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A Lower Bound on the Separation between two Solutions of a Scalar Riemann-Liouville Fractional Differential Equation

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Abstract. In this paper, we establish a lower bound on the separation between two distinct solutions of a scalar Riemann-Liouville fractional differential equation. As a consequence, we show that the Lyapunov exponent of an arbitrary non-trivial solution of a bounded linear scalar Riemann-Liouville fractional differential equation is always non-negative.

Introduction

In recent years, fractional differential equations have become an attractive research areas. The main reason for the increasing interest in the theory of fractional differential systems comes from the fact that many mathematical problems in science and engineering can be modeled by fractional differential equations, see e.g., [1, 2, 3].

So far, the problem on the existence and uniqueness of solutions of Riemann-Liouville fractional differential equations has been solved, see e.g., [4, 5] and the references therein. However, to the best our knowledge, there have been only few published results dealing with the qualitative theory of solutions.

In the qualitative theory, investigating the separation between two differential solutions is an important task and an important quantity measuring this separation is the Lyapunov exponent.

Our main result in this paper is to establish a lower bound on the difference between two solutions of a scalar Riemann-Liouville fractional differential equation. This lower bound implies that the Lyapunov exponent of an arbitrary non-trivial solution of linear scalar Riemann-Liouville fractional differential equation with bounded coefficient is always non-negative. This kind of surprising observation (in comparison with the Lyapunov exponent of solutions of ordinary differential equations) in a different model of fractional systems (Caputo fractional differential equations) was already achieved, see [6].

The paper is structured as follows: In Section II, we recall some background on Riemann-Liouville fractional differential equation and state the main theorem of the paper (Theorem 1). As a result of this theorem, we also show that the Lyapunov exponent of an arbitrary non-trivial solution to a linear scalar Riemann–Liouville fractional differential equation with bounded coefficient is always non-negative. The proof of the main result is shifted to Section III.

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Preliminaries and the statement of the main result

Let $\alpha \in (0,1)$ and $T > 0$. Consider a scalar Riemann–Liouville fractional equation of the following form on the interval $[0,T]$

$$D^{\alpha}_{0+}x(t) = f(t,x(t)), \tag{1}$$

where $D^{\alpha}_{0+}x(t)$ is the Riemann–Liouville derivative of the function $x : [0,T] \to \mathbb{R}$, i.e.,

$$D^{\alpha}_{0+}x(t) := \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}x(\tau)\,d\tau$$

with $\Gamma : (0,\infty) \to \mathbb{R}$ is the Gamma function defined by

$$\Gamma(\gamma) := \int_0^{\infty} t^{\gamma-1}e^{-t}\,dt, \qquad \gamma \in (0,\infty).$$

Assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a measurable function satisfies the following conditions:

(H1) there is a constant $L > 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$, for almost all $t \in [0,T]$ and all $x, y \in \mathbb{R}$;

(H2) $\int_0^T |f(t,0)|\,dt < \infty$.

By virtue of [4, Theorem 3.2], for any initial condition $x \in \mathbb{R}$, the Equation (1) with the initial condition

$$\lim_{t\to 0} t^{1-\alpha}x(t) = x$$

has a unique solution denoted by $\varphi(\cdot, x)$ on $(0,T]$.

In what follows, we state the main result of this paper in which a lower bound on the separation of two differential solutions of (1) is established. Recall that for any $\beta \in \mathbb{R}$, the Mittag-Leffler function $E_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is defined as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Theorem 1 (A lower bound on the separation between two solutions)**.** *Consider Equation (1) and assume that the function f satisfies the assumptions (H1) and (H2). Then for any $x, y \in \mathbb{R}$, the following estimate*

$$|\varphi(t,x) - \varphi(t,y)| \ge t^{\alpha-1}E_{\alpha,\alpha}(-Lt^{\alpha})|x-y|$$

holds for all $t \in (0,T]$.

Next, we establish an application of the preceding result about non-negativity of Lyapunov exponent of an arbitrary non-trivial solution of a bounded linear Riemann-Liouville equation. Consider the following linear scalar equation

$$D^{\alpha}_{0+}x(t) = a(t)x(t), \tag{2}$$

where $a : [0,\infty) \to \mathbb{R}$ is bounded, i.e. there is a constant $M > 0$ such that $|a(t)| \le M$ for almost all $t \in [0,\infty)$ and measurable. According to [4, Theorem 3.2], for any $x \in \mathbb{R}$ the Equation (2) has a unique solution on $[0,\infty)$, denoted by $\Phi(\cdot, x)$, satisfying the initial value condition

$$\lim_{t\to 0+} t^{1-\alpha}\varphi(t,x) = x.$$

Corollary 1. *The Lyapunov exponent of any non-trivial solution of (2) is non-negative, i.e., for any $x \in \mathbb{R}\setminus\{0\}$ we have*

$$\limsup_{t\to\infty}\frac{1}{t}\log(|\varphi(t,x)|) \ge 0.$$

Proof. Using Theorem 1, we obtain that

$$|\varphi(t,x)| \geq t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha})|x|$$

for all $t > 0$. According to [3, Theorem 1.3, p. 32], there is a positive constant C such that

$$\lim_{t\to\infty} t^{2\alpha}E_{\alpha,\alpha}(-Mt^{\alpha}) = C.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}\log(|\varphi(t,x)|) \geq \limsup_{t\to\infty}\frac{1}{t}\log(\frac{C}{t^{\alpha+1}}) = 0,$$

which completes the proof. □

Proof of the main result

The main ingredient in the proof of the main result is the fact that two differential solution of scalar systems cannot intersect. Note that in contrast to ordinary differential equations, the problem of intersection of two trajectories of fractional systems is not trivial, see [7] for a more detail. To show non-intersection of two trajectories, we rewrite Equation (1) in the following form

$$D^{\alpha}_{0+}x(t) = -Lx + g(t,x(t)), \tag{3}$$

where the positive constant L is given in the content of (H1) and the function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ is defined by

$$g(t,x) := Lx + f(t,x). \tag{4}$$

Remark 1. *(i) An advantage of rewriting equation (1) as (3) is that the function $g(\cdot,\cdot)$ is nondecreasing in the second argument, i.e., for all $x < y$*

$$g(t,x) \leq g(t,y) \tag{5}$$

for almost all $t \in [0,T]$. Indeed, by (4) we have

$$g(t,y) - g(t,x) = -L(y-x) + f(t,y) - f(t,x),$$

which together with (H1), i.e.,

$$|f(t,y) - f(t,x)| \leq L(y-x),$$

shows (5).

(ii) According to [4, Theorem 4.2], the solution $\varphi(\cdot,x)$ of (1) (and also (3)) satisfies

$