On asymptotic properties of discrete Volterra equations of convolution type

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***Abstract*—This paper discusses dynamic properties of discrete Volterra equations of convolution type. The asymptotic separation of solutions is studied. More precisely, a polynomial lower bound for the norm of differences between two different solutions of discrete Volterra equations of convolution type is presented. We apply this result to the theory of fractional difference equations.**

I. Introduction and notation

In this paper we consider discrete Volterra equations of convolution type. This type of equations has been widely used as a mathematical model in population dynamics [1]. Some of the first qualitative results about asymptotic behaviour of discrete Volterra equations were presented in [2]. A systematic study of Volterra difference equations may be traced back to paper [3] by Elaydi that appeared in 1994. Independently, Kolmanovskii and his collaborators developed a parallel theory [4]–[7]. Interesting results on stability, boundedness and existence of solutions of Volterra difference equations may be found in [8]–[14]. Very deep and general results about the exact rates of decay of the solutions of a Volterra difference equation are obtained in [15]. Yet another type of qualitative results about discrete Volterra difference equations, such as oscilation, convergence and stability are presented in the papers [16]–[18]. The main objective of this paper is to present an asymptotic lower bound on separation between two solutions of Volterra difference equations, i.e. we will present an asymptotic lower bound for the norm of the difference of two solutions. Results of this kind may be found in [19], however, the assumptions of this paper are too strong to apply the results to fractional difference equations. Such an application was one of our objective and motivation to study this problem.

In this paper we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geq 0}$ the set of natural numbers $\{0, 1, 2, \dots\}$ including 0, and by $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$ we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \dots\}$. The symbol $\mathbb{R}^d$ is used for the d-dimensional Euclidean space with Euclidean norm $\|\cdot\|$. We will use the same symbol to denote the operator norm generated by the Euclidean norm. By $\lceil x \rceil := \min\{k \in \mathbb{Z} \colon k \geq x\}$ we denote the least integer greater or equal to x and by $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \colon k \leq x\}$ the greatest integer less or equal to x.

II. Main result

In the paper [20] it has been shown that if a solution of the multidimensional Volterra equation of convolution type

$$R(n+1) = AR(n) + \sum_{k=0}^{n} B(n-k)R(k) \quad (n \in \mathbb{N}),$$

tends to zero, then it decays exponentially if and only if the sequence $(B(n))_{n\in\mathbb{N}}$ does so. An open problem is to describe the decay rate for sequences $(B(n))_{n\in\mathbb{N}}$ tending to zero with another rate of convergency. A partial answer to this question is the content of this paper.

Let us consider the following discrete convolution Volterra equation

$$x(n+1) = x(0) + \sum_{k=0}^{n} a(n-k)A(k)x(k) \quad (n \in \mathbb{N}), \tag{1}$$

where $(A(n))_{n\in\mathbb{N}}$ is a bounded sequence of $d \times d$ matrices

$$\sup_{n\in\mathbb{N}} \|A(n)\| = M < \infty, \tag{2}$$

$(a(n))_{n\in\mathbb{N}}$ is a decreasing sequence of positive numbers satisfying

$$a(n) \leq \frac{\overline{M}}{n^{\alpha}} \tag{3}$$

for $\alpha \in (0,1)$, for all $n \in \mathbb{N} \setminus \{0\}$, and $\overline{M} > 0$. Observe that the following inequality

$$\sum_{k=1}^{n} \frac{1}{k^{\alpha}} \leq \int_{1}^{n+1} x^{-\alpha} dx + 1 = \frac{(n+1)^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} + 1$$

implies that there exists a positive constant C such that

$$\sum_{k=1}^{n} a(k) \leq C n^{1-\alpha}. \tag{4}$$

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For an initial value $x_0 \in \mathbb{R}^n$, (1) has a unique solution $x\colon \mathbb{N} \to \mathbb{R}^d$ which satisfies the initial condition $x(0) = x_0$. We denote it by $\varphi(\cdot, x_0)$.

The next theorem contains the main result of our paper, which shows that the norm of the difference of two solutions of equation (1) tends to infinity slower than n^λ with a certain positive λ.

