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Optimization in an asymmetric Lanchester (n, 1) model

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Abstract

Counter-terrorism is a global task that every nation is concerned about. To improve operations against terrorism, many nations carry out counter-terroristic operations not only by themselves but also in cooperation with other nations. In this paper, we propose an extended the Kaplan-Kress-Szechtman model to cope with multi-party counter-terrorism. The optimal control problem for this model is studied. Our main tool is Pontryagin's maximal principle. The optimal intelligence level and individual reinforcement of each party are found. The numerical results show that counter-terrorism operations in cooperative models are more effective than that in single models.

Keywords

Lanchester model, multi-party counter-terrorism, optimal control, intelligence, counter-terrorism, Kim–Kim–Suzuki model

1. Introduction

Mathematical models have been widely used for the theoretical studies, prediction, planning, etc., of warfare. In combat modeling, they are used to illustrate a combating process and to simulate a weapon system or events happening on a battlefield. Combating models have become the main tools for military policy planners in the North Atlantic Treaty Organization (NATO) and many countries with a developed army (see Caulkins et al.[1]). A mathematical model as a system of differential equations for a combat was introduced by Lanchester in 1916. In this model, the combat was supposed to be between two forces of the same type; the attrition effects are represented by constant attrition rate coefficients. Lanchester proposed two models corresponding to aimed and unaimed firepower. The attrition rates follow square and linear laws for aimed and unaimed firepower, respectively.

In 1962, Deitchman[2] introduced a model for guerilla warfare, which is an asymmetric Lanchester-type model. In this model, the firepower of guerilla forces is aimed, whereas that of government forces is unaimed. Schaffer[3] extended Deitchmann's model to investigate the optimal distribution of firepower. Schaffer[3] is believed to be the first to include the intelligence level in the attrition rates. Recently, Kress and Szechtmann[4] and Kaplan et al.[5] studied anti-insurgent operations by using a Lanchester-like model including the intelligence level. In these works, the "double-edged sword" effect was introduced for the first time and the authors managed to explain why the terrorists cannot be eliminated completely. The impact of the intelligence level on the outcome of a combat between a government force and an insurgent force in an asymmetric model has been studied in the works of Kress and Szechtmann[4] and Kaplan et al.[5] The higher the intelligence level is, the more effective the firepower. In addition, problems of optimal control for dynamical systems modeling of a combat have been studied for a long time. In 1974, Taylor[6] studied the optimal distribution of firepower for some combat models. Lin and Mackay[7] studied the optimal distribution of firepower in a Lanchester model with respect to time. Chen et al.[8] investigated the optimal

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control problem by setting reinforcements as control variable for a Lanchester (1,1) model and Chen et al.[9] extended this to a Lanchester (2,2) model. On the other hand, Feichtinger et al.[10] studied the optimal control problem for an asymmetric Lanchester model with the reinforcement and intelligence level being control variables. In this paper, we study an asymmetric Lanchester $(n, 1)$ model representing a combat between a terroristic force and n government forces sharing intelligence. The combat process is modeled as a system of differential equations. Based on this dynamic, an optimal control problem is set up and studied by Pontryagin's maximal principle. It is shown that the interior and boundary steady states exists under some mild assumptions. In these states, the forces are mutually suppressing. The preliminary numerical results justify the theoretical studies in many cases under investigation. The rest of article is organized as follows. Section 2 is devoted to the description of our model. Some results, including the existence of interior steady states and numerical illustrations, are presented in Section 3. In the last section we make conclusions.

