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Variation of Constant Formulas of Linear Autonomous Grünwald-Letnikov-type Fractional Difference Equations

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Abstract. In this paper, the variation of constant formulas for Grünwald-Letnikov-type fractional difference equations are established. As an application, we prove a stability result for solutions of a class of autonomous scalar fractional difference equations.

Introduction

It is well-known that the Laplace transform method can be utilized to derive a variation of constants formula for linear fractional differential equations [1]. In this paper we use the $\mathcal{Z}$-transform to establish variation of constant formulas for Grünwald-Letnikov-type fractional difference equations in Section 2. In Section 3 we prove some stability results for solutions of a class of autonomous scalar fractional difference equations.

Let $\alpha \in (0,1)$, $h > 0$. Consider a fractional difference equation of the form

$$({}_0\widetilde{\Delta}_h^\alpha x)(t+h) = A(t)x(t) + f(t) \qquad (t \in h\mathbb{N}), \tag{1}$$

where $x\colon h\mathbb{N} \to \mathbb{R}^d$, ${}_0\widetilde{\Delta}_h^\alpha$ is the Grünwald-Letnikov-type fractional h-difference operator of order α, $f\colon h\mathbb{N} \to \mathbb{R}^d$ and $A\colon h\mathbb{N} \to \mathbb{R}^{d\times d}$. For an initial value $x_0 \in \mathbb{R}^n$, (1) has a unique solution $x\colon h\mathbb{N} \to \mathbb{R}^d$ which satisfies the initial condition $x(0) = x_0$. We denote x by $\varphi_{\text{G-L}}(\cdot, x_0)$.

Remark 1 (Solving linear homogeneous fractional difference equations (1))**.** *The linear homogeneous initial value problem,* $({}_0\widetilde{\Delta}_h^\alpha x)(t+h) = Ax(t), x(0) = x_0 \in \mathbb{R}^d$, $t \in h\mathbb{N}$, $A \in \mathbb{R}^{d\times d}$, *has a unique solution given by the formula (see e.g. [2, Proposition 32])*

$$\varphi_{\text{G-L}}(t, x_0) = E_{(\alpha,\alpha)}\left(Ah^\alpha, \tfrac{t}{h}\right) x_0 \qquad (t \in h\mathbb{N}).$$

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A reader who is familiar with fractional difference equations may very well skip the remainder of the introduction in which we recall notation to keep the paper self-contained. Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geq 0}$ the set $\{0, 1, 2, \dots\}$ of natural numbers including 0, and by $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$, $h > 0$, we denote by $(h\mathbb{N})_a$ the set $h\mathbb{N} + a = \{a, a + h, \dots\}$. We write $h\mathbb{N}$ instead of $(h\mathbb{N})_0$. By $\Gamma \colon \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{R}$ we denote the Euler Gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \qquad (\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}). \tag{2}$$

Note that

$$\Gamma(\alpha) = \begin{cases} \int_0^\infty x^{\alpha-1} e^{-x} dx & \text{if } \alpha > 0, \\ \frac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}. \end{cases} \tag{3}$$

Binomial coefficients $\binom{r}{m}$ can be defined for any $r, m \in \mathbb{C}$ as described in [3, Section 5.5, formula (5.90)]. For $r \in \mathbb{R}$ and $m \in \mathbb{Z}$ the binomial coefficient satisfies [3, Section 5.1, formula (5.1)]

$$\binom{r}{m} = \begin{cases} \frac{r(r-1)\cdots(r-m+1)}{m!} & \text{if } m \in \mathbb{Z}_{\geq 1}, \\ 1 & \text{if } m = 0, \\ 0 & \text{if } m \in \mathbb{Z}_{\leq -1}. \end{cases}$$

Definition 2. *[2, Definition 27] Let $\alpha, a \in \mathbb{R}$. The Grünwald-Letnikov-type h-difference operator ${}_a\widetilde{\Delta}_h^\alpha$ of order α for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by*

$$({}_a\widetilde{\Delta}_h^\alpha x)(t) = \sum_{k=0}^{\frac{t-a}{h}} a_k^{(\alpha)} x(t - kh) \qquad (t \in (h\mathbb{N})_a),$$

where $a_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} \frac{1}{h^\alpha}$.

For $\beta \in \mathbb{C}$, we define a discrete-time Mittag-Leffler function $E_{(\alpha,\beta)}(A, \cdot)$ by

$$E_{(\alpha,\beta)}(A, n) = \sum_{k=0}^{\infty} A^k \binom{n - k + k\alpha + \beta - 1}{n - k} \qquad (n \in \mathbb{Z}), \tag{4}$$

see [2, 4, 5] for this and similar definitions.

