International Journal of Mechanical Sciences 163 (2019) 105117

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

A novel differential kinematics model to compare the kinematic performances of 5-axis CNC machines

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ARTICLE INFO

Keywords:
5-axis machine comparison
Kinematic modeling
Machine manipulability
Machine dexterity
Non-linear kinematic error

ABSTRACT

A 5-axis CNC machine is similar to two cooperating robots, one robot carrying the workpiece and one robot carrying the tool. The 5-axis CNC machines are designed in a large variety of kinematic configurations and structures. Comparing different 5-axis kinematic configurations plays an important role in machine selection and optimal machine design. In this sense, the present study proposes a new mathematical model to analyze and compare the kinematic performances of the 5-axis machines. First, a generalized kinematic chain of 5-axis machine is treated as a unified kinematic chain of two collaborative robots in order to formulate a generalized differential kinematics model of the machines. Second, four important properties of the kinematics model are proved in a generalized case so that quantitative parameters characterizing the kinematic performances of the machines can be evaluated effectively. Last, six typical groups of 5-axis CNC configurations are compared through the evaluated parameters. In addition, it has been shown that, by using the properties of the kinematics model, the forward and inverse kinematic equations for the rotary axes of any 5-axis machine can be formulated in an effective and simplified manner that could be useful for developing the postprocessors for any 5-axis machine.

1. Introduction

Recently, 5-axis CNC machining has been one of the most modern and effective material removal technologies used in manufacturing industries. The 5-axis CNC machines have been used for machining typical complex parts such as molds, turbine blades, automotive and aerospace parts whose geometries are typically defined by complex surfaces.

A 5-axis CNC machine is similar to two cooperating robots [1], one robot carrying the workpiece and one robot carrying the tool. Fig. 1 shows the structure and kinematic chain diagrams of a 5-axis CNC machine Maho 600e.

A typical 5-axis mechanism consists of three prismatic joints (translational axes) X, Y and Z, and two revolute joints (rotary axes) AB, AC or BC. The three translational axes X, Y and Z represent the three orthogonal movements along with three axes of a machine coordinate system (a Cartersian coordinate system fixed to the machine base) whose Z axis is always coincides with the tool axis of a machine. The rotary axes A, B and C characterize the rotations of the machine table or the machine spindle head around the axis X, Y and Z of the machine coordinate system, respectively. Note that when a rotary axis whose centerline is parallel with one of the axes X, Y and Z is called the orthogonal rotary axis. In contrast, if the centerline of a rotary axis is inclined at an angle, it is called the non-orthogonal rotary axis.

Theoretically, by taking into account the order of joints, there will be 5! possible combinations of joint sequence for each type of 5-axis mechanism. Furthermore, each combination of joint sequence has 6 possible configurations that consist of two cooperative kinematic chains. Consequently, the number of possible configurations of 5-axis mechanism is $3 \times 5!x6=2160$. However, in practice, the rotary axes of a machine are usually implemented nearest either to the workpiece or to the tool. Therefore, the number of the possible configurations of each machine type is reduced to six: (i) two configurations consisting of both revolute joints on the workpiece carrying chain (e.g. XYZAB and XYZBA), (ii) two configurations consisting of both revolute joints on the tool carrying chain, and (iii) two configurations consisting of one rotary axis on the tool carrying chain and one rotary axis on the workpiece carrying chain. As a consequence, the number of feasible configurations of 5-axis mechanism is recalculated as $3 \times 6 \times 6 = 108$.

It is clear that the 5-axis machines can be designed in a large variety of kinematic configurations and structures. Therefore, comparison of the machines plays an important role in selecting suitable machines for applications in manufacturing industries and in developing new 5-axis CNC machines.

In recent years, some efforts have been taking place to synthesize, analyze and compare multi-axis CNC machines [3,1,4]. Yan and Chen [3] presented a general method to generate all possible configurations

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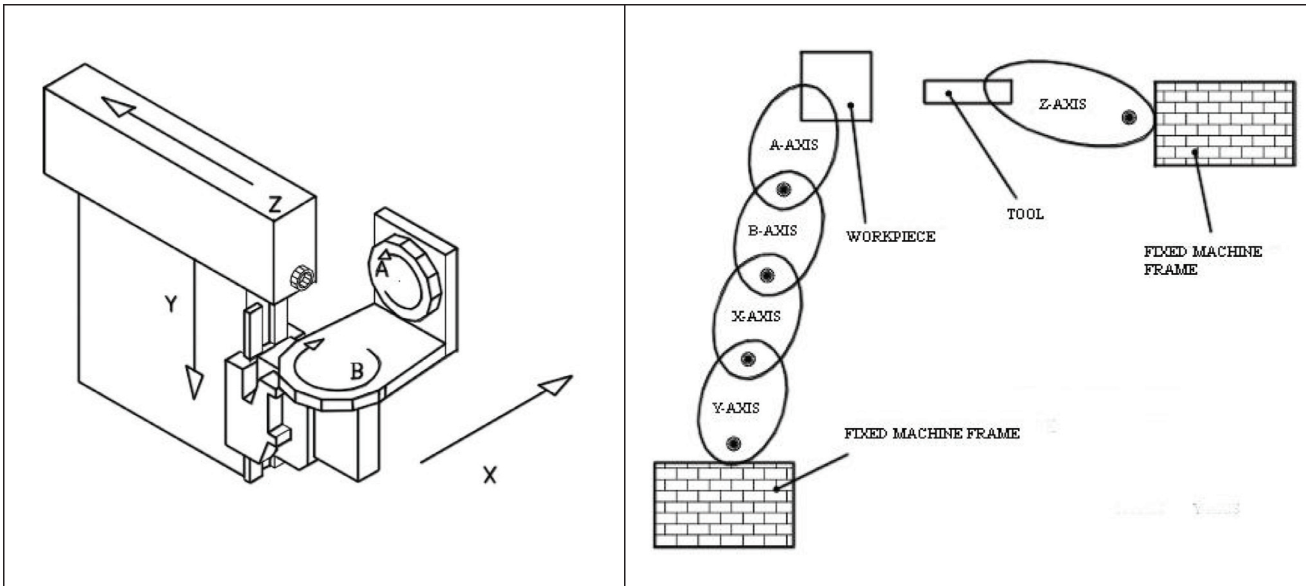
https://doi.org/10.1016/j.ijmecsci.2019.105117
Received 20 May 2019; Received in revised form 27 August 2019; Accepted 27 August 2019
Available online 29 August 2019
0020-7403/



a) A machine structure diagram [1] b) A kinematic chain diagram [1,2]

Fig. 1. The 5-axis CNC machine Maho 600e [1,2].

of machining centers which have up to seven DOFs. Bohez [1] classified, in general, the 5-axis machines into four main groups and investigated their advantages and disadvantages. However, these investigations have not discussed about the kinematic performances of the machines yet. Tutunea-Fatan and Bhuiya [4] compared the 5-axis CNC machines through the nonlinearity errors. However, this study mainly focused on the 5-axis machine type with two orthogonal rotary axes implemented on the spindle head only. Comparison of the 5-axis machines with respect to important kinematic characteristics such as the manipulability of a machine and the flexibility of the tool-workpiece orientation has been overlooked. Note that the manipulability of a machine and the flexibility of the tool-workpiece orientation play an important role in comparing the performances of the 5-axis CNC machines. The manipulability of a machine characterizes the tendency of changes in dexterity characteristics along with the variance of the motion of the axes, and it indicates of how close the machine configuration is to the singularity. The flexibility of the tool-workpiece orientation implies how and under which angle the tool can orient relative to the workpiece in the workspace of a machine. The more the manipulability and the flexibility of the tool-workpiece orientation of a machine are, the more complex parts the machine can machine with a high performance. A 5-axis machine of high dexterity is capable of machining complex parts consisting numerous sculptured surfaces with reduced setup times so that it can increase the productivity of a manufacturing system.

To compare the kinematic performances of the 5-axis machines effectively, a generalized differential kinematics model of the machines is often required. However, most of the previous investigations on the kinematic modeling of 5-axis machine mainly focused on individual types of the machines.

Decades ago, there have been efforts working on the kinematic modeling of the 5-axis machines with two orthogonal rotary axes. Xu et al. [5] analyzed the 5-axis kinematics model with the purpose of minimizing the angular acceleration of the rotary axes. Munlin et al. [6] and Munlin and Makhanov [7] focused on a kinematics model of the 5-axis machine Maho 600e when investigating the optimization of the cutter rotations near singular points. Lee and She [8] formulated kinematic equations for the table- tilting machines, the spindle - tilting machines and the table/spindle - tilting machines. Xu et al. [9] developed a kinematics model incorporated with the tool inclinations for the machine type XYZAC. Farouki et al. [10] investigated the optimal tool orientation control with the use of the inverse kinematics for the rotary axes AC on the workpiece carrying chain, and AB on the tool carrying chain. Lavernhe et al. [11] concentrated on the kinematic behavior of the Mikron milling center (XYZAC). Wu et al. [12] and Yun et al. [13] investigated the kinematic modeling of individual 5-axis machines with both orthogonal rotary axes. With the purpose of postprocessor development, Jung et al. [14] and, Boz and Lazoglu [15] formulated the kinematic equations for the table-tilting type 5-axis machines as well.

In recent years, there have also been some studies that focused on the kinematic modeling of the 5-axis machines which consist of non-orthogonal rotary axes. My [2] and, My and Bohez [16] investigated a kinematics model of the nutating table 5-axis machines DMU 50e and DMU 70e for the postprocessor development and the kinematic error minimization. Sørby [17] and, She and Huang [18] studied the forward and inverse kinematic equations for the nutating table and nutating spindle 5-axis machines also. Liu et al. [19] formulated the kinematic equations with the purpose of identification of geometric errors of rotary axes in 5-axis machine tools with non-orthogonal rotary axes on the table. Wang et al. [20] investigated the kinematic modeling of a 5-axis CNC machine with one orthogonal rotary axis on the table and one non-orthogonal rotary axis on the tool chain.

Apart from the aforementioned works, there have been attempts that emphasized on the generalization of the kinematics model for the 5-axis CNC machines [21–25]. She and Lee [25] proposed a postprocessor for general 5-axis machines, using the kinematics module, which added two rotary movements on the workpiece table and two rotary movements on the spindle. Tutunea-Fatan and Feng [24] derived a general coordinate transformation matrix for all 5-axis machines with two rotary axes. The model was then used to verify the feasibility of the two rotary joints within the kinematics chain of three main types of 5-axis CNC machines. She and Chang [22] did further research on the basis of [23] by extending the inverse kinematics solution for translational motions in a unified form. Yang and Altintas [23], and Liu et al. [21] presented a generalized kinematics model using Screw theory. Note that the kinematic equations proposed in [25,22] were expressed in terms of seven generalized coordinates since two more revolute joints were added on the kinematic chain of a general 5-axis CNC mechanism; the kinematic equations in [21,23] were represented in the form of a product of exponential functions. With the purpose of comparing the kinematic efficiency of the 5-axis CNC machines, the use of such kinematics models to formulate the differential kinematic equations and evaluate the kinematic performances of a general 5-axis configuration is challenging.

The above raised critical issues lead to the motivations of developing a new method to evaluate and compare the kinematic performances of the 5-axis CNC machines. In this paper, a generalized differential kinematics model of the 5-axis CNC machines is formulated, where a general mechanism of 5-axis CNC machines is treated as a closed loop mechanism of two cooperating robot arms. Four important properties of the generalized kinematics model of the 5-axis CNC machines are proved in a generalized case so that the manipulability index, the non-singular range of the five joint variables, the dexterity index, the condition number, and the non-linear kinematic error of the machines can be evaluated effectively. Based on such the evaluation of the indicators, six typical types of 5-axis CNC configurations are compared. It was demonstrated that, with the four important properties proved, the kinematics model formulated in this study is advantageous and effective when compared with the previous models. It was also shown that the comparison of the machines is useful when selecting suitable machines for given applications, especially when analysing new conceptual designs of a 5-axis CNC machine. In addition, by using the properties of the proposed kinematics model, the inverse kinematic equations for any 5-axis CNC machines are derived in a simplified and generalized manner. This is very useful and effective when working on the kinematic performance analysis as well as the postprocessor development for any 5-axis CNC machines.