Theorem 1: Let $\lambda > \frac{1-\alpha}{\alpha}$, $x, y \in \mathbb{R}^d$ and $x \neq y$. Then

$$\limsup_{n\to\infty} n^\lambda \|\varphi(n,x) - \varphi(n,y)\| = \infty. \tag{5}$$

Proof. Let $x, y \in \mathbb{R}^d$ with $x \neq y$ and $\lambda > \frac{1-\alpha}{\alpha}$. Suppose the contrary, i.e. there exists $K \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} n^\lambda \|\varphi(n,x) - \varphi(n,y)\| < K,$$

which implies that

$$\lim_{n\to\infty} \|\varphi(n,x) - \varphi(n,y)\| = 0 \tag{6}$$

and therefore

$$L := \sup_{n\in\mathbb{N}} \|\varphi(n,x) - \varphi(n,y)\| < \infty. \tag{7}$$

Furthermore, there exists $N \in \mathbb{N}$ such that

$$\|\varphi(n,x) - \varphi(n,y)\| \le Kn^{-\lambda} \qquad (n \ge N). \tag{8}$$

We have

$$\varphi(n,x) - \varphi(n,y) =$$

$$x - y + \sum_{k=0}^{n} a(n-k)A(k)(\varphi(k,x) - \varphi(k,y)) =$$

$$x - y + \sum_{k=0}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)),$$

where

$$B(n,k) := a(n-k)A(k).$$

Thus,

$$\|x - y\| \le$$

$$\|\varphi(n,x) - \varphi(n,y)\| + \left\| \sum_{k=0}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\|.$$

Letting $n \to \infty$ and using (6), we obtain that

$$\limsup_{n\to\infty} \left\| \sum_{k=0}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| > 0. \tag{9}$$

Since $\lambda > \frac{1-\alpha}{\alpha}$, there exists $\delta \in (\frac{1-\alpha}{\lambda}, \alpha)$. Thus, to gain a contradiction to inequality (9), it is sufficient to show that

$$\limsup_{n\to\infty} \sum_{k=0}^{\lceil n^\delta \rceil - 1} B(n,k)(\varphi(k,x) - \varphi(k,y)) = 0 \tag{10}$$

and

$$\limsup_{n\to\infty} \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)) = 0. \tag{11}$$

By definition of $B(n,k)$ and non-negativity of the sequence $a(n)$, we have

$$\left\| \sum_{k=0}^{\lceil n^\delta \rceil - 1} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| \le$$

$$\sum_{k=0}^{\lceil n^\delta \rceil - 1} \|B(n,k)\| \|\varphi(k,x) - \varphi(k,y)\| \le$$

$$\sum_{k=0}^{\lceil n^\delta \rceil - 1} Ma(n-k) \|\varphi(k,x) - \varphi(k,y)\| \le$$

$$ML \sum_{k=0}^{\lceil n^\delta \rceil - 1} a(n-k) = ML \sum_{k=n-\lceil n^\delta \rceil + 1}^{n} a(k),$$

where we used (7) to obtain the last inequality. Because the sequence $(a(n))_{n\in\mathbb{N}}$ is decreasing,

$$\left\| \sum_{k=0}^{\lceil n^\delta \rceil - 1} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| \le$$

$$ML \lceil n^\delta \rceil a(n - \lceil n^\delta \rceil).$$

Using (3), we obtain that

$$\left\| \sum_{k=0}^{\lceil n^\delta \rceil - 1} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| \le$$

$$ML(n^\delta + 1) \frac{\overline{M}}{(n - n^\delta)^\alpha},$$

which, together with the fact that $\delta < \alpha$, proves (10). To conclude the proof we show (11). For this purpose, we use the estimate

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| \le$$

$$\sum_{k=\lceil n^\delta \rceil}^{n} \|B(n,k)\| \|\varphi(k,x) - \varphi(k,y)\| \le$$

$$M \sum_{k=\lceil n^\delta \rceil}^{n} a(n-k) \|\varphi(k,x) - \varphi(k,y)\|.$$