2. The model

In current anti-terrorism wars nations should cooperate to launch anti-terrorism operations, especially when terroristic forces can simultaneously operate in many nations' territories. This requires countries to cooperate for a joint anti-terrorist plan. In other words, anti-terrorism should be carried out collectively. We extend the Kaplan-Kress-Szechtman (KKS) model[10] to the case in which two counter-terrorism forces fight against a terroristic force. Based on the KKS model, we denote by $X_1(t), X_2(t), \ldots, X_n(t)$ numbers of n counter-terrorism forces facing a group of terrorists whose quantity is represented by a portion $Y(t)$ (out of a population P). Since P is assumed to be constant, we, for simplicity, let $P = 1$. Therefore, one has $0 \leq Y(t) \leq 1$. Constant attrition rates of the terroristic forces against n counter-terrorism ones are denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively. Similarly, by $\gamma_1, \gamma_2, \ldots, \gamma_n$ we respectively represent attrition rates of counter-terrorism forces against the terroristic one. μ represents the joint intelligence operation of n anti-terrorist forces. It represents, for example, the level at which counter-terrorism forces can pinpoint the exact location of terrorist forces. To sum up, we propose our model in the form of differential equations as follows:

$$\begin{cases} \dot{X}_1 = -\alpha_1 Y - \delta_1 X_1 + \beta_1, \\ \dot{X}_2 = -\alpha_2 Y - \delta_2 X_2 + \beta_2, \\ \ldots \\ \dot{X}_n = -\alpha_n Y - \delta_n X_n + \beta_n, \\ \dot{Y} = -\left(\sum\limits_{i=1}^{n} \gamma_i X_i\right)(\mu + (1-\mu)Y) + \theta(C). \end{cases} \tag{1}$$

Here, $\beta_1, \beta_2, \ldots, \beta_n$ denotes the reinforcing rates of the counter-terrorism forces, while $\theta(C)$ reflects the double-edged sword effect with the following:

$$C = \left(\sum_{i=1}^{n} \gamma_i X_i\right)(1-\mu)(1-Y). \tag{2}$$

Conventionally, cf. Feichtinger et al.,[10] we assume that the cost of collecting intelligence is a convex function satisfying the following:

$$I(0) = 0,\ I'(\mu) > 0,\ I''(\mu) > 0,\ I(1) = +\infty. \tag{3}$$

The damage function $D(Y)$ is also a convex function, with the following constraints:

$$D(0) = 0,\ D'(Y) > 0,\ D''(Y) > 0. \tag{4}$$

The costs to maintain the military of the n counter-terrorism forces, denoted by $A(X_1), A(X_2), \ldots, A(X_n)$, can be assumed to be linear or concave functions satisfying the following:

$$A_i(0) = 0,\ A_i'(X_i) > 0,\ A_i''(X_i) \leq 0, i = \overline{1:n}.$$

$K_i(\beta_i), i = \overline{1:n}$ denotes the cost of the reinforcing process. It is conventional in human resource planning that such costs are often assumed to be quadratic, and, for simplicity, they are $\frac{\beta_1^2}{2}, \frac{\beta_2^2}{2}, \ldots, \frac{\beta_n^2}{2}$, respectively.

So the task of the military planner is to solve the following problem:

$$\min_{\mu, \beta} \int_0^{\infty} e^{-rt}(D(Y) + I(\mu) + A(X) + K(\beta))dt, \tag{5}$$

where

$$A(X) = \sum_{i=1}^{n} A_i(X_i);\ K(\beta) = \sum_{i=1}^{n} K_i(\beta_i).$$

In this article, we also presume that the damage caused by terrorists follows quadratic laws, $\frac{fY^2}{2}$ (with $f > 0$). The costs to maintain the warfare are linear functions, which are $c_1X_1, c_2X_2, \ldots, c_nX_n$ (with $c_1, c_2, \ldots, c_n > 0$). The cost of collecting intelligence is assumed to be $I(\mu) = -\log(1-\mu)$ and corresponding control costs are $\frac{\beta_1^2}{2}, \frac{\beta_2^2}{2}, \ldots, \frac{\beta_n^2}{2}$. Finally, we assume that $\theta(C) = \theta C^2$, with θ being a positive real number.