2. Formulation of a generalized kinematics model of the 5-axis machines

In this section, the kinematic equations at position level and velocity level are formulated for a generalized kinematic chain of the 5-axis CNC machines. In particular, four important properties of the kinematic equations are proved. These useful properties will be taken full advantages when evaluating and comparing the kinematic performances of the 5-axis CNC machines.

Let us consider a general 5-axis mechanism that consists of three prismatic joints (X, Y and Z) and two revolute joints (AB/AC/BC). This mechanism is a type of 5 DOFs closed-loop mechanism which is synthesized with two collaborative kinematic chains. The two chains are constrained via a planned tool path during the machining process. A general kinematic diagram of the 5-axis CNC machines is presented in Fig. 2.

Let $\mathbf{q} = [\ q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5\ ]^T$ ($q_i \in \{X, Y, Z, A, B, C\}$) denote a vector of the generalized coordinates of the system, $O_0x_0y_0z_0$ is a reference coordinate system, $O_tx_ty_tz_t$ is a tool coordinate system, and $O_wx_wy_wz_w$ is a workpiece coordinate system. The tool coordinate system $O_tx_ty_tz_t$ is located at the tooltip and oriented parallel with the machine coordinate system $O_0x_0y_0z_0$. The workpiece coordinate system $O_wx_wy_wz_w$ is usually placed on the workpiece and oriented parallel with $O_0x_0y_0z_0$ as well.

In the viewpoint of the multibody system dynamics, the motion of each joint of a 5-axis CNC machine can be described by a homogeneous transformation matrix as follows:

$$\mathbf{H}_i(q_i) = \begin{cases} \begin{bmatrix} \mathbf{E} & \mathbf{t}_i \\ 0 & 1 \end{bmatrix}, & \text{for a prismatic joint } i \\ \begin{bmatrix} \mathbf{S}_i & \boldsymbol{\tau}_i \\ 0 & 1 \end{bmatrix}, & \text{for a revolute joint } i \end{cases} \tag{1}$$

where $\mathbf{E}$ is a 3×3 identity matrix, and $\mathbf{t}_i$ is a translation vector describing the motion of a prismatic joint i. When$q_i = X$, the vector $\mathbf{t}_i$ can be written as $\mathbf{t}_i = [\ X + X_0 \quad 0 \quad 0\ ]^T$. When $q_i = Y$ or $q_i = Z$ the vector $\mathbf{t}_i$is represented as $\mathbf{t}_i = [\ 0 \quad Y + Y_0 \quad 0\ ]^T$or $\mathbf{t}_i = [\ 0 \quad 0 \quad Z + Z_0\ ]^T$, respectively. X_0, Y_0 and Z_0 are the initial values of the prismatic joint

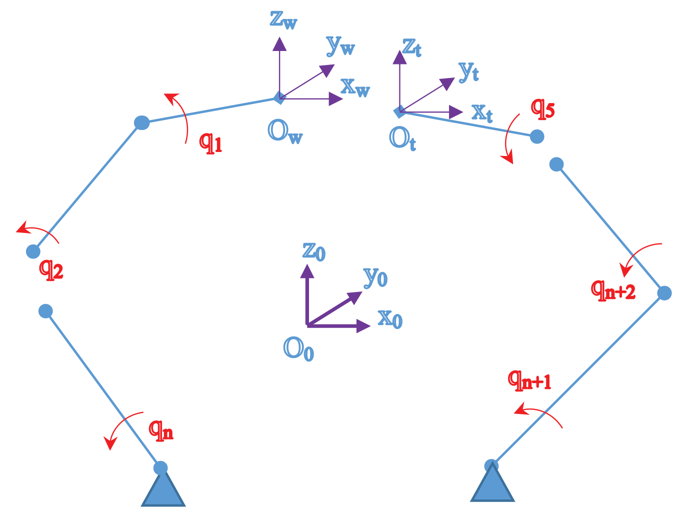


Fig. 2. A general kinematic diagram of the 5-axis CNC machines.

variables (the initial position of the pointO_t in the coordinate system $O_0x_0y_0z_0$). $\boldsymbol{\tau}_i$ represents the offset distance between the centerline of a revolute joint q_i (A, B or C) and a corresponding axis of the machine coordinate system (O_0x_0, O_0y_0 or O_0z_0). The rotation matrix $\mathbf{S}_i$ characterizing the motion of a revolute joint i can be written as follows:

$$\mathbf{S}_i(q_i = A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos A & -\sin A \\ 0 & \sin A & \cos A \end{bmatrix} \tag{2}$$

$$\mathbf{S}_i(q_i = B) = \begin{bmatrix} \cos B & 0 & \sin B \\ 0 & 1 & 0 \\ -\sin B & 0 & \cos B \end{bmatrix} \tag{3}$$

$$\mathbf{S}_i(q_i = C) = \begin{bmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Note that if the centerline of a revolute joint q_i is inclined at an angle α (the non-orthogonal rotary axisq_i), and a matrix $\mathbf{S}_R(\alpha)$ represents the rotation of the centerline, the matrix $\mathbf{S}_i(q_i)$must be additionally multiplied by $\mathbf{S}_R(\alpha)$ and $\mathbf{S}_R(-\alpha)$ in the left and right sides of $\mathbf{S}_i(q_i)$, respectively.

In the reference frame $O_0x_0y_0z_0$, the cumulative transformation matrices for the tool carrying chain and for the workpiece carrying chain are calculated as follows, respectively:

$$\mathbf{H}_{0t} = \mathbf{H}_{n+1}(q_{n+1})\mathbf{H}_{n+2}(q_{n+2})...\mathbf{H}_5(q_5) \tag{5}$$

$$\mathbf{H}_{0w} = \mathbf{H}_n(q_n)\mathbf{H}_{n-1}(q_{n-1})...\mathbf{H}_1(q_1) \tag{6}$$

For the cases in which the last joint q_5 is a revolute joint, the transformation matrix $\mathbf{H}_5(q_5)$ must be multiplied by a transformation matrix characterizing the distance between the tooltip and the centerline of the joint q_5.

One more interesting feature of the mechanism under consideration is that if the two kinematic chains are unified through the reference frame $O_0x_0y_0z_0$, the closed loop mechanism of the two chains becomes a serial open mechanism of a single kinematic chain. The joint sequence of the unified mechanism is q_1,q_2,q_3,q_4 andq_5. The motion of the tool relative to the workpiece is thus described by the following kinematic relationship:

$$\begin{aligned} \mathbf{H}_{wt} &= (\mathbf{H}_{0w})^{-1}\mathbf{H}_{0t} \\ &= \mathbf{H}_1^{-1}(q_1)....\mathbf{H}_n^{-1}(q_n)\mathbf{H}_{n+1}(q_{n+1})...\mathbf{H}_5(q_5) \end{aligned} \tag{7}$$

Since $n \in \{0, ..., 5\}$, the matrix $\mathbf{H}_{wt}$ can be rewritten as follows:

$$\mathbf{H}_{wt} = \boldsymbol{\Theta}_1(q_1)\boldsymbol{\Theta}_2(q_2)\boldsymbol{\Theta}_3(q_3)\boldsymbol{\Theta}_4(q_4)\boldsymbol{\Theta}_5(q_5) \tag{8}$$

where

$$\mathbf{\Theta}_i(q_i) = \left|\begin{array}{l}\begin{bmatrix}\mathbf{E} & \mathbf{T}_i\\ 0 & 1\end{bmatrix}, \text{ for a prismatic joint } i\\ \begin{bmatrix}\mathbf{R}_i & \boldsymbol{\tau}_i\\ 0 & 1\end{bmatrix}, \text{ for a revolute joint } i\end{array}\right. \tag{9}$$

With respect to all the joints of the tool carrying chain,

$$\begin{array}{l}\mathbf{T}_i = \mathbf{t}_i, \quad \text{for a prismatic joint } i\\ \mathbf{R}_i = \mathbf{S}_i, \quad \text{for a revolute joint } i\end{array} \tag{10}$$

With respect to all the joints of the workpiece carrying chain,

$$\begin{array}{l}\mathbf{T}_i = -\mathbf{t}_i, \quad \text{for a prismatic joint } i\\ \mathbf{R}_i = \mathbf{S}_i^T, \quad \text{for a revolute joint } i\end{array} \tag{11}$$

Eq. (11) implies that all the transformation matrices describing the motion of the joints of the workpiece carrying chain must be inversed because the unified kinematic chain starts at the workpiece and ends at the tool.

Finally, the kinematics model of the machine is formulated as follows:

$$\mathbf{H}_{wt} = \begin{bmatrix}\mathbf{R}_{wt} & \mathbf{p}_T\\ 0 & 1\end{bmatrix} \tag{12}$$

where $\mathbf{p}_T = [\ x \quad y \quad z\ ]^T$ is the position of the tooltip in the workpiece coordinate system $O_w x_w y_w z_w$. The three direction cosines of the tool axis vector i, j and k are the three entries of the last column of the matrix $\mathbf{R}_{wt}$.

Let's denote $\mathbf{X} = [\ x \quad y \quad z \quad i \quad j \quad k\ ]^T$ as the tool posture, the forward kinematic equation of the system can be written as follows:

$$\mathbf{X} = \mathbf{f}(\mathbf{q}) \tag{13}$$

In Eq. (13), $\mathbf{X}$ is the so called the cutter location point (CL point) which is usually calculated by CAD/CAM systems when planning a tool path for the 5-axis CNC machining. To produce a G-codes file for controlling an individual machine, the following inverse kinematic equation must be solved for $\mathbf{q}$.

$$\mathbf{q} = \mathbf{f}^{-1}(\mathbf{X}) \tag{14}$$

Note that Eqs. (13 and 14) are the kinematic equations at position level that have often used for postprocessor development for the 5-axis machines. With the purpose of evaluating the kinematic performances of the 5-axis machines, both the kinematic equations at position level and the kinematic equations at velocity level are required. Unfortunately, with the six dependent equations in Eq. (13), it is impossible to formulate the inverse differential kinematic equations that can be used to evaluate the kinematic performances of the machines. Eq. (13) is composed of five independent equations and one dependent equation, since $i^2+j^2+k^2=1$. Therefore, Eq. (13) needs to be transformed into a set of all five independent kinematic equations.

Let q_u and q_v be the joint variables of the primary revolute joint and the secondary revolute joint, where $u < v$ and $(u, v) \in \{1, 2, 3, 4, 5\}$. Note that the condition $u < v$ implies that the revolute joint q_u always precedes the revolute joint q_v in any joint orders of the unified kinematic chain. For all configurations of the unified kinematic chain, the joint $q_u$is always closer to the wotkpiece than the joint q_v, and it is called the primary revolute joint. The joint $q_v$is the secondary revolute joint.

Let

$$\mathbf{q}_T = [\ X \quad Y \quad Z\ ]^T,$$

and

$$\mathbf{q}_R = [\ q_u \quad q_v\ ]^T$$

be the vectors of the three prismatic joint variables and the two revolute joint variables, respectively.

Table 1
Constant matrix $\mathbf{\Phi}$.

q_u	A	B	C
$\mathbf{\Phi}$	$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

Let

$$\mathbf{p}_R = [\ \phi \quad \varphi\ ]^T$$

denote the orientation of the tool axis.

