau(1-\tau^{-1})^{\alpha}-A)^{-1}$ is causal.

Remark 23 Let $A \in L(H)$, $\varrho > 1$ and $\alpha \in (0,1)$. By Lemma 21 and Proposition 22, we can always choose ϱ large enough such that $\tau(1-\tau^{-1})^{\alpha}-A$ is invertible with causal inverse. As a consequence the linear fractional difference Eq. (7) or (8) has a unique solution $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$. Moreover, from the previous Theorem it follows that (7) or (8) has a unique solution in $\ell_{2,\mu}(\mathbb{Z}; H)$ for $\mu \geq \varrho$ which coincides with the solution u, since $\ell_{2,\varrho}(\mathbb{N}; H) \subseteq \ell_{2,\mu}(\mathbb{N}; H)$. Therefore we can speak of the solution operator $(\tau(1-\tau^{-1})^{\alpha}-A)^{-1}$.

The difference equation for an initial value $x \in H$ and $A \in L(H)$

$$(\Delta^{\alpha}u)_n = Au_n, \qquad u_0 = x,$$

or

$$(\Delta_C^{\alpha}u)_n = Au_n, \qquad u_0 = x,$$

can be solved algebraically with a unique solution $u \in H^{\mathbb{N}}$ (cf. Proposition 17(iii), (vi)). Recall the embedding ι of Proposition 6. Since A has bounded spectrum, when by the previous theorem, there is $\varrho > 1$ such that $\iota u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ is the unique solution of (7) or (8).

Asymptotic Stability

We discuss asymptotic stability of linear fractional difference equations. For an analysis of rates of convergence, see also [5, 7].

Definition 24 (*Asymptotic stability*) Let $A \in L(H)$. The zero equilibrium of Eq. (7) or (8), i.e., the solution $u = 0$ for the inital value 0, is said to be asymptotically stable if for every $\varrho > 1$, every solution $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ of (7) or (8) with spt $u \subseteq \mathbb{N}$ satisfies $\lim_{n\to\infty} u_n = 0$ in H.

Remark 25 If a sequence $u \in H^{\mathbb{Z}}$ satisfies spt $u \subseteq \mathbb{N}$ and $\lim_{n\to\infty} u_n = 0$, then necessarily for all $\varrho > 1$ we have $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$. One could say that the spaces $\ell_{2,\varrho}(\mathbb{Z}; H)$, $\varrho > 1$, are large enough to look for asymptotically stable solutions of a linear sequence equation.

Proposition 26 (Necessary condition for asymptotic stability) *Let $A \in L(H)$ such that the zero equilibrium of Eq.* (7) *or* (8) *is asymptotically stable and let f_{μ} ($\mu > 1$) be as in Lemma 21. Then for all $\mu > 1$, $\tau(1-\tau^{-1})^{\alpha}-A$ is invertible in $\ell_{2,\mu}(\mathbb{Z}; H)$ with causal inverse, i.e., for each $\mu > 1$, $\sigma(A) \cap \operatorname{ran} f_{\mu} = \emptyset$.*

Proof Assume by contradiction there is $z' \in \operatorname{ran} f_{\varrho} \cap \sigma(A) \neq \emptyset$ where $\varrho > 1$. We may assume that $\operatorname{ran} f_{\mu} \cap \sigma(A) = \emptyset$ for $\mu > \left|z'\right|$. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ with $|z_n| > \left|z'\right|$ such that $z_n(1-z_n^{-1})^{\alpha}$ is in the resolvent set of A $(n \in \mathbb{N})$ and such that $z_n \to z'$ $(n \to \infty)$. Using the resolvent estimate, we have $\lim_{n\to\infty} \left\|(z_n(1-z_n^{-1})^\alpha - A)^{-1}\right\|_{L(H)} = \infty$. Using the Banach–Steinhaus theorem, there is $x \in H$ with $\lim_{n\to\infty} \left\|(z_n(1-z_n^{-1})^\alpha - A)^{-1}x\right\|_H = \infty$. By Lemma 21 and Proposition 22, for $\mu > |z'|$, we know that $\tau(1-\tau^{-1})^\alpha - A$ is invertible in $\ell_{2,\mu}(\mathbb{Z}; H)$ and $v := (\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_0 x$ satisfies $\operatorname{spt} v \subseteq \mathbb{N}$. Since the zero equilibrium is asymptotically stable, we have $v \in \ell_{2,\varrho'}(\mathbb{Z}; H)$ for some $\varrho' \in (1, |z'|)$ by Remark 25. Then the mapping $F : \mathbb{C}_{|\cdot|>\varrho'} \to H, z \mapsto \sum_{k=-\infty}^{\infty} v_k z^{-k}$ is analytic and equals $G : \mathbb{C}_{|\cdot|>|z'|} \to H, z \mapsto (z(1-z^{-1})^\alpha - A)^{-1}\delta_0 x$ on $\mathbb{C}_{|\cdot|>|z'|}$ by Lemma 13. Therefore we have $\lim_{n\to\infty} F(z_n) < \infty$, since F is analytic which contradicts $\lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} G(z_n) = \infty$.