3. Interior steady states and stability analysis

3.1 Interior steady states

Using Pontryagin's optimal principle, we set up the Hamilton function as follows:

$$H=\left[-\frac{fY^2}{2}-\sum_{i=1}^{n}c_iX_i+\log(1-\mu)-\sum_{i=1}^{n}\frac{\beta_i^2}{2}\right]+ \\ +\sum_{i=1}^{n}\lambda_i(-\alpha_iY-\delta_iX_i+\beta_i)+ \\ +\lambda_{n+1}\left(-\gamma X(\mu+(1-\mu)Y)+\theta(\gamma X)^2(1-\mu)^2(1-Y)^2\right),$$

with $\lambda_1, \lambda_2, ..., \lambda_{n+1}$ being adjoint variables and, for the sake of brevity, γX represents the sum $\sum_{i=1}^{n}\gamma_iX_i$.

The partial derivatives with respect to control variables are as follows:

$$\frac{\partial H}{\partial \beta_i}=-\beta_i+\lambda_i=0\Rightarrow \beta_i=\lambda_i,\ \ =\overline{1:n}$$

$$\frac{\partial H}{\partial \mu}=\frac{-1}{1-\mu}+\lambda_{n+1}\left[-\gamma X(1-Y)-2\theta\gamma X^2(1-\mu)(1-Y)^2\right].$$

If $\lambda_{n+1}>0$, then the Hamiltonian is monotonically decreasing. It leads to an optimal level of intelligence $\mu=0$. For $\lambda_{n+1}<0$, the following holds:

$$\frac{\partial H}{\partial \mu}=\frac{-1}{1-\mu}+\lambda_{n+1} \\ \left[-\gamma X(1-Y)-2\theta(\gamma X)^2(1-\mu)(1-Y)^2\right], \tag{6}$$

where $x=\gamma X(1-\mu)(1-Y)$. The solution of this equation is as follows:

$$x=\frac{-1+\sqrt{1-\frac{8\theta}{\lambda_{n+1}}}}{4\theta}.$$

Since $x\leq 1$, it follows that $\lambda_{n+1}=-\frac{1}{x(1+2\theta x)}\leq-\frac{1}{1+2\theta}$, and the corresponding intelligence level is given by the following:

$$\mu=1-\frac{x}{(\gamma X)(1-Y)}. \tag{7}$$

The adjoint variables satisfy the system of differential equations:

$$\begin{cases}\dot{\lambda}_i=r\lambda_i-\frac{\partial H}{\partial X_i},\ i=\overline{1:n} \\ \dot{\lambda}_{n+1}=r\lambda_{n+1}-\frac{\partial H}{\partial Y}\end{cases}. \tag{8}$$

The concrete form of this system is as follows:

$$\dot{\lambda}_1=(r+\delta_1)\lambda_1+c_1-\lambda_{n+1}[-\gamma_1(\mu+(1-\mu)Y) \\ +2\theta\gamma_1(\gamma X)(1-\mu)^2(1-Y)^2],$$
$$\dot{\lambda}_2=(r+\delta_2)\lambda_2+c_2-\lambda_{n+1}[-\gamma_2(\mu+(1-\mu)Y) \\ +2\theta\gamma_2(\gamma X)(1-\mu)^2(1-Y)^2],$$
$$\dots$$
$$\dot{\lambda}_n=(r+\delta_n)\lambda_n+c_n-\lambda_{n+1}[-\gamma_n(\mu+(1-\mu)Y) \\ +2\theta\gamma_n(\gamma X)(1-\mu)^2(1-Y)^2],$$
$$\dot{\lambda}_{n+1}=r\lambda_{n+1}+fY+\sum_{i=1}^{n}\alpha_i\lambda_i-\lambda_{n+1}[-(\gamma X)(1-\mu) \\ -2\theta(\gamma X)^2(1-\mu)^2(1-Y)].$$

Note the following:

$$-\gamma_i(\mu+(1-\mu)Y)+2\theta\gamma_i(\gamma X)(1-\mu)^2(1-Y)^2 \\ =-\gamma_i\left[1+\frac{1}{\lambda_{n+1}(\gamma X)}\right],$$

$$-(\gamma X)(1-\mu)-2\theta(\gamma X)^2(1-\mu)^2(1-Y) \\ =\frac{1}{\lambda_{n+1}(1-Y)},$$

and

$$-(\gamma X)(\mu+(1-\mu)Y)+\theta(\gamma X)^2(1-\mu)^2(1-Y)^2 \\ =-(\gamma X)+x(1+\theta x).$$