The parameters $\varphi$and ϕ are the two independent direction cosines selected from three dependent direction cosines (i,j and$k$) of the tool axis vector. The parameters $\varphi$and ϕ must be selected so that both of them are expressed in terms of both the variables $q_u$and q_v.

$$\mathbf{p}_R = \mathbf{\Phi}[\ i \quad j \quad k\ ]^T, \tag{15}$$

where $\mathbf{\Phi}$ is a constant matrix in Table 1.

For example, when the primary revolute joint $q_u$is the A-axis, and the secondary revolute joint q_v is the B-axis, $\varphi = k = \cos q_u \cos q_v$, and $\phi = j = -\sin q_u \cos q_v$.

Rewriting Eq. (13) in a form of five independent equations yields

$$\mathbf{p} = \mathbf{g}(\mathbf{q}), \tag{16}$$

where

$$\mathbf{p} = [\ \mathbf{p}_T \quad \mathbf{p}_R\ ]^T, \tag{17}$$

and

$$\mathbf{q} = [\ \mathbf{q}_T \quad \mathbf{q}_R\ ]^T. \tag{18}$$

Thus, the differential kinematic equation for the 5-axis machines can be written as follows:

$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}}, \tag{19}$$

where $\mathbf{J}_{5\times 5}$ is the Jacobian matrix.

$$\mathbf{J} = \begin{bmatrix}\frac{\partial \mathbf{p}_T}{\partial \mathbf{q}_T} & \frac{\partial \mathbf{p}_T}{\partial \mathbf{q}_R}\\ \frac{\partial \mathbf{p}_R}{\partial \mathbf{q}_T} & \frac{\partial \mathbf{p}_R}{\partial \mathbf{q}_R}\end{bmatrix} = \begin{bmatrix}\mathbf{J}_{TT} & \mathbf{J}_{TR}\\ \mathbf{J}_{RT} & \mathbf{J}_{RR}\end{bmatrix} \tag{20}$$

Eq. (19) is the differential kinematic equation that relates the joint velocities, the tool velocity and the Jacobian matrix which characterize the structure of the machines.

It is worth noting that all the 5-axis mechanisms have some important common features as follows.

The first feature is that the three translational axes X, Y and Z are orthogonal each other, which are aligned with three axes of the machine coordinate system $O_0x_0y_0z_0$. Particularly, in this study, the defined coordinate systems $O_tx_ty_tz_t$ and $O_wx_wy_wz_w$ are always parallel with the machine coordinate system $O_0x_0y_0z_0$.

The second one is that the tool axis vector always points in the direction of the axis $O_tz_t$of the tool coordinate system $O_tx_ty_tz_t$.

The third one is that the rotary axes A, B and C imply the rotations of the machine table or the machine spindle head around the axes O_0x_0, O_0y_0 and $O_0z_0$of the machine coordinate system $O_0x_0y_0z_0$, respectively.

With respect to the three features above mentioned, some important properties of the kinematics model of the general machine can be proved that are very useful when working on the modeling and analysis of the machine kinematic performances.

Property 1. For all configurations of 5-axis CNC machine, the forward kinematic equations for the two rotary axes can be formulated directly with the two rotation matrices, regardless of where the primary and the secondary rotary joints are in the generalized kinematic chain of the 5-axis CNC machines.

In other words, the tool orientation vector $\mathbf{p}_R$ can be calculated with Eq. (21), and the direction cosines *i, j* and *k* are calculated with Eq. (22). Both the calculations are independent of the three prismatic joints variables X, Y and Z.

$$\mathbf{p}_R = \boldsymbol{\Phi}\mathbf{R}_u\mathbf{R}_v\boldsymbol{\Gamma} \tag{21}$$

$$\begin{bmatrix} i & j & k \end{bmatrix}^T = \mathbf{R}_u\mathbf{R}_v\boldsymbol{\Gamma} \tag{22}$$

where $\mathbf{R}_u$ and $\mathbf{R}_v$are the two rotation matrices describing the motion of the primary revolute joint (q_u) and the secondary revolute joint (q_v), accordingly. $\boldsymbol{\Gamma} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

The use of this property will reduce the computational complexity when formulating and analyzing the Jacobian matrix **J**, the manipulability index, the dexterity index, and the conditioning index for the 5-axis CNC machines that will be presented in the next section.

Proof. Based on the rules of the block matrix multiplication, the following block matrix multiplications can be obtained:

$$\begin{bmatrix} \mathbf{E} & \mathbf{T}_{i-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{E} & \mathbf{T}_{i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{T}_{T} \\ 0 & 1 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \mathbf{R}_i & \boldsymbol{\tau}_{i} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{E} & \mathbf{T}_{i+1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i & \mathbf{R}_i\mathbf{T}_{i+1} + \boldsymbol{\tau}_i \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} \mathbf{E} & \mathbf{T}_{i-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_i & \boldsymbol{\tau}_{i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i & \mathbf{T}_{i-1} + \boldsymbol{\tau}_i \\ 0 & 1 \end{bmatrix} \tag{25}$$

It can be seen from Eq. (23) that multiplying two translational transformation matrices results in a transformation matrix in the same form of the multiplied matrices. Eqs. (24 and 25) show that multiplying a translational transformation matrix with a rotational transformation matrix, in a different order, the rotation block matrix $\mathbf{R}_i$ in the rotational transformation matrix is not transformed. Consequently, Eq. (12) can be rewritten as follows:

$$\begin{aligned} \mathbf{H}_{wt} &= \begin{bmatrix} \mathbf{R}_u & \mathbf{T}_{Ru} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_v & \mathbf{T}_{Rv} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_u\mathbf{R}_v & \mathbf{R}_u\mathbf{T}_{Rv} + \mathbf{T}_{Ru} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_u\mathbf{R}_v & \mathbf{p}_T \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{26}$$

In Eq. (26) $\mathbf{T}_{Ru}$ and $\mathbf{T}_{Rv}$ are the vectors yielded by the multiplications of a rotational transformation matrix with the translational transformation matrices, respectively.

Since the tool axis vector points in the axis $z_t$of the coordinate system $O_tx_ty_tz_t$, its direction cosines *i, j* and *k* are the three entries of the last row of the rotation matrix $\mathbf{R}_u\mathbf{R}_v$. Therefore,

$$\begin{bmatrix} i & j & k \end{bmatrix}^T = \mathbf{R}_u\mathbf{R}_v\boldsymbol{\Gamma} \tag{27}$$

Substituting Eq. (27) into Eq. (15) yields Eq. (21) and completes the proof.

It is also important to note that, $\mathbf{p}_R$ calculated with Eq. (21) is a function of only $\mathbf{q}_R = \begin{bmatrix} q_u & q_v \end{bmatrix}^T$, and it is independent of $\mathbf{q}_T = \begin{bmatrix} X & Y & Z \end{bmatrix}^T$. As a consequence of Property 1,

$$\mathbf{J}_{RT} = \frac{\partial \mathbf{p}_R}{\partial \mathbf{q}_T} = \mathbf{0} \tag{28}$$

Eq. (28) is an important consequence of Property 1 that is very useful when calculating the Jacobian determinant presented later on.

Property 2. The determinant of the Jacobian matrix **J**of the generalized 5-axis kinematics model can be directly calculated by using only the kinematics sub-model of the two rotary axes. In other words,

$$Det(\mathbf{J}) = Det(\mathbf{J}_{RR}). \tag{29}$$

To evaluate the kinematic performances of a 5-axis machine, the Jacobian determinant is often needed. However, formulation of the Jacobian determinant in a generalized case is challenging, since the matrix **J** has a dimension of 5×5, and the determinant is a function of all five joint variables. Hence, this property is important which makes it possible to formulate the Jacobian determinant for the machines. By using Property 2, $Det(\mathbf{J}_{5\times5})$can be formulated as the determinant of a matrix with a dimension of 2×2 only.

Workpiece → Tool

S1	q_u	q_v	q_3	q_4	q_5
S2	q_u	q_2	q_3	q_v	q_5
S3	q_u	q_2	q_3	q_4	q_v
S4	q_1	q_2	q_u	q_4	q_v
S5	q_1	q_2	q_3	q_u	q_v
S6	q_1	q_u	q_3	q_v	q_5
S7	q_1	q_2	q_u	q_v	q_5

Fig. 3. Seven sequences of the joint variables.

Proof. As discussed in the Proof of Property 1, it is clearly seen that all the transformation matrices $\boldsymbol{\Theta}_i(q_i)$in Eq. (8) are expressed as particular block matrices. Hence, multiplying two or three successive translational transformation matrices, in different orders, yields a matrix in the same form of the multiplied matrices. However, multiplication of a translational transformation matrix with a rotational transformation matrix is not commutative. Thus, the result of the matrix chain multiplication in Eq. (8) depends on where the two rotational transformation matrices are in the matrix chain. Theoretically, there exist seven different sequences of the joint variables (Fig. 3) that correspond to seven different results of the matrix chain multiplication.

The family of the 5-axis machines with both rotary axes on the table is represented by the sequence of the joint variables S1. The sequence S3 represents the group of 5-axis machines with the primary rotary axis on the table, and the secondary one on the spindle head. The sequence S5 denotes the 5-axis machines with both rotary axes on the tool chain. The sequence S2 represents for all the cases in which the rotational transformation matrix $\boldsymbol{\Theta}_v(q_v)$ is in between two translational transformation matrices, meanwhile the matrix $\boldsymbol{\Theta}_u(q_u)$ is the first matrix of the matrix chain. Similarly, as for the sequence S6, both the matrices $\boldsymbol{\Theta}_u(q_u)$ and $\boldsymbol{\Theta}_v(q_v)$ are in between a couple of translational transformation matrices; the sequence S7 implies that $\boldsymbol{\Theta}_u(q_u)$ and $\boldsymbol{\Theta}_v(q_v)$ are adjacent, but both the matrices are in the middle of the matrix chain.

With the seven chains of the transformation matrices, the seven results of the matrix chain multiplication can be archived accordingly, where the tooltip position $\mathbf{p}_T = \begin{bmatrix} x & y & z \end{bmatrix}^T$ is expressed as follows:

$$
\begin{aligned}
&\text{S1. } \mathbf{p}_T = \mathbf{R}_u\mathbf{R}_v\mathbf{q}_T + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u\\
&\text{S2. } \mathbf{p}_T = \mathbf{R}_u\mathbf{R}_v\mathbf{q}_T - \mathbf{R}_u\mathbf{R}_v(\mathbf{T}_2+\mathbf{T}_3) + \mathbf{R}_u(\mathbf{T}_2+\mathbf{T}_3) + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u\\
&\text{S3. } \mathbf{p}_T = \mathbf{R}_u\mathbf{q}_T + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u + \mathbf{R}_u\mathbf{R}_v\boldsymbol{\tau}_t\\
&\text{S4. } \mathbf{p}_T = \mathbf{R}_u\mathbf{q}_T + (\mathbf{E}-\mathbf{R}_u)(\mathbf{T}_1+\mathbf{T}_2) + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u + \mathbf{R}_u\mathbf{R}_v\boldsymbol{\tau}_t\\
&\text{S5. } \mathbf{p}_T = \mathbf{q}_T + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u + \mathbf{R}_u\mathbf{R}_v\boldsymbol{\tau}_t\\
&\text{S6. } \mathbf{p}_T = \mathbf{R}_u\mathbf{R}_v\mathbf{q}_T - \mathbf{R}_u\mathbf{R}_v(\mathbf{T}_1+\mathbf{T}_3) + \mathbf{R}_u\mathbf{T}_3 + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u + \mathbf{T}_1\\
&\text{S7. } \mathbf{p}_T = \mathbf{R}_u\mathbf{R}_v\mathbf{q}_T + (\mathbf{E}-\mathbf{R}_u\mathbf{R}_v)(\mathbf{T}_1+\mathbf{T}_2) + \mathbf{R}_u\boldsymbol{\tau}_v + \boldsymbol{\tau}_u
\end{aligned}
\tag{30}
$$

It is clear that, except from the Jacobian determinant of the first term of all the expressions (S1-S7), the Jacobian determinant of all the other terms equal to zero because the Jacobian matrix of a constant vector is a zero matrix, and the determinant of a Jacobian matrix containing one zero-column equals to zero.