Let $n \in \mathbb{N}$ satisfy that $n^\delta \ge N$. Using (8), we obtain that

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi(k,x) - \varphi(k,y)) \right\| \le$$

$$MK \lceil n^\delta \rceil^{-\lambda} \sum_{k=\lceil n^\delta \rceil}^{n} a(n-k) \le MKn^{-\lambda\delta} \sum_{k=0}^{n-\lceil n^\delta \rceil} a(k).$$

By (4) we have

$$\left\| \sum_{k=\lceil n^\delta \rceil}^{n} B(n,k)(\varphi(k,x)-\varphi(k,y)) \right\| \le$$

$$MKn^{-\lambda\delta} \sum_{k=0}^{n-\lceil n^\delta \rceil} a(k) \le$$

$$MKn^{-\lambda\delta} \sum_{k=0}^{n} a(k) \le MKn^{-\lambda\delta} C n^{1-\alpha}.$$

Note that $-\lambda\delta + 1 - \alpha < 0$, (11) is proved and the proof is complete. ■

III. A theoretical application

In this section we will show that, using Theorem 1 with an appropriately defined sequence $(a(n))_{n\in\mathbb{N}}$, we obtain results about nonautonomous fractional difference systems [21].

Before we present the main result of this section we introduce some necessary notions concerning fractional calculus. By $\Gamma\colon \mathbb{R}\setminus\mathbb{Z}_{\le 0} \to \mathbb{R}$ we denote the Euler Gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \quad (\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\le 0}), \quad (12)$$

which is well-defined, since the limit exists (see e.g. [22, p. 156], and

$$\Gamma(\alpha) = \begin{cases} \int_0^\infty x^{\alpha-1}e^{-x}dx & \text{if } \alpha > 0, \\ \frac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R}\setminus\mathbb{Z}_{\le 0}. \end{cases} \quad (13)$$

Note that $\Gamma(\alpha) > 0$ for all $\alpha > 0$.

For $s \in \mathbb{R}$ with $s+1, s+1-\alpha \notin \mathbb{Z}_{\le 0}$ the falling factorial power $(s)^{(\alpha)}$ is defined by

$$(s)^{(\alpha)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)} \quad \big(s \in (\mathbb{R}\setminus\mathbb{Z}_{\le -1}) \cap (\mathbb{R}\setminus \alpha + \mathbb{Z}_{\le -1})\big). \quad (14)$$

For $\nu \in \mathbb{R}_{>0}$ and a function $f : \mathbb{N} \to \mathbb{R}$, the ν-th delta fractional sum $\Delta^{-\nu} f\colon \mathbb{N}_\nu \to \mathbb{R}$ of f is defined as

$$(\Delta^{-\nu} f)(t) := \frac{1}{\Gamma(\nu)} \sum_{k=0}^{t-\nu} (t-k-1)^{(\nu-1)} f(k) \quad (t \in \mathbb{N}_\nu).$$

Let $\alpha \in (0,1)$ and $f\colon \mathbb{N} \to \mathbb{R}$. The Caputo forward difference ${}_{\mathrm{C}}\Delta^\alpha f\colon \mathbb{N}_{1-\alpha} \to \mathbb{R}$ of f of order α is defined as the composition ${}_{\mathrm{C}}\Delta^\alpha := \Delta^{-(1-\alpha)} \circ \Delta$ of the $(1-\alpha)$-th delta fractional sum with the classical difference operator $t \mapsto \Delta f(t) := f(t+1) - f(t)$, i.e.

$$({}_{\mathrm{C}}\Delta^\alpha f)(t) := (\Delta^{-(1-\alpha)} \Delta f)(t) \quad (t \in \mathbb{N}_{1-\alpha}).$$

Consider equation (1) with the sequence $(a(n))_{n\in\mathbb{N}}$ of the following form

$$a(n) = (-1)^n \binom{-\beta}{n}, \quad (15)$$

where

$$\binom{\beta}{n} := \frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!} \quad (\beta \in \mathbb{R},\ n \in \mathbb{N}).$$

At first, let us notice certain properties of the sequence.