For a sufficient condition of asymptotic stability we observe that if $u \in \ell_{2,1}(\mathbb{Z}; H)$ with $\operatorname{spt} u \subseteq \mathbb{N}$ then $\lim_{n\to\infty} u_n = 0$.

Proposition 27 (Sufficient condition for asymptotic stability) *Let $A \in L(H)$. For all $\varrho > 1$ let $\tau(1-\tau^{-1})^\alpha - A$ be invertible in $\ell_{2,\varrho}(\mathbb{Z}; H)$ with causal inverse. If for all $x \in H$ the mapping $\mathbb{C}_{|\cdot|>1} \to H, z \mapsto \sum_{k=-\infty}^{\infty}[(\tau(1-\tau^{-1})^\alpha - A)^{-1}\delta_{-1}x]_k z^{-k}$ has a continuous continuation to the unit circle S_1, then the zero equilibrium of Eq. (7) or (8) is asymptotically stable.*

Proof Let g be the continuous continuation. Then $g|_{S_1} \in L_2(S_1, H)$ and $v := \mathscr{Z}_1^{-1} g|_{S_1} \in \ell_{2,1}(\mathbb{Z}; H)$. Moreover, $u = v$ that is $u \in \ell_{2,1}(\mathbb{Z}; H)$.

Remark 28 We believe that the necessary conditions for stability in Proposition 26 are not sufficient, neither are the sufficient conditions for stability in Proposition 27 necessary. Already for semigroups the asymptotic stability can in general not be characterized by spectral conditions solely. The shift operator on continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$ which decay at infinity, for example, is asymptotically stable although its spectrum consists of all complex numbers with non-positive real part [2, Example 2.5(c)]. The characterization of asymptotic stability for linear fractional difference equations is an intricate problem which still needs to be addressed.

Example 29 Let $H = \mathbb{C}$, $A : \mathbb{C} \to \mathbb{C}, z \mapsto \lambda z$ where $\lambda \in \mathbb{R}$ and $\alpha \in (0, 1)$. We study the asymptotic behavior of the linear fractional equations (7) and (8) on $\ell_{2,\varrho}(\mathbb{Z}; H)$ $(\varrho > 1)$ in view of Proposition 26 and Proposition 27 and therefore want to apply the $\mathscr{Z}$ transform to Eq. (7) and (8). In order to obtain an asymptotically stable zero equilibrium by Proposition 26, we must have $\sigma(A) \cap \operatorname{ran} f = \emptyset$ where $f : \mathbb{C}_{|\cdot|>1} \to \mathbb{C}, z \mapsto z(1-z^{-1})^\alpha$ is defined as in Lemma 21 and $\sigma(A) = \{\lambda\}$. We remark that for $z \in \mathbb{C}_{|\cdot|>1}$, $f(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$ since f is injective and since $f(\overline{z}) = \overline{f(z)}$. Moreover $f(\mathbb{C}_{|\cdot|>1} \cap \mathbb{R}) = (-\infty, -2^\alpha) \cup (0, \infty)$ and so $\lambda \notin \operatorname{ran} f$ if and only if $\lambda \in [-2^\alpha, 0]$. By Proposition 26, we necessarily have $\lambda \in [-2^\alpha, 0]$ if the zero equilibrium of (7) or (8) is asymptotically stable. Let $\lambda \in [-2^\alpha, 0]$, and for $u \in \ell_{2,\varrho}(\mathbb{Z}; H)$ we denote $\hat{u} := \mathscr{Z}u$.

We consider (7) with $x \in \mathbb{C}$ first. Also for $z \in S_\varrho$ we have $(\mathscr{Z}\delta_{-1}x)(z) = zx$. Applying the $\mathscr{Z}$ transform to Eq. (7), we obtain for $z \in S_\varrho$

$$z(1-z^{-1})^{\alpha}\hat{u}(z) = A\hat{u}(z) + zx.$$

If $\lambda \in (-2^{\alpha}, 0)$, the mapping $\mathbb{C}_{|\cdot|>1} \to H, z \mapsto \frac{zx}{z(1-z^{-1})^{\alpha}-\lambda}$ has a continuous continuation to S_1 and by Proposition 27, we obtain that the zero equilibrium of (7) is asymptotically stable.

We now consider Eq. (8) where $x \in \mathbb{C}$. For $z \in S_{\varrho}$, we have $(\mathscr{Z}\chi_{\mathbb{Z}_{\geq 1}}x)(z) = \frac{zx}{1-z^{-1}}$. Applying the $\mathscr{Z}$ transform to Eq. (8), we obtain for $z \in S_{\varrho}$

$$z(1-z^{-1})^{\alpha}\hat{u}(z) = A\hat{u}(z) + z(1-z^{-1})^{\alpha-1}x.$$

If $\lambda \in (-2^{\alpha}, 0)$, the mapping $\mathbb{C}_{|\cdot|>1} \to H, z \mapsto \frac{z(1-z^{-1})^{\alpha-1}x}{z(1-z^{-1})^{\alpha}-\lambda}$ has a continuous continuation to S_1 and using Proposition 27, we obtain that the zero equilibrium of (8) is asymptotically stable.

The cases $\lambda = 0$ and $\lambda = -2^{\alpha}$ are discussed in [5].

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