Consequently, for the expressions S1, S2, S6 and S7, the Jacobian $\mathbf{J}_{TT}$ can be calculated with Eq. (31). For S3 and S4, $\mathbf{J}_{TT}$ is calculated with Eq. (32), and for S5, $\mathbf{J}_{TT}$ is calculated with Eq. (33).

$$
\begin{aligned}
\mathbf{J}_{TT} &= \frac{\partial \mathbf{p}_T}{\partial \mathbf{q}_T} = \frac{\partial(\mathbf{R}_u\mathbf{R}_v\mathbf{q}_T)}{\partial \mathbf{q}_T}\\
&= \mathbf{R}_u\mathbf{R}_v
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\mathbf{J}_{TT} &= \frac{\partial \mathbf{p}_T}{\partial \mathbf{q}_T} = \frac{\partial(\mathbf{R}_u\mathbf{q}_T)}{\partial \mathbf{q}_T}\\
&= \mathbf{R}_u
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\mathbf{J}_{TT} &= \frac{\partial \mathbf{p}_T}{\partial \mathbf{q}_T} = \frac{\partial(\mathbf{q}_T)}{\partial \mathbf{q}_T}\\
&= \mathbf{E}
\end{aligned}
\tag{33}
$$

Note that $Det(\mathbf{R}_u) = Det(\mathbf{R}_v) = Det(\mathbf{E}) = 1$. Hence, for all the cases

$$
Det(\mathbf{J}_{TT}) = 1 \tag{34}
$$

On the other hand, the determinant of the block matrix $\mathbf{J}$ can be calculated as follows:

$$
Det(\mathbf{J}) = Det(\mathbf{J}_{TT}\mathbf{J}_{RR} - \mathbf{J}_{TR}\mathbf{J}_{RT}) \tag{35}
$$

Substituting Eqs. (28) and(34) into Eq. (35) yields Eq. (29) and completes the proof.

Property 3. The Jacobian determinant $Det(\mathbf{J})$, a function $f_J(\mathbf{q})$of a multi-variables vector $\mathbf{q}$, can be transformed and expressed in terms of only one joint variable q_v. In other words,

$$
Det(\mathbf{J}) = f_J(q_v). \tag{36}
$$

Property 3 is important when solving $Det(\mathbf{J}(\mathbf{q})) = 0$ for $\mathbf{q}$.

Proof. It is worth to note that $\mathbf{R}'_u = \mathbf{R}_u\boldsymbol{\Lambda}_u$, and $\mathbf{R}'_v = \mathbf{R}_v\boldsymbol{\Lambda}_v$, where $\boldsymbol{\Lambda}_u$and$\boldsymbol{\Lambda}_v$ are constant matrices which can be looked up in the following Table 2.

Based on Eq. (21), the Jaocobian matrix $\mathbf{J}_{RR} = [\frac{\partial \mathbf{p}_R}{\partial \mathbf{q}_R}]_{2\times 2}$ can be formulated as a block matrix multiplication as follows:

$$
\mathbf{J}_{RR} = [\boldsymbol{\Phi}\mathbf{R}_u]_{2\times 3}[\ \boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\Gamma} \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\Gamma}\ ]_{3\times 2} \tag{37}
$$

Since $\boldsymbol{\Phi}\boldsymbol{\Phi}^T = \mathbf{E}_{2\times 2}$,

$$
\begin{aligned}
\mathbf{J}_{RR} &= [\boldsymbol{\Phi}\mathbf{R}_u\boldsymbol{\Phi}^T]_{2\times 2}[\ \boldsymbol{\Phi}\boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}\mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\Gamma}\ ]_{2\times 2}\\
&= \mathbf{H}_u\mathbf{K}_v
\end{aligned}
\tag{38}
$$

Note that $\mathbf{H}_u = \boldsymbol{\Phi}\mathbf{R}_u\boldsymbol{\Phi}^T = [\begin{smallmatrix}\cos q_u & \mp\sin q_u\\ \pm\sin q_u & \cos q_u\end{smallmatrix}]$ is an orthogonal matrix formulated with respect to $q_u$only. Therefore, $Det(\mathbf{H}_u) = 1$.

Table 2
Constant matrices $\boldsymbol{\Lambda}_u$ and $\boldsymbol{\Lambda}_v$.

q_i	A	B	C
$\boldsymbol{\Lambda}_i$	$\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\-1&0&1\end{bmatrix}$	$\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$

$\mathbf{K}_v = [\ \boldsymbol{\Phi}\boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}\mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\Gamma}\ ]$ is a matrix formulated with respect to $\mathbf{R}_v(q_v)$ only. Therefore, the determinant $Det(\mathbf{K}_v)$ is a function of only one variable q_v, $Det(\mathbf{K}_v) = f_J(q_v)$.

Finally, Eq. (29) can be transformed as follows:

$$
\begin{aligned}
Det(\mathbf{J}) &= Det(\mathbf{H}_u)\,Det(\mathbf{K}_v)\\
&= f_J(q_v)
\end{aligned}
\tag{39}
$$

Eq. (39) completes the proof.

Property 4. For all 5-axis mechanisms, the condition number of the Jacobian matrix, $\kappa(\mathbf{J})$, does not depends on the primary revolute joint variable q_u. Particularly, the condition number for all the machines with both rotary axes implemented on the tool carrying chain (the spindle – tilting machines) depends on only one joint variable q_v.

With the purpose of comparing the kinematic performances of different 5-axis CNC machines, the condition number $\kappa(\mathbf{J})$ must be taken into account. Generally, $\kappa(\mathbf{J})$ depends on the variation of all five joint variables of a machine. Hence, comparing $\kappa(\mathbf{J})$ of all different 5-axis machines is challenging. Therefore, the use of this important property will makes possible the evaluation and comparison of the condition numbers for the machines.

Proof. Due to the fact that if the matrix $\mathbf{J}$ can be factorized into two matrices, where the first matrix is an orthogonal matrix, and the second one is a matrix independent ofq_u, the eigenvalues of the second matrix will be the singular values of $\mathbf{J}$ ($\sigma_1 \div \sigma_5$), and the condition number $\kappa(\mathbf{J}) = \sigma_{\max}/\sigma_{\min}$ is thus independent ofq_u.

Revisiting Eq. (30), for the expressions S1, S2, S3, S4, S6 and S7, the block matrix $\mathbf{J}$ can be derived and factorized with Eq. (40). For S5, $\mathbf{J}$ can be factorized with Eq. (41).

$$
\mathbf{J} = \begin{bmatrix}\mathbf{R}_u & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_u\end{bmatrix}\begin{bmatrix}\mathbf{R}_v & \mathbf{L}_v\\ \mathbf{0} & \mathbf{K}_v\end{bmatrix} \tag{40}
$$

$$
\mathbf{J} = \begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_u\end{bmatrix}\begin{bmatrix}\mathbf{E} & (\mathbf{R}_u\mathbf{M}_v - \mathbf{E}\mathbf{M}_v)\mathbf{N}_v^{-1}\\ \mathbf{0} & \mathbf{E}\end{bmatrix}\begin{bmatrix}\mathbf{E} & \mathbf{M}_v\\ \mathbf{0} & \mathbf{N}_v\end{bmatrix} \tag{41}
$$

The block $\mathbf{L}_v$ in Eq. (40) is calculated as follows:

$$
\begin{aligned}
&\text{S1. } \mathbf{L}_v = [\ \boldsymbol{\Lambda}_u\mathbf{R}_v\mathbf{q}_T + \boldsymbol{\Lambda}_u\boldsymbol{\tau}_v \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\mathbf{q}_T\ ]\\
&\text{S2. } \mathbf{L}_v = \begin{bmatrix}\boldsymbol{\Lambda}_u\mathbf{R}_v\mathbf{q}_T - \boldsymbol{\Lambda}_u\mathbf{R}_v(\mathbf{T}_2+\mathbf{T}_3) + \boldsymbol{\Lambda}_u(\mathbf{T}_2+\mathbf{T}_3) \\ +\boldsymbol{\Lambda}_u\boldsymbol{\tau}_v \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\mathbf{q}_T - \mathbf{R}_v\boldsymbol{\Lambda}_v(\mathbf{T}_2+\mathbf{T}_3)\end{bmatrix}\\
&\text{S3. } \mathbf{L}_v = [\ \boldsymbol{\Lambda}_u\mathbf{q}_T + \boldsymbol{\Lambda}_u\boldsymbol{\tau}_v + \boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\tau}_t \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\tau}_t\ ]\\
&\text{S4. } \mathbf{L}_v = [\ \boldsymbol{\Lambda}_u\mathbf{q}_T + \boldsymbol{\Lambda}_u(\mathbf{T}_1+\mathbf{T}_2) + \boldsymbol{\Lambda}_u\boldsymbol{\tau}_v + \boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\tau}_t \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\tau}_t\ ]\\
&\text{S6. } \mathbf{L}_v = \begin{bmatrix}\boldsymbol{\Lambda}_u\mathbf{R}_v\mathbf{q}_T - \boldsymbol{\Lambda}_u\mathbf{R}_v(\mathbf{T}_1+\mathbf{T}_3) + \boldsymbol{\Lambda}_u\mathbf{T}_3 \\ +\boldsymbol{\Lambda}_u\boldsymbol{\tau}_v \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\mathbf{q}_T - \mathbf{R}_v\boldsymbol{\Lambda}_v(\mathbf{T}_1+\mathbf{T}_3)\end{bmatrix}\\
&\text{S7. } \mathbf{L}_v = \begin{bmatrix}\boldsymbol{\Lambda}_u\mathbf{R}_v\mathbf{q}_T - \boldsymbol{\Lambda}_u\mathbf{R}_v(\mathbf{T}_1+\mathbf{T}_2) \\ +\boldsymbol{\Lambda}_u\boldsymbol{\tau}_v \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\mathbf{q}_T - \mathbf{R}_v\boldsymbol{\Lambda}_v(\mathbf{T}_1+\mathbf{T}_2)\end{bmatrix}
\end{aligned}
\tag{42}
$$

The blocks $\mathbf{M}_v$ and $\mathbf{N}_v$ in Eq. (41) are calculated as follows:

$$
\mathbf{M}_v = [\ \boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\tau}_t \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\tau}_t\ ] \tag{43}
$$

$$
\mathbf{N}_v = [\ \boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\Gamma} \quad \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\Gamma}\ ] \tag{44}
$$

It is clearly seen that, in the matrix multiplication Eq. (40), the first matrix is an orthogonal matrix, and the last matrix is independent of q_u. Since$\kappa(\mathbf{J})$is calculated with only the eigenvalues of the last matrix, it is independent of $q_u$as well. The Eqs. (41), (43) and (44) show that the last matrix in the matrix multiplication Eq. (41) is dependent on q_v only, not dependent on q_u, *X*, *Y*and *Z*. Moreover, because the first matrix is an orthogonal matrix, and the second one is a triangular matrix with ones on the main diagonal, the singular values of $\mathbf{J}$ is only affected by the last matrix. Therefore $\kappa(\mathbf{J})$ is dependent on q_v only. This completes the proof.

3. Comparison of the kinematic performances of the 5-axis CNC machines

In this section, all the proved properties of the kinematics model are made full use to evaluate and compare the kinematic performances of the six main types of 5-axis CNC machines.

The first machine type (Type I) includes all the machines whose both rotary axes are implemented on the workpiece carrying chain, and the axes are orthogonal [1,5,8,11,22,23].

Type II consists of the machines with both the rotary axes on the workpiece carrying chain. One rotary axis is orthogonal and the other one is a non-orthogonal rotary axis [1,2,6,7,16–19].