Lemma 1: Let $\beta > 0$. Then the following statements hold:

I) $a(n) > 0$ for $n \in \mathbb{N}$;
II) if $0 < \beta < 1$, then $(a(n))_{n\in\mathbb{N}}$ is a decreasing sequence;
III) there exists $\overline{M} > 0$ such that

$$a(n) < \frac{\overline{M}}{n^{1-\beta}} \quad (n \in \mathbb{N}\setminus\{0\}).$$

Proof. I) Let $\beta > 0$. To prove this point let us notice that $a(0) = 1$ and for $n \in \mathbb{N}\setminus\{0\}$,

$$a(n) = (-1)^n \frac{(-\beta)(-\beta-1)\cdots(-\beta-n+1)}{n!} =$$

$$\frac{\beta(\beta+1)\cdots(\beta+n-1)}{n!}.$$

Thus, $a(n) > 0$ for $n \in \mathbb{N}$.

II) From the assumption that $\beta \in (0,1)$ we get

$$a(n+1) = \frac{\beta(\beta+1)\cdots(\beta+n)}{(n+1)!} =$$

$$\frac{\beta+n}{n+1} a(n) < a(n) \quad (n \in \mathbb{N}),$$

i.e. $(a(n))_{n\in\mathbb{N}}$ is a decreasing sequence.

III) To prove this point we will consider the Euler Gamma function $\Gamma : \mathbb{R}\setminus\mathbb{Z}_{\le 0} \to \mathbb{R}$. From its definition we obtain

$$\begin{aligned} \Gamma(\beta) &= \lim_{n\to\infty} \frac{1}{(-1)^n n^{-\beta}(\beta+n)\frac{(-\beta)(-\beta-1)\cdots(-\beta-n+1)}{n!}} \\ &= \lim_{n\to\infty} \frac{1}{n^{1-\beta}(\frac{\beta}{n}+1)(-1)^n\binom{-\beta}{n}} \\ &= \lim_{n\to\infty} \frac{1}{n^{1-\beta}(\frac{\beta}{n}+1)a(n)}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \Gamma(\beta)\left(\frac{\beta}{n}+1\right) n^{1-\beta} a(n) = 1.$$

Using I) and the fact that $\Gamma(\beta) > 0$, there exists $M > 0$ such that

$$\Gamma(\beta)\left(\frac{\beta}{n}+1\right) n^{1-\beta} a(n) \le M \quad (n \in \mathbb{N}).$$

Hence

$$a(n) \le \frac{M}{\Gamma(\beta)(\frac{\beta}{n}+1)n^{1-\beta}} \le \frac{\overline{M}}{n^{1-\beta}} \quad (n \in \mathbb{N}\setminus\{0\})$$

with

$$\overline{M} := \frac{M}{\Gamma(\beta)}.$$

■

In the next theorem we will show that equation (1) with $a(n)$ given by (15) is equivalent to a linear bounded nonautonomous fractional difference system

$$({}_{\mathrm{C}}\Delta^{\beta}x)(n+1-\beta) = A(n)x(n) \quad (n \in \mathbb{N}), \tag{16}$$

where $x\colon \mathbb{N} \to \mathbb{R}^d$, ${}_{\mathrm{C}}\Delta^{\beta}$ is the Caputo forward difference operator of a real order $\beta \in (0,1)$ and $A\colon \mathbb{N} \to \mathbb{R}^{d\times d}$ is bounded.

More precisely the following theorem holds. In addtion, we present the proof for clarity below.

Theorem 2: Let $\beta \in (0,1)$ then $x\colon \mathbb{N} \to \mathbb{R}^d$ is a solution of (16) with initial condition $x(0) \in \mathbb{R}^d$ if and only if

$$x(n+1) = x(0) + \sum_{k=0}^{n} (-1)^{n-k} \binom{-\beta}{n-k} A(k)x(k) \quad (n \in \mathbb{N}). \tag{17}$$

Proof. Using the identity from [23, Thm. 2.4] we get

$$x(t) = x(0) +$$

$$\frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} (\Delta^{\beta}x)(s) \quad (t \in \mathbb{N}\setminus\{0\}),$$

where $(s)^{(\beta)}$ is the falling factorial power (see (14)).