Proposition 3. *[2, Proposition 9] Let $\alpha \in (0, 1]$, $\lambda, \beta \in \mathbb{R}$ and $\beta < \alpha + 1$. If all solutions $z \in \mathbb{C}$ of the equation $(z - 1)^\alpha = \lambda z^{\alpha-1}$ satisfy $|z| < 1$, then $\lim_{n\to\infty} E_{(\alpha,\beta)}(\lambda, n) = 0$.*

Corollary 4. *[2, Corollary 10] Let $\alpha \in (0, 1]$, $\lambda, \beta \in \mathbb{R}$ and $\beta < \alpha + 1$. All solutions $z \in \mathbb{C}$ of the equation $(z - 1)^\alpha = \lambda z^{\alpha-1}$ satisfy $|z| < 1$ if and only if $-2^\alpha < \lambda < 0$.*

Variation of constants formulas

Theorem 5 (Variation of constant formulas for Grünwald-Letnikov-type fractional difference equations)**.** *Let $\alpha \in (0, 1)$, $A \in \mathbb{R}^{d\times d}$ and $f \colon h\mathbb{N} \to \mathbb{R}^d$. The solution of* (1)*, $({}_0\widetilde{\Delta}_h^\alpha x)(t + h) = Ax(t) + f(t)$, $x(0) = x_0 \in \mathbb{R}^n$ is given by*

$$\varphi_{\text{G-L}}(t, x_0) = E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h})x_0 + \sum_{k=0}^{\frac{t}{h}} E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h} - k) f(k) \qquad (t \in h\mathbb{N}). \tag{5}$$

Remark 6 (Variation of constants formula applied to nonlinear equation)**.** *Let* $\alpha \in (0,1)$ *and* $x\colon h\mathbb{N} \to \mathbb{R}^d$ *be a solution of the nonlinear fractional difference equation* $({}_0\widetilde{\Delta}_h^\alpha x)(t+h) = Ax(t) + g(x(t))$*, where* ${}_0\widetilde{\Delta}_h^\alpha$ *is the Grünwald-Letnikov-type difference operator of order* α*,* $g\colon \mathbb{R}^d \to \mathbb{R}^d$ *and* $A \in \mathbb{R}^{d\times d}$*. Then* x *is also a solution of the (nonautonomous) linear fractional difference equation* (1) *with* $f\colon h\mathbb{N} \to \mathbb{R}^d$*,* $t \mapsto g(x(t))$*. By Theorem 5,* x *satisfies the implicit equation*

$$x(t) = E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h})x_0 + \sum_{k=0}^{\frac{t}{h}} E_{(\alpha,\alpha)}(A, \tfrac{t}{h} - k)g(x(k)) \qquad (t \in h\mathbb{N}).$$

To prepare the proof of Theorem 5, we summarize some results about the $\mathcal{Z}$-transform of a sequence $x\colon \mathbb{N} \to \mathbb{R}$, which is defined by $\mathcal{Z}[x](z) = \sum_{i=0}^{\infty} x(i)z^{-i}$ for $z \in \mathbb{C}$, $|z| > R := \limsup_{i\to\infty} |x(i)|^{1/i}$, see e.g. [6, Chapter 6] and [7]. The $\mathcal{Z}$-transform of $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$ valued sequences is defined component-wise. The proof of the following lemma follows from [8, Lemma 3] and [2, Proposition 28].

Lemma 7 ($\mathcal{Z}$-transform of Mittag-Leffler functions and fractional differences)**.** *Let* $a \in \mathbb{R}$*,* $\alpha \in (0,1)$*.*

(i) *Let* $A \in \mathbb{R}^{d\times d}$*,* $\beta \in \mathbb{R}$*. Then* $\mathcal{Z}[E_{(\alpha,\beta)}(A,\cdot)](z) = \left(\frac{z}{z-1}\right)^\beta \left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^\alpha A\right)^{-1}$*.*

(ii) *Let* $x\colon (h\mathbb{N})_a \to \mathbb{R}^d$ *and* $y(n) := ({}_a\widetilde{\Delta}_h^\alpha x)(t)$ *for* $t \in (h\mathbb{N})_\alpha$ *and* $n \in \mathbb{N}_0$ *such that* $t = a + nh$*. Then* $\mathcal{Z}[y](z) = \left(\frac{zh}{z-1}\right)^{-\alpha} \mathcal{Z}[\overline{x}](z)$*, where* $X(z) = \mathcal{Z}[\overline{x}](z)$ *and* $\overline{x}(n) := x(a+nh)$*.*

Proof of Theorem 5. By Lemma 7(ii), applying the $\mathcal{Z}$-transform to equation (1) with the Grünwald-Letnikov-type forward difference operator, we get

$$\left(\frac{zh}{z-1}\right)^{-\alpha} \mathcal{Z}[\varphi_{\text{G-L}}(\cdot, x_0)](z) = A\mathcal{Z}[\varphi_{\text{G-L}}(\cdot, x_0)](z) + \mathcal{Z}[f](z).$$