Type III is a set of the 5-axis CNC machines consisting of both rotary axes implemented on the tool carrying chain, and both the axes are orthogonal [4,8,10,22–24].

Type IV is a family of the machines with the two rotary axes on the tool carrying chain, but only one rotary axis is orthogonal [4,22,18].

Type V covers all machines consisting of one rotary axis on the tool carrying chain and one rotary axis on the workpiece carrying chain. Both the axes are orthogonal [1,8,10,22,23].

Type VI includes the machines with one rotary axis on the tool carrying chain and one rotary axis on the workpiece carrying chain. However, the rotary axis on the tool carrying chain is non-orthogonal [18,20,22,23].

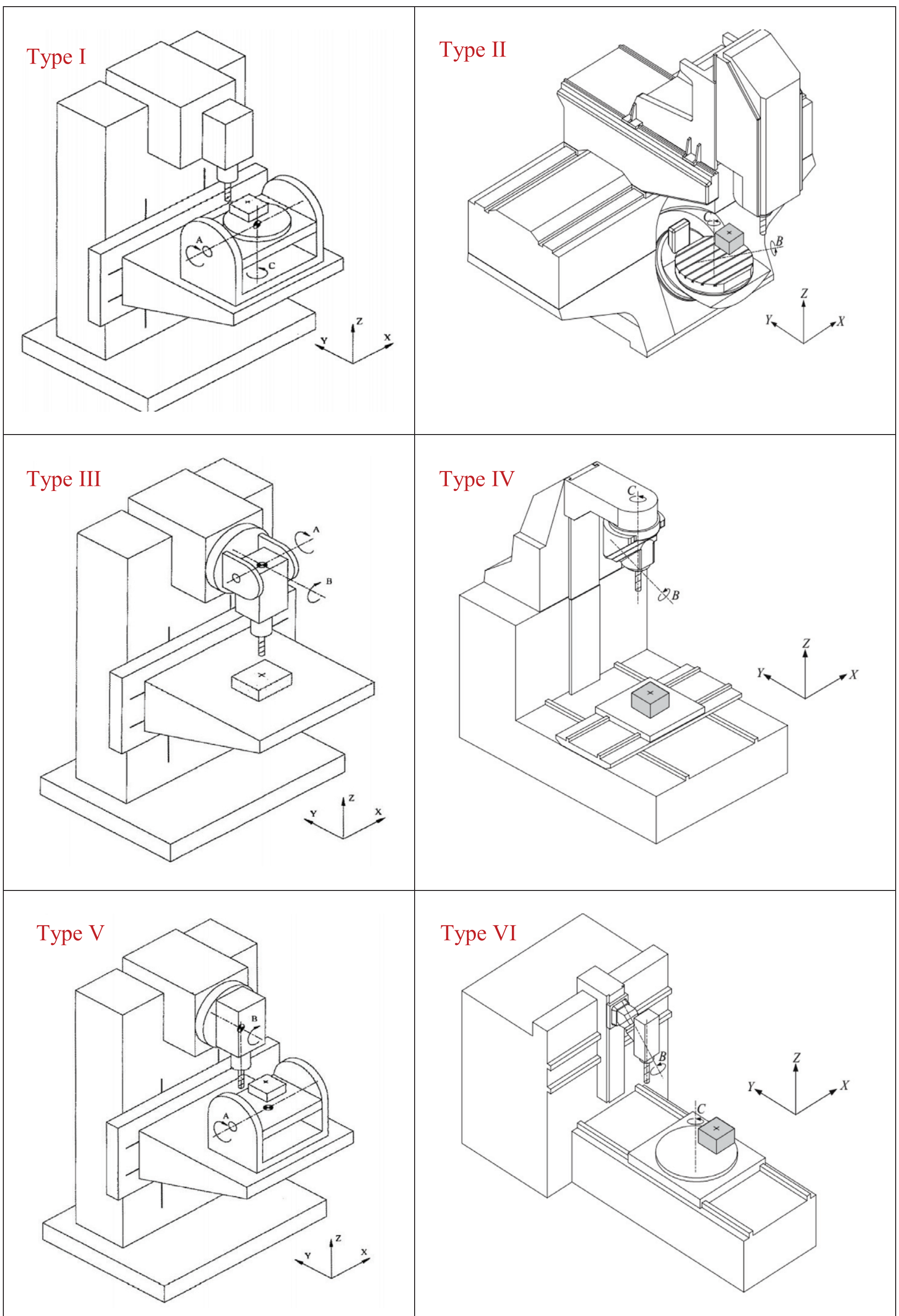


Fig. 4. The six machine types [8,18].

The six types of the machines are shown in Fig. 4.

To evaluate the kinematic performances of a machine, the manipulability index, the dexterity index, the condition number, the non-singular range of the joint variables and the non-linear kinematic error are evaluated as common indicators to compare the machines.

3.1. Manipulability index

In order to quantify the kinematic efficiency of an industrial robot, the tendency of changes in dexterity characteristics along with the variance of the revolute joint variables is of importance and should be analysed. In particular, the kinematic manipulability index, $\omega = \sqrt{Det(\mathbf{J}\mathbf{J}^T)}$ plays an essential role in the kinematical performance analysis since it indicates of how close the machine configuration is to the singularity. In this study, the manipulability index is evaluated for all the types of 5-axis CNC machines.

By using Property 2, the manipulability index ω for a 5-axis CNC machine can be formulated as follows:

$$\begin{aligned} \omega &= \sqrt{Det\left(\mathbf{J}\mathbf{J}^T\right)} \\ &= Det\left(\mathbf{J}_{RR}\right) \end{aligned} \tag{45}$$

Recalling Eq. (37), the manipulability index is formulated as follows:

$$\omega = Det\Big(\left[\boldsymbol{\Phi}\mathbf{R}_u\right]_{2\times3}\left[\begin{array}{cc}\boldsymbol{\Lambda}_u\mathbf{R}_v\boldsymbol{\Gamma} & \mathbf{R}_v\boldsymbol{\Lambda}_v\boldsymbol{\Gamma}\end{array}\right]_{3\times2}\Big) \tag{46}$$

It is shown that ω can be calculated directly with only rotation matrices $\mathbf{R}_u$ and $\mathbf{R}_v$ of a given machine, regardless of other joint variables X, Y and Z. Therefore, the manipulability index $\omega$of different machines can be evaluated in an effective and simplified manner.

Fig. 5 shows the manipulability index of three different machines. The first machine is a machine Type IV with the orthogonal rotary axis C_t and non-orthogonal rotary axis $B_t$on the tool chain (C_tB_t- nutating machine). The second machine is a machine Type I (Spinner U5-620) which has both orthogonal rotary axes on the table (B_wC_w - orthogonal). The third one is a machine Type II (DMU 50E) whose the axis B_w is non-orthogonal (B_wC_w- nutating). The inclination angle of the non-orthogonal rotary axis is 45°.

In this manner, for all the six machine types, the maximum value of the manipulability index $\omega_{\max}$ can be calculated and compared effectively. The comparison is presented in Section 3.6.

3.2. Non-singular range of joint variables

In order to provide more insightful information on the manipulability of a machine, the non-singular range of the joint variables $\mathbf{q} = \left[\begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array}\right]^T$ needs to be taken into account. In a non-singular region of the joint variables, a machine operates under the desirable dexterity condition, without singularities. In other words, the larger the non-singularity range of the joint variables is, the more flexible the tool of a machine can be oriented, and the kinematic efficiency of the machine is increased.

In this study, we define Π as the non-singular range of the joint variables of a 5-axis CNC machine. Actually, $\Pi$is the solution of the following inequality.

$$Det(\mathbf{J}) > 0 \tag{47}$$

In the general case, evaluation of $\Pi$is challenging since Eq. (47) is a multi-variables inequality. Fortunately, by applying Property 3, the inequality is dependent on only one joint variable q_v. Therefore, Π of a given 5-axis CNC machine can be obtained by solving the following inequality forq_v.

$$f_J\left(q_v\right) > 0 \tag{48}$$

For instance, the non-singular range of the joint variables Π of the machine Spinner U5-620 (Type I) are $\frac{-\pi}{2} < \Pi < 0$ and $0 < \Pi < \frac{\pi}{2}$.

Thus, for all the machine types, the indicator Π can be evaluated and compared effectively. The comparison is detailed in Section 3.6.

3.3. Dexterity index

The dexterity index is a measure of a 5-axis machine to achieve different orientations for each point within the workspace. Similar to a robot manipulator, the orientation of the tool of a 5-axis machine can be described by a rotation matrix using parameters such as Euler angles, Roll-Pitch-Yaw angles, etc. It can be observed that, when a 5-axis machine operates, the orientation of the tool axis is archived by the rotation of the two rotary axes. For this reason, a tilt angleα and a roll angle β should be considered to characterize the tool orientation relative to the workpiece since the tilt angle α can be calculated easily with respect to the displacement of the second rotary joint only. The angle β is determined with the direction cosines of the tool axis in the workpiece coordinate system. Fig. 6 shows the tilt and roll angles in all three cases:

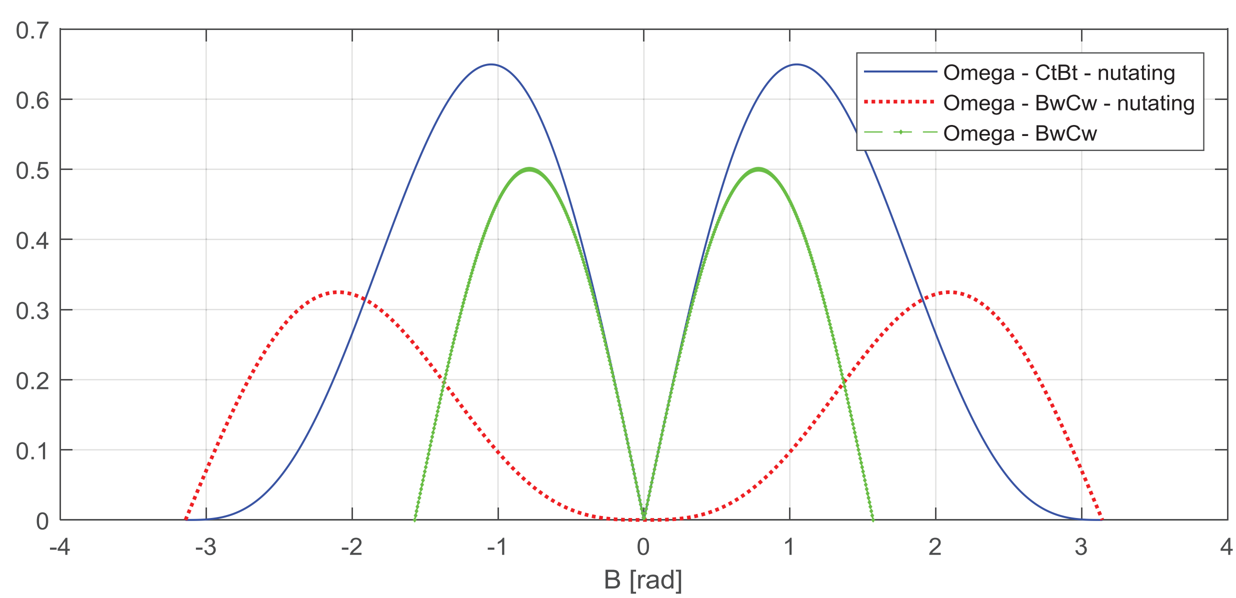


Fig. 5. The index ω of the nutating head configuration (C_tB_t- nutating), the machine Spinner U5-620 (B_wC_w), and the machine DMU 50E (B_wC_w- nutating).