The coupled canonical system consisting of the dynamics and the adjoint equations is as follows:

$$\dot{X}_i=-\alpha_iY-\delta_iX_i+\lambda_i,\ i=\overline{1;n}, \tag{9}$$

$$\dot{Y}=-(\gamma X)+x(1+\theta x), \tag{10}$$

$$\dot{\lambda}_i=(r+\delta_i)\lambda_i+c_i+\gamma_i\lambda_{n+1}+\frac{\gamma_i}{\gamma X},\ i=\overline{1;n}, \tag{11}$$

$$\dot{\lambda}_{n+1}=r\lambda_{n+1}+fY+\sum_{i=1}^{n}\alpha_i\lambda_i-\frac{1}{1-Y}. \tag{12}$$

Interior steady states of this system satisfy the following equations (by letting the derivatives in (9), (10), and (11) vanish):

$$\gamma X=x(1+\theta x), \tag{13}$$

$$\lambda_i=-\frac{1}{r+\delta_i}\left(c_i+\frac{\theta\gamma_i}{(1+\theta x)(1+2\theta x)}\right),\ i=\overline{1:n}, \tag{14}$$

$$\alpha_n\delta_iX_i-\alpha_i\delta_nX_n=\alpha_i\lambda_n-\alpha_n\lambda_i,\ i=\overline{1:n-1}. \tag{15}$$

One obtains the following linear system:

$$\begin{cases}\gamma_1X_1+\gamma_2X_2+\dots+\gamma_nX_n=x(1+\theta x), \\ \alpha_n\delta_1X_1-\alpha_1\delta_nX_n=\alpha_1\lambda_n-\alpha_n\lambda_1 \\ \alpha_n\delta_2X_2-\alpha_2\delta_nX_n=\alpha_2\lambda_n-\alpha_n\lambda_2 \\ \dots \\ \alpha_n\delta_{n-1}X_{n-1}-\alpha_{n-1}\delta_nX_n=\alpha_{n-1}\lambda_n-\alpha_n\lambda_{n-1}\end{cases}. \tag{16}$$

The determinant of this system is as follows:

$$D = (-\alpha_n)^{n-1}\left[\delta_n \sum_{i=1}^{n-1} \gamma_i \alpha_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \delta_j + \gamma_n \prod_{j=1}^{n} \delta_j\right].$$

Obviously, we see that, $D > 0$ if n-odd and $D < 0$ if n-even, so the system has a unique solution $(X_1, X_2, ..., X_n)$. The following thus follows:

$$Y = \frac{\lambda_1 - \delta_1 X_1}{\alpha_1}.$$

Since $\lambda_i < 0, i = \overline{1:n}$, it follows that $Y < 0$. For the existence of the interior states it requires that $c_i < 0, i = \overline{1:n}$.

Substituting λ_i, X_i, Y into equation $r\lambda_{n+1} + fY + \sum_{i=1}^{n} \alpha_i \lambda_i - \frac{1}{1-Y} = 0$, one obtains a one-dimensional equation $F(x) = 0$. Solving this equation for x, one can compute the corresponding interior steady states. Numerical illustrations can be found in the upcoming section. Now, let us discuss the stability of the steady states.

3.2 Stability of the steady states

Provided that the interior steady states are found, the corresponding Jacobian is given by the following:

$$\begin{bmatrix} G_1 \; G_2 \\ G_3 \; G_4 \end{bmatrix}, \tag{17}$$

where:

$$G_1 = \begin{bmatrix} -\delta_1 & 0 & ... & 0 & -\alpha_1 \\ 0 & -\delta_2 & ... & 0 & -\alpha_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & -\delta_n & -\alpha_n \\ -\gamma_1 & -\gamma_2 & ... & -\gamma_n & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & (1+2\theta x)\frac{\partial x}{\partial \lambda_{n+1}} \end{bmatrix}$$