Using Lemma 7(i), we obtain

$$\mathcal{Z}[\varphi_{\text{G-L}}(\cdot, x_0)](z) = \mathcal{Z}[E_{(\alpha,\alpha)}(Ah^\alpha, \cdot)(z)x_0] + \mathcal{Z}[E_{(\alpha,\alpha)}(Ah^\alpha, \cdot)(z)x_0]\mathcal{Z}[f](z).$$

Applying the inverse of the $\mathcal{Z}$-transform yields

$$\varphi_{\text{G-L}}(\cdot, x_0) = E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h})x_0 + \mathcal{Z}^{-1}\left[\mathcal{Z}[E_{(\alpha,\alpha)}(Ah^\alpha, \cdot)(z)x_0]\mathcal{Z}[f](z)\right].$$

Hence, we get

$$\varphi_{\text{G-L}}(t, x_0) = E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h})x_0 + \sum_{k=0}^{\frac{t}{h}} E_{(\alpha,\alpha)}(Ah^\alpha, \tfrac{t}{h} - k)f(k) \qquad (t \in h\mathbb{N}).$$

□

Boundedness of scalar linear Grünwald-Letnikov-type difference equations

Theorem 8 (Stability of linear homogeneous Grünwald-Letnikov-type scalar difference equation)**.** *Let* $\alpha \in (0,1)$*,* $\lambda \in \mathbb{R}$*. If all solutions* $z \in \mathbb{C}$ *of*

$$1 - \frac{1}{z}\left(\frac{z}{z-1}\right)^\alpha \lambda h^\alpha = 0 \tag{6}$$

satisfy $|z| < 1$*, then*

$$({}_0\widetilde{\Delta}_h^\alpha x)(t+h) = \lambda x(t) \qquad (t \in h\mathbb{N}) \tag{7}$$

is asymptotically stable.

Proof. From Theorem 5, we have $\varphi_{\text{G-L}}(t, x_0) = E_{(\alpha,\alpha)}(\lambda h^\alpha, \frac{t}{h})x_0$. From (8), we have $(z-1)^\alpha = \lambda h^\alpha z^{\alpha-1}$. Hence, from Proposition 3, we have $\lim_{\frac{t}{h}\to\infty} E_{(\alpha,\alpha)}(\lambda^\alpha, \frac{t}{h}) = 0$. □

Theorem 9 (Boundedness of nonautonomous linear Grünwald-Letnikov-type scalar difference equation)**.** *Let* $\alpha \in (0,1)$*,* $\lambda \in \mathbb{R}$ *and* $(\lambda_n)_{n\in\mathbb{N}}$ *a sequence in* l^1*. If all solutions* $z \in \mathbb{C}$ *of*

$$1 - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} \lambda h^{\alpha} = 0. \tag{8}$$

satisfy $|z| < 1$*, then the solution of*

$$({}_0\widetilde{\Delta}_h^{\alpha} x)(t+h) = \lambda x(t) + \lambda_{t/h} \qquad (t \in h\mathbb{N}), \tag{9}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$*, is bounded.*

Proof. By Theorem 5, the solution of (9), $x(0) = x_0 \in \mathbb{R}^n$, is given by

$$\varphi_{\text{G-L}}(t, x_0) = E_{(\alpha,\alpha)}(\lambda h^{\alpha}, \tfrac{t}{h})x_0 + \sum_{k=0}^{\frac{t}{h}} E_{(\alpha,\alpha)}(\lambda h^{\alpha}, \tfrac{t}{h} - k)\lambda_k \qquad (t \in h\mathbb{N}).$$

By Theorem (8), there exists $M \geq 0$ such that $|E_{(\alpha,\alpha)}(\lambda^{\alpha}, \frac{t}{h})| \leq M$ for $t \in h\mathbb{N}$. Then,

$$\left|\sum_{k=0}^{\frac{t}{h}} E_{(\alpha,\alpha)}(\lambda h^{\alpha}, \frac{t}{h} - k)\lambda_k\right| \leq M \sum_{k=0}^{\frac{t}{h}} |\lambda_k| \qquad (t \in h\mathbb{N}),$$

proving that $|\varphi_{\text{G-L}}(t, x_0)| \leq M|x_0| + M \sum_{k=0}^{\infty} |\lambda_k|$. □

Using Corollary 4, the following Corollary follows from Theorem 9.

Corollary 10. *Let* $\alpha \in (0,1)$*,* $\lambda \in \mathbb{R}$*,* $(\lambda_n)_{n\in\mathbb{N}}$ *a sequence in* l^1*. Then the solution of* (9) *is bounded if* $-\left(\frac{2}{h}\right)^{\alpha} < \lambda < 0$*.*

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