For $n \in \mathbb{N}$ and $t := n+1$, the substitution $s = k + (1-\beta)$ yields

$$x(n+1) = x(0) +$$

$$\frac{1}{\Gamma(\beta)} \sum_{k=0}^{n} (n-k+\beta-1)^{(\beta-1)} (\Delta^{\beta}x)(k+1-\beta) \quad (n \in \mathbb{N}).$$

From the definition of falling factorial power we have

$$\frac{1}{\Gamma(\beta)}(n-k+\beta-1)^{(\beta-1)} = \frac{\Gamma(n-k+\beta)}{\Gamma(\beta)\Gamma(n-k+1)} =$$

$$\binom{n-k+\beta-1}{n-k} = \binom{\beta+(n-k)-1}{n-k} \quad (k \in \{0,\dots,n\}).$$

Using the identity (see [24, pp. 165])

$$\binom{\beta+k-1}{k} = (-1)^k \binom{-\beta}{k} \quad (\beta \in \mathbb{R},\ k \in \mathbb{N})$$

we obtain

$$\frac{1}{\Gamma(\beta)}(n-k+\beta-1)^{(\beta-1)} =$$

$$(-1)^{n-k}\binom{-\beta}{n-k} \quad (k \in \{0,\dots,n\})$$

and therefore

$$x(n+1) = x(0) +$$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{-\beta}{n-k} (\Delta^{\beta}x)(k+1-\beta) \quad (n \in \mathbb{N})$$

which completes the proof. ■

From III) of Lemma 1 we see that the sequence $(a(n))_{n\in\mathbb{N}}$ given by (15) satisfies (3) and therefore from Theorem 1 and then by Theorem 2, we obtain the following corollary.

Corollary 1: If $\lambda > \frac{1-\beta}{\beta}$, $x,\ y \in \mathbb{R}^d$ and $x \neq y$, then

$$\limsup_{n\to\infty} n^{\lambda} \|\varphi_{\mathrm{C}}(n,x) - \varphi_{\mathrm{C}}(n,y)\| = \infty, \tag{18}$$

where $(\varphi_{\mathrm{C}}(n,z))_{n\in\mathbb{N}}$ is the solution of (16) with initial condition $\varphi_{\mathrm{C}}(0,z) = z$.

The characterization of asymptotic properties of solutions of linear equations Lyapunov exponents is frequently used in control theory [25]–[27] and in particular within model predictive control [28]–[30]. The Lyapunov exponent of a solution $(y(n))_{n\in\mathbb{N}}$ is defined as

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|y(n)\|$$

and, in particular, the stability analysis is performed according to its sign. A consequence of Corollary 1 is that the Lyapunov exponent of an arbitrary non-trivial solution of (16) is non-negative, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|\varphi_{\mathrm{C}}(n,x_0)\| \geq 0 \tag{19}$$

for all $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$ and therefore, this classical definition of the Lyapunov exponents cannot be applied to the stability analysis of nonautonomous fractional difference systems of the form (16)

IV. Conclusions

In this paper we investigated dynamic properties of discrete Volterra equations of convolution type. For this equation we presented a polynomial lower bound for the norm of differences between two different solutions. In particular, this theorem implies that the classical Lyapunov exponent is not an appropriate tool for stability analysis of fractional equations. An appropriate modification of the definition of Lyapunov exponents for discrete fractional equations is an important challenge of the theory of Lyapunov exponents. We also showed that for a particular sequence of coefficients the Volterra equation specializes to a discrete time-varying fractional equation and in this way we applied our main result to the context of fractional linear systems.

Acknowledgment

The research of the second and third authors was funded by the National Science Centre in Poland granted according to decisions DEC-2015/19/D/ST7/03679 and DEC-2017/25/B/ST7/02888, respectively. The research of the fourth author was supported by the Polish National Agency for Academic Exchange according to the decision PPN/BEK/2018/1/00312/DEC/1. The research of the last author was partially supported by an Alexander von Humboldt Polish Honorary Research Fellowship.

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