$$G_3 = \begin{bmatrix} -\frac{\gamma_1^2}{(\gamma X)^2} & -\frac{\gamma_1\gamma_2}{(\gamma X)^2} & ... & -\frac{\gamma_1\gamma_n}{(\gamma X)^2} & 0 \\ -\frac{\gamma_2\gamma_1}{(\gamma X)^2} & -\frac{\gamma_2^2}{(\gamma X)^2} & ... & -\frac{\gamma_2\gamma_n}{(\gamma X)^2} & 0 \\ ... & ... & ... & ... & ... \\ -\frac{\gamma_n\gamma_1}{(\gamma X)^2} & -\frac{\gamma_n\gamma_1}{(\gamma X)^2} & ... & -\frac{\gamma_n^2}{(\gamma X)^2} & 0 \\ 0 & 0 & ... & 0 & \frac{f(1-Y)^2-1}{(1-Y)^2} \end{bmatrix}$$

$$G_4 = \begin{bmatrix} r+\delta_1 & 0 & ... & 0 & \gamma_1 \\ 0 & r+\delta_2 & ... & 0 & \gamma_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & r+\delta_n & \gamma_n \\ \alpha_1 & \alpha_2 & ... & \alpha_r & r \end{bmatrix},$$

with $\frac{\partial x}{\partial \lambda_{n+1}} = -\frac{1}{\lambda_{n+1}\sqrt{\lambda_{n+1}^2 - 8\theta\lambda_{n+1}}}$. By studying the real parts of the eigenvalues of this matrix, one can derive the stability of the underlying interior steady states.

4. Numerical illustrations

4.1 2 versus 1

In Feichtinger et al.,[10] the parameters $r = 2.2$; $\alpha = 2.23$; $\delta = 0.34$; $\gamma = 1.19$; $\theta = 1.86$; $c = -2.3734$; $f = 1.12$ were chosen. The obtained numerical results were reported as follows (with appropriate changes of notation): the interior state is $X = 1.39523; Y = 0.16034$, the optimal reinforcement rate is $\lambda = \beta = 0.83193$; and $\mu = 0.48822$ is the corresponding optimal intelligence level. In order to make a comparison, we reuse these parameters for our model with some duplications, thus:

$$\textbf{Case 1 :} \; r = 2.2, \alpha_1 = \alpha_2 = 2.23; \delta_1 = \delta_2 = 0.34; \\ \gamma_1 = \gamma_2 = 1.19; c_1 = c_2 = -2.3734; f = 1.2; \theta = 1.86.$$

With these parameters, our results are as follows.

Interior steady state:

$$X_1 = X_2 = 0.2948; Y = 0.2844;$$

$$\lambda_1 = \lambda_2 = 0.7344; \lambda_3 = -0.9984;$$

optimal intelligence level: $\mu = 0.20$.

Also in this case, the eigenvalues of the Jacobian are as follows:

$$4.3313; 2.54; -2.131; -0.34; 1.1 \pm 1.6790i.$$

These results show that the steady states can only be reached by two stable manifolds. These facts are illustrated in Figure 1.

We also modify the parameters in Case 1 to make the following:

$$\textbf{Case2 :} \; r = 2.5; \alpha_1 = 2.0; \alpha_2 = 1.9; \delta_1 = 0.34; \delta_2 = 0.37; \\ \gamma_1 = \gamma_2 = 1.3; c_1 = c_2 = -2.5; f = 1.2; \theta = 1.86.$$

The results are as follows.

Interior steady state:

$$X_1 = 0.355214; \; X_2 = 0.387704; \; Y = 0.302679;$$
$$\lambda_1 = 0.726132; \; \lambda_2 = 0.718542; \; \lambda_3 = -0.698659.$$

optimal intelligence level: $\mu = 0.257162$.