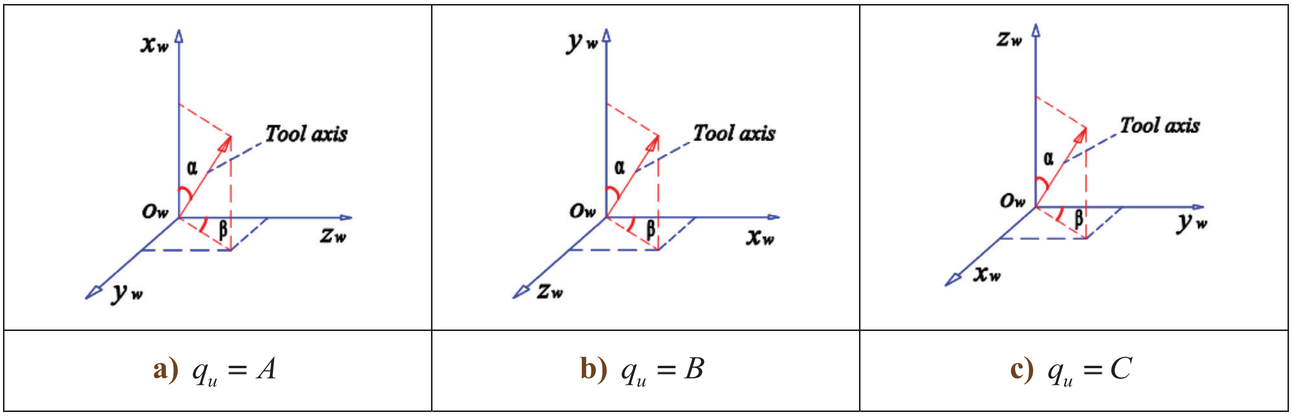


Fig. 6. The tilt angle α and roll βangle of a tool axis.

Table 3
The formulations of α and β.

q_u	α	β
A	arccos(i)	arctan2(j, k)
B	arccos(j)	arctan2(k, i)
C	arccos(k)	arctan2(i, j)

$q_u = A$, $q_u = B$ and $q_u = C$. Table 3 presents the formulation of α and β for the three general cases correspondingly.

It is clearly seen that, for a 5-axis machine, the larger the range of α and β is, the more flexible the tool orientation relative to the workpiece can be obtained. The more the flexibility of the tool orientation of a machine is, the more complex the parts can be machined with the machine. The smaller range of α and β decreases the orientation capability of a machine.

The angles α and β can vary within the range of $(0 \div 2\pi)$. Thus the dexterity index can be defined as follows:

$$D = \frac{1}{2}\left(\frac{\Delta\alpha}{2\pi} + \frac{\Delta\beta}{2\pi}\right), \tag{49}$$

where $\Delta\alpha = \alpha_{\max} - \alpha_{\min}$ and $\Delta\beta = \beta_{\max} - \beta_{\min}$ are the possible range of variation of the angles α and β for each point of the workspace.

The dexterity index D can vary within the range of $(0 \div 1)$. If the dexterity index is equal to unity we will say that the manipulator has full dexterity at a particular point or an area. For example, the dexterity index D of the machine Spinner U620 (Type I) is 0.75 since the possible range of the tilt angle $\Delta\alpha = \pi$and the range $\Delta\beta = 2\pi$. However, for the machine DMU 50e (Type II), the dexterity index D is 1 since the machine has the same possible range of roll angle, but a lager range of tilt angle$\Delta\alpha = 2\pi$.

3.4. Condition number

The condition number $\kappa(\mathbf{J}) \in [1, +\infty)$is an important index which is often used to describe first the accuracy/dexterity of a manipulator and, second, the closeness of a pose to a singularity. The condition number approaches to infinity when a machine operates near a singularity. The condition number is a local property for any 5-axis CNC machine as it depends on Jacobian matrix $\mathbf{J}$ which is a structural property. Since the condition number characterizes a norm of the Jacobian matrix, it is a measure of the relative amplification of the computed cutter positioning error $\delta\mathbf{p}$, upon a linear transformation, Eq. (50), with respect the systematic positioning errors $\delta\mathbf{q}$.

$$\delta\mathbf{p} = \mathbf{J}\delta\mathbf{q} \tag{50}$$

$$\boldsymbol{\tau} = \mathbf{J}^T\mathbf{F} \tag{51}$$

As can be seen from Eq. (51), the Jacobian matrix relates the input forces/torques $\boldsymbol{\tau}$ and output forces/torques $\mathbf{F}$of a 5-axis machine. Thus, the so called mechanical advantage of a machine can be investigated through the condition number as well.

When performing an identical machining task, which machines having a larger value of the condition number $\kappa(\mathbf{J})$ will have a larger positioning error $\delta\mathbf{p}$ at the tooltip and require more input forces/torques $\boldsymbol{\tau}$. In this sense, the condition number should be minimized in order to maintain a suitable positioning accuracy and the mechanical advantage. When the condition number equals an optimal value of one, the manipulator is described as isotropic. Isotropic configurations have a number of advantages, including good servo accuracy, noise rejection, and singularity avoidance. At any point in the non-singular range of the joint variables, the lower the condition number is, the better operating condition of a 5-axis machine is reached.

As for the 5-axis CNC machines, evaluating and comparing the condition number $\kappa(\mathbf{J})$ is a challenging task since $\mathbf{J}$ is a Jacobian matrix of five joint variables. Nevertheless, by taking full advantages of Property 4, the condition number for the 5-axis machines can be formulated and evaluated effectively.

According to Property 4, the condition number can be calculated with the largest and smallest eigenvalues of the last matrix of the matrix multiplications Eqs. (41) and (40) for the machines Type III and IV, and for other machine types, respectively.

For the machines Type III and IV, the computed condition numbers exhibits a periodic behavior shown in Fig. 7, which does not depend on the displacements of the machine axes X, Y, Z and q_u. It depends on q_v only. Note that the machines with different constant distance L_t from the tooltip to the pivot point of the rotary axis q_v will have different condition number curves as shown in Fig. 6 also. As the value of $\kappa(\mathbf{J})$ is dependent on only q_v, for any point (X,Y,Z) in the domain of the linear joint variables, the condition number of all machines Type III and IV does not change.

Figs. 8 and 9 show the curves of the condition number of a machine Type I and Type II respectively. The shape of the curves is similar to that of the previous ones in Fig. 7. One important thing is that the value of $\kappa(\mathbf{J})$ is rapidly increased when X, Y and Z are increased (Figs. 7 and 8). When the machine operates near the zero point ($X = Y = Z = 0$) in the joint space, the value of $\kappa(\mathbf{J})$ is minimized.

In comparison with the machines Type III and IV, the condition number of the other machine types is much larger. Moreover, the condition number of the machines Type III and IV does not change within the whole domain of linear axes X, Y and Z. For other machines, when the

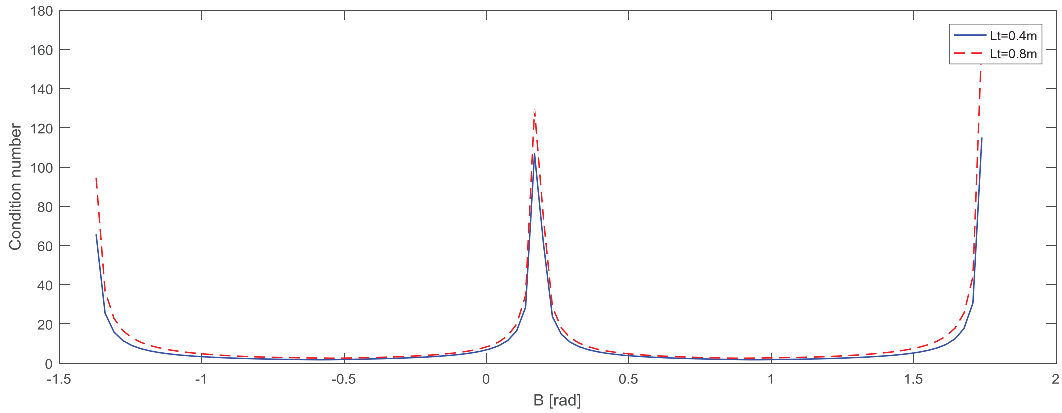


Fig. 7. The condition number $\kappa(\mathbf{J})$ for a machine Type III.

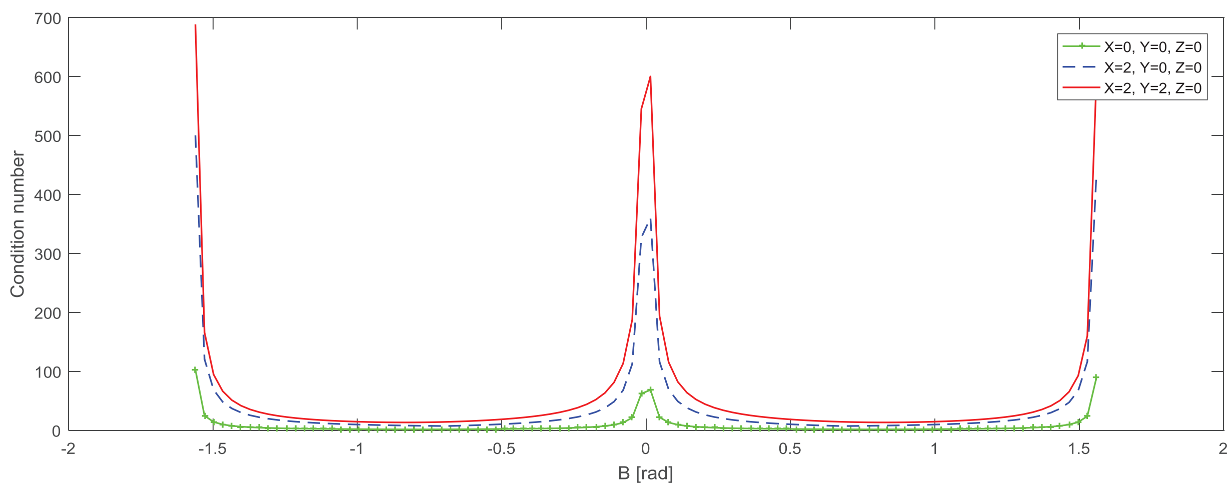


Fig. 8. The condition number $\kappa(\mathbf{J})$ for a machine Type I (Spinner U620).

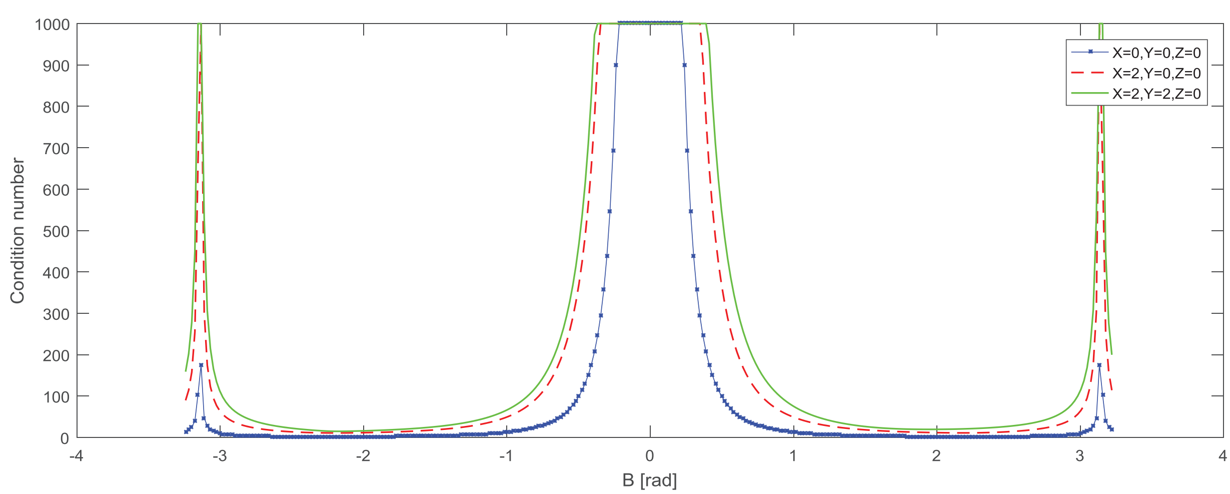


Fig. 9. The condition number $\kappa(\mathbf{J})$ for a machine Type II (DMU 50e).

cutter operates father and farther from the zero point, the condition number is larger and larger.