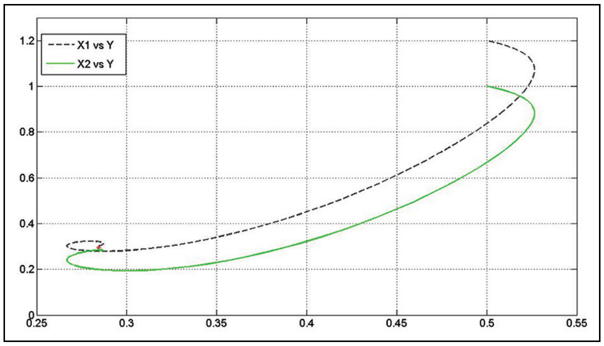

Figure 1. Phase portrait in Case 1.

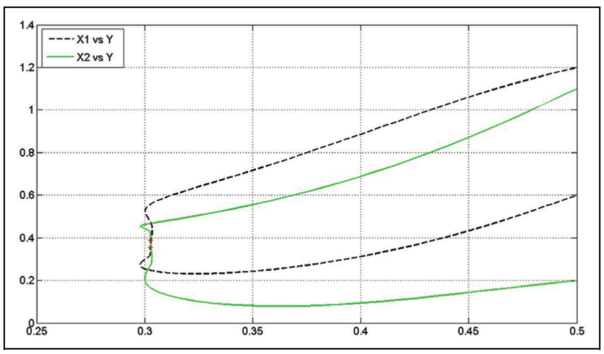

Figure 2. Phase portrait in Case 2.

Also in this case, the eigenvalues of the Jacobian are as follows:

$$4.7469; 2.8553; -2.2469; -0.3553; 1.25 \pm 1.6240i.$$

The steady states can only be reached by two stable manifolds. This fact is illustrated in Figure 2.

4.2 3 versus 1

In order to get further insights into the $(n, 1)$ model, we make two other (3, 1) test cases with similar parameters as the (2, 1) case above. Thus we consider the following:

$$\textbf{Case 3 :}\ r = 2.5; \alpha_1 = 2.0, \alpha_2 = 1.9, \alpha_3 = 2.0; \delta_1 = 0.34; \delta_2 = 0.37, \delta_3 = 0.32; \gamma_1 = 1.0, \gamma_2 = 1.3; \gamma_3 = 1.1; c_1 = c_2 = c_3 = -2.5; f = 1.2; \theta = 1.86.$$

The results are as follows.
Interior steady state:

$$X_1 = 0.14635; X_2 = 0.18627; X_3 = 0.19253; Y = 0.33134;$$

optimal intelligence level: $\mu = 0.10393$.

$$\textbf{Case 4 :}\ r = 2.3; \alpha_1 = 2.2, \alpha_2 = 2.1, \alpha_3 = 2.3; \delta_1 = 0.34; \delta_2 = 0.37, \delta_3 = 0.32; \gamma_1 = 1.5, \gamma_2 = 1.4; \gamma_3 = 1.2; c_1 = -2.6, c_2 = -2.4, c_3 = -2.5; f = 1.2; \theta = 1.86.$$

The results in Case 4 are as follows.
Interior steady state:

$$X_1 = 0.09208; X_2 = 0.16590; X_3 = 0.10893; Y = 0.29424;$$

Optimal intelligence level: $\mu = 0.10729$.

The interior steady states are states where the counter-terrorism forces and terroristic one are mutually suppressed. In these states, the terrorism force cannot expand its operation, and meanwhile the counter-terrorism forces stay the same. After investigating four cases with four interior steady states, it is apparent that the sum of X_i decreases and Y increases as n increases. This fact is an argument for the idea of cooperation in counter-terrorism, viz., counter-terrorism forces need not be too many while the terroristic one need not be eliminated to an excessively small quantity. Moreover, the optimal intelligence level is reduced, which means that the cost for collecting intelligence can be cut down. These facts suggest that the more anti-terrorism parties or nations get involved in the combat, the lower the overall costs will be.

5. Conclusions

The model under consideration is an extension of the KKS model. The results of Feichtinger et al.[10] are a particular case of ours. It is clear that cooperation in counter-terrorism is much better, at least when the costs are considered. The costs are expected to be lower if more parties take part in the combat.

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