3.5. Non-linear kinematic tool path error

When CL data are generated by a CAM system it is assumed that the tool path between two successive CL points is a straight line relative to the workpiece. However, due to the rotary axes of the machine, the actual tool path between two blocks in the NC program will be non-linear relative to the workpiece, reducing the accuracy of the tool path. If the deviation between the straight line and the actual tool path is greater than the allowable limitation, more cutter contact points need to be inserted in between the two points. The deviation due to the non-linear kinematic behavior of a machine is called the non-linear kinematic error of the tool path. To reduce the non-linear kinematic error in 5-axis freeform surface machining, a number of blocks G01 is

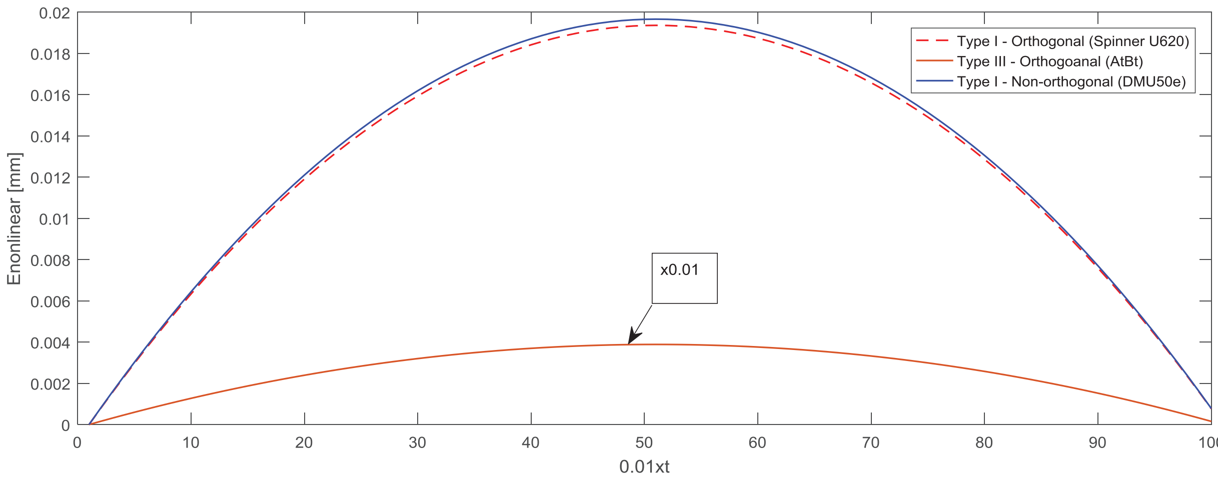


Fig. 10. The non-linear kinematic errors of three 5-axis CNC machines cutting a common straight line between two CL points.

additionally interpolated so that the number of the commands of a NC file is usually greater than the number of CL points generated by a CAM system. The non-linear kinematic error is a structural and local property of an individual 5-axis machine. Five-axis machines with different architectures have different non-linear kinematic error levels while cutting a similar path. Therefore, the non-linear kinematic error is also an important measure of a 5-axis CNC machine that needs to be analysed.

Let's consider $\mathbf{X}_1$ and $\mathbf{X}_2$ as two successive CL points represented in the workpiece space $O_w x_w y_w z_w$. To machine a desired straight line $\mathbf{X}_d = (1-t)\mathbf{X}_1 + t\mathbf{X}_2$ $(0 \le t \le 1)$, the displacements of the five axes of a machine are calculated with Eq. (14) as follows:

$$\mathbf{q}_1 = \mathbf{f}^{-1}(\mathbf{X}_1) \tag{52}$$

$$\mathbf{q}_2 = \mathbf{f}^{-1}(\mathbf{X}_2) \tag{53}$$

In the joint space, the displacement of the joints is interpolated in the parametric domain $0 \le t \le 1$ as follows:

$$\mathbf{q}(t) = (1-t)\mathbf{q}_1 - t\mathbf{q}_2 \tag{54}$$

The actual machined path $\mathbf{X}_a$ is calculated with the kinematic equation:

$$\mathbf{X}_a(t) = \mathbf{f}(\mathbf{q}(t)) \tag{55}$$

Thus, the non-linear error $E_{nonlinear}$ is calculated as follows:

$$E_{nonlinear}(t) = \left\| \mathbf{X}_a(t) - \mathbf{X}_d(t) \right\| \tag{56}$$

Fig. 10 shows the non-linear kinematic errors $E_{nonlinear}$ when a machine Type I (Spinner U620), a machine Type III and a machine Type II (DMU 50e) cut a similar tool path starting from $\mathbf{X}_1 = [\ 60.0\ \ 50.0\ \ 108.8\ \ -0.3106\ \ 0.0661\ \ 0.9482\ ]^T$ to $\mathbf{X}_2 = [\ 63.0\ \ 50.0\ \ 110.4\ \ -0.2843\ \ 0.0627\ \ 0.9567\ ]^T$. The maximum value $E^{max}_{nonlinear}$ is 0.0194 mm, 0.0000388 mm and 0.0197 mm respectively for the three machines. It can be seen that, the non-linear kinematic error $E^{max}_{nonlinear}$ of the machines with both the rotary axes on the table (Type I and II) is much larger than that of the machines with the rotary axes on the tool chain (Type III and IV).

To investigate the variation of the non-linear kinematic error along with the variation of the five joint variables, the Taylor series expansion of the function $\mathbf{p}$(Eq. (16)) is formulated in a vicinity of a point $\mathbf{q}$ in the domain of joint variables as follows:

$$\Delta\mathbf{p} = \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \Delta\mathbf{q} + \ldots \tag{57}$$

$$\Delta\mathbf{p} \simeq \mathbf{J}\Delta\mathbf{q} \tag{58}$$

Obviously, the non-linear kinematic error $E_{nonlinear} = \|\Delta\mathbf{p}\|$ is amplified by the norm of the Jacobian matrix $\mathbf{J}$ which is characterized by the condition number of $\mathbf{J}$ as well.

As shown in Property 4 and discussed in Section 3.4, for the machines with two rotary axes on the tool chain (Type III and IV), the condition number of $\mathbf{J}$ does not change along with the prismatic joint variables X, Y and Z. Therefore, the non-linear kinematic error $E_{nonlinear}$ of these machines depends on only one variable q_v, and does not change in the whole domain of the linear axes X, Y and Z. In contrast, for other machines, e. g the machines Type I and II, the condition number of $\mathbf{J}$ is rapidly increased when the values of X, Y and Z are increased. Hence, the error $E_{nonlinear}$ is increased when the machines operate with increasing values of X, Y and Z.

Fig. 11 shows the non-linear kinematic error curves for the machine Spinner U620 (Type I) and DMU 50e (Type II) cutting two different straight lines. For the first line, the values of the joint variables are calculated as X = −0.1679, Y = −0.1455 and Z = 0.1687 which are closer to the zero point (near the center point of the table). For the second line, the calculated values X = −80.3403, Y = −61.3940 and Z = 87.8774 are far from the zero point in the joint space. It is shown that when the machines cut farther and farther from the zero point, the non-linear kinematic error is larger and larger.

3.6. Comparison of the 5-axis CNC machines

Through the evaluation of the manipulability index ω, the non-singular range of joint variables Π, and the dexterity index D, the condition number $\kappa(\mathbf{J})$, and the non-linear kinematic error $E_{nonlinear}$, the kinematic performances of the 5-axis CNC machines can be directly compared. Table 4 shows a quantitative comparison of the six machine types.

It is clearly seen that, the machines Type II, Type IV and Type VI have the largest non-singular range of the joint variables Π as compared with other machines. The machines Type I and Type II have the largest dexterity index D. The value of D for the machines Type III and V is very limited. The maximum value of the manipulability index ω_{max} of the machines Type II is lower than that of other machines. The condition number of the machines Type III and IV is very small and does not change in the joint space. However, the condition number of the machines Type I and II is rapidly increased when the displacement of the linear axes is increased.

With the highest manipulability index $\omega_{max} = 0.66$, the machines Type IV has a maximum manipulability as compared with other machines. This implies that, for these machines, the tendency of changes in dexterity characteristics along with the variance of the revolute joint variables is maximized. With the lowest manipulability

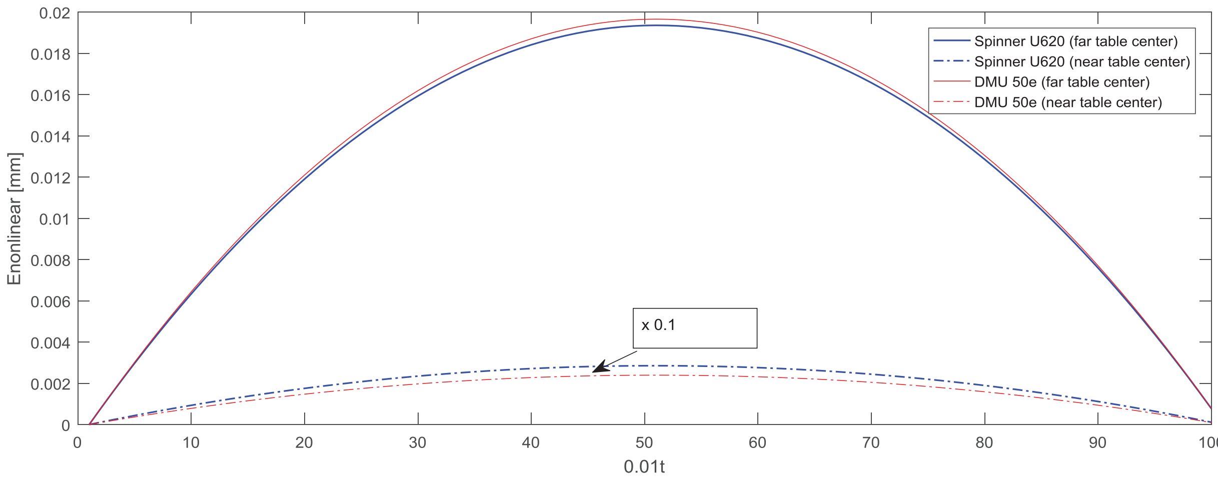


Fig. 11. The non-linear kinematic errors of the machines Type I and II cutting two different straight lines.

Table 4
A comparison of the six machine types.

Indicators		ω_{max}	Π	D	$\kappa(\mathbf{J})$	$E_{nonlinear}$
Two rotary axes on the table	Type I *(Orthogonal rotary axes)*	0.5	$0 < \lvert\Pi\rvert < \frac{\pi}{2}$	0.75	Large	Large and varying
	Type II *(Non-orthogonal rotary axes)*	0.33	$0 < \lvert\Pi\rvert < \pi$	1.0	Large	Large and varying
Two rotary axes on the spindle	Type III *(Orthogonal rotary axes)*	0.5	$0 < \lvert\Pi\rvert < \frac{\pi}{2}$	0.5	Small	Small and unvarying
	Type IV *(Non-orthogonal rotary axes)*	0.66	$0 < \lvert\Pi\rvert < \pi$	1.0	Small	Small and unvarying
One rotary axis on the spindle, and one rotary axis on the spindle	Type V *(Orthogonal rotary axes)*	0.5	$0 < \lvert\Pi\rvert < \frac{\pi}{2}$	0.5	Medium	Medium and varying
	Type VI *(Non-orthogonal rotary axes)*	0.33	$0 < \lvert\Pi\rvert < \pi$	0.75	Medium	Medium and varying

index$\omega_{max} = 0.33$, the tool of the machines Type II and VI have the lowest manipulability when manipulating the workpiece.

It has shown that the machines with a non-orthogonal rotary axis (Type II, IV and VI) have the largest value of the non-singular range of the joint variables $0 < |\Pi| < \pi$. Note that the 5-axis machines with a large non-singular range of the revolute joint variables are generally desirable when machining complex parts. This is because in the non-singular region of the joint variables, the manipulability index ω is large enough to avoid singularities, and within this region, the machine operates under the desirable dexterity condition.

The machines Type III and V have the lowest value of the dexterity index $D = 0.5$ that means that orientation ability of the tool relative to the workpiece is limited. For these machines, changing the orientation of the tool axis is less dexterous as compared with other machines. Note that, the machines Type V have a rotary table which can rotate unlimitedly, so that they have a better tool orientation capability when compared with the machines Type III. With the largest value of the dexterity index D, the machines Type II and IV are the most dexterous ones. Since one rotary axis on the machine table can rotate unlimitedly, the machines Type II and I are excellent when machining sculptured surfaces with high curvature.

Based on the evaluation of the condition number $\kappa(\mathbf{J})$ for the machines, it has shown that the machines Type I and II have the poorest mechanical advantage since their condition index is very large. Moreover, the value of the condition number is dramatically increased when the machines of these types operate farther and farther from the zero point of the linear axes X, Y and Z. This implies that the machines of these types have a larger positioning error at the tooltip and require more driving torques/forces (more power consumption) than other machine types when cutting an identical tool path. Based on this criterion, the machines Type III and IV are the best, because their condition number is small and unvarying along with the linear axes. With a small and stable deviation of the tool tip relative to the ideal tool path due to the rotary axes errors and any geometrical deviations caused in the joint space, the advantages of these machine types additionally include the good servo control accuracy, noise rejection capability, and stable machining accuracy.

With respect to the evaluation of the non-linear kinematic error, it has also shown that the magnitude of the non-linear kinematic error of a 5-axis CNC machine is considerable, and it significantly decreases the accuracy of machined parts. When a 5-axis machine operates in a vicinity of the zero point of the linear axes, the non-linear kinematic error is trivial and inconsiderable. However, when the machines cut an area near to a singular point, the non-linear kinematic error is dramatically increased, since at that points the Jacobian matrix is nearly degenerated, and the condition number of the Jacobian matrix approaches to infinity. As compared with other machines, the machines Type III and IV have a very small and stable non-linear kinematic error for the whole domain of the linear axes. This is a very important advantage of the machines, especially for the case that a very large workpiece is machined with a very large number of NC blocks. With respect to this criterion, the machines Type I and II suffer from a critical disadvantage. For these machines, the non-linear kinematic error is not only large but also significantly increased when the absolute value of the machine coordinates increased. To machine an identical large surface (an identical number of CL points generated by a CAM system), a much larger number of NC blocks is required for the machines Type I and II, in comparison with the machines Type III and IV. This is because much more interpolated points must be inserted in between the CL points to keep the machining error within a permitted tolerance.

In summary, each machine type has both advantages and disadvantages. The machines Type I and II have some main advantages over other machines such as the higher manipulability, the higher dexterity and the larger non-singular range of the machine coordinates. Especially, the machines Type II have the highest dexterity index. It is clear that these machine configurations with two rotary axes on the table tend to move more flexibly than those having the two rotational axes on the spindle head. These machines can operate with a larger range of the tilt angle, roll angle and joint variables without singularity. However, they also have several disadvantages such as the large and unstable non-linear kinematic error, the low level of the mechanical advantage, the high level of the positioning error amplification, the large number of NC blocks in a NC program. These machines, especially the machines Type II, are strongly recommended to machine parts consisting of complex sculptured surfaces of high curvature, and complex parts with small volume. Nevertheless, owing to the structural advantage, the machines Type II is capable of machining larger workpieces. Note that the non-linear kinematic error of these machines are large and increased along with the increase of the movements X, Y and Z. Therefore, when machining surfaces with high accuracy requirement, it is necessary to implement the tool path linearization procedure to interpolate and insert intermediate NC blocks in the NC programs. In addition, the positioning error of the cutter needs to be checked frequently since the effect of the positioning errors of the five joint displacements on the volumetric positioning error at the tool tip is a noticeable concern.

The machines Type III and IV have important advantages. The first one is that the non-linear kinematic error of these machines is very small and unvarying along with the entire domain of the linear axes that ensures the accuracy of the machining process. The second advantage is that they consume less power, since less driving torques/forces are required (small condition number $\kappa(\mathbf{J})$) when compared with other machine types. The third advantage of these machines is that the amplification of the positioning errors of the five displacements is small and unchanged for the whole domain of the linear displacements. Therefore, the positioning error at the tooltip due to the positioning errors of the five individual motions of a machine Type III and IV is lower than that of other machines. In addition, for these machine types, the higher control resolution in the tasks pace can be archived. All types of large workpieces comprised of surfaces with not very high curvature should be recommended to be machined with these machine types. However, the orientation of the tool of these machines relative to the workpiece is not so flexible. Moreover, the design and implementation of two rotary axes on the tool chain with a nutating spindle head for a 5-axis CNC machine could be very complex. The stiffness of an inclined spindle head is also a critical issue for the machines.

Note that the machines Type V and VI which incorporate one rotary axis on the tool carrying chain and another one rotary axis on the workpiece carrying chain are, however, combining most of the disadvantages of both previous types of machines and are often used for the production of smaller workpieces. The application range of this machine type is about the same as with machines with two rotation axes implemented on the table.

4. Discussions and conclusion

Owing to the particular architecture of the 5-axis CNC machines, the generic kinematics model of the machines has four special properties, as compared with other CNC machine types and industrial robots. In this paper, these properties have proved in a generalized case so as to make possible the formulation, evaluation and comparison of the kinematic performances of different 5-axis machines. It was shown that the comparison of the machines plays an important role in selecting suitable machines for specific applications, and in synthesizing optimal 5-axis mechanisms for new machine designs.

It has also shown that, the four properties of the generic kinematics model proved in this study are useful not only for the purpose of comparing the machines, but also for other important purposes. The use of Properties 1 and 2 is very helpful for modeling the kinematic tool path error around singularities of the 5-axis CNC machines that was presented in [2]. By applying the properties, the singular points of any 5-axis CNC machine can be effectively identified and analysed with $Det(\mathbf{J}_{RR}) = 0$, instead of $Det(\mathbf{J}) = 0$. Moreover, since $Det(\mathbf{J}_{RR})$ is a function of merely one variable q_v, the computational complexity of the algorithm constructed for minimizing the kinematic tool path error in [2] is significantly reduced.

Another important application of the formulated kinematics model and its properties is that the inverse kinematic equations for the two rotary axes of 5-axis machines, which are necessary for the postprocessor development can be established in an effective and generalized manner.

Table 5 presents all possible combinations of the primary and secondary rotary axes q_u and q_v. Note that for the cases $q_u = A$ or $q_u = B$, the combinations$CA$and $CB$are not practical because the movement of the axis Ccoincides with the rotation of the spindle. The forward kinematic equations for the two rotary axes can be immediately formulated with Eq. (27). Thus, the inverse kinematic equations in a closed form is effectively calculated accordingly. The kinematic equations for all the machines are presented in Table 5. It is clear that these kinematic equations are explicitly expressed in a closed form. Therefore, the method proposed in this study for formulating the kinematic equations for the two rotary axes is more effective and advantageous, when compared with other methods [19,23–25]. When developing an individual postprocessor for a given 5-axis CNC machine, the user just looks up the inverse kinematic equations for the two rotary axes in the last column of Table 5.

Note that, the following notes should be carefully considered when looking up the inverse kinematic equations in Table 5.

a) The coordinate systems$O_t x_t y_t z_t, O_w x_w y_w z_w$ and $O_0 x_0 y_0 z_0$ must be parallel, and their corresponding axes must point in the same direction.

Table 5
The kinematic equations for the two rotary axes of 5-axis CNC machines.

q_u	q_v	Forward kinematics	Inverse kinematics
A	B	$i = \sin B$ $j = -\sin A \cos B$ $k = \cos A \cos B$	$A = \arctan 2(-j, k)$ $B = \arcsin(i)$
B	A	$i = \sin B \cos A$ $j = -\sin A$ $k = \cos A \cos B$	$B = \arctan 2(i, k)$ $A = \arcsin(-j)$
C	A	$i = \sin C \sin A$ $j = -\cos C \sin A$ $k = \cos A$	$C = \arctan 2(-i, j)$ $A = \arccos(k)$
	B	$i = \cos C \sin B$ $j = \sin C \sin B$ $k = \cos B$	$C = \arctan 2(j, i)$ $B = \arccos(k)$
	B *(inclined at α)*	$i = -\sin\alpha \cos C \sin B + \sin\alpha \cos\alpha \sin C(1 - \cos B)$ $j = \sin\alpha \sin C \sin B + \sin\alpha \cos\alpha \cos C(1 - \cos B)$ $k = \cos^2\alpha(1 - \cos B) + \cos B$	$B = \arccos((k - \cos^2\alpha)/\sin^2\alpha)$ $C = \arctan 2\left(\begin{matrix} j \sin\alpha \sin B + i \sin\alpha \cos\alpha(1 - \cos B), \\ -i \sin\alpha \sin B + j \sin\alpha \cos\alpha(1 - \cos B)\end{matrix}\right)$

b) If the positive direction of a rotary axis of the machine under consideration does not comply with the Right-hand rule, the sign of the corresponding joint variable in Table 5 must be changed. For example, with the machine DMU 50e presented in [17], the sign of the variables must be changed as $B=-B$ and $C=-C$ when looking for the kinematic equations in the last row of Table 5.

In conclusion, a new differential kinematics model of the 5-axis CNC machines was successfully formulated in this paper. In particular, four important properties of the kinematics model have been proved in a generalized case, so that the kinematic performances of the 5-axis CNC machines can be evaluated effectively. It was shown that, the generalized kinematics model and its properties presented in this study are advantageous and useful for several purposes such as the comparison of the kinematic performances of the 5-axis CNC machines, the minimization of the tool path error around singularities, and the development of 5-axis postprocessors.

The evaluation and comparison of the kinematic performances of the six basic types of 5-axis machines have revealed that the machines (Type I and II) with both rotary axes on the table have some main advantages over other machines such as the higher manipulability, the higher dexterity and the larger non-singular range of the machine coordinates. Especially, the machines Type II have the highest dexterity. Hence, these machines tend to move more flexibly than those having the two rotational axes on the spindle head. These machines can operate with a larger range of the tilt angle, roll angle and joint variables without singularity. However, they also have critical disadvantages such as the large and unstable non-linear kinematic error, the low level of the mechanical advantage, the high level of the positioning error amplification, and the large number of NC blocks in a NC program. In contrast, the machines with two rotary axes on the spindle head (Type III and IV) possess a lower flexibility when orientating the tool axis relative to the workpiece. However, these machines have some special advantages. The first advantage is that the non-linear kinematic error is very small and unchanged along with the entire domain of the linear axes that ensures the accuracy of the machining process. The second advantage is that the machines consume less energy, since less driving torques/forces are required, as compared with other machine types. The third advantage is that the positioning error at the tooltip due to the amplification of the positioning errors of the five axes of a machine is smaller than that of other machines. For these machine types, the higher control resolution in the workspace can be archived as well.

Finally, it was shown that the comparison of the machines is useful when selecting suitable machines for given applications, especially when analysing new conceptual designs of a 5-axis CNC machine.

Experimental investigation on the non-linear kinematic tool path error of all typical types of 5-axis CNC machines will be the future work of this research.

Declaration of Competing Interest

The authors declare that there is no conflict of interest.

Funding statement

Research under Vingroup Innovation Foundation(VINIF) annual research support program in project code VINIF.2019.DA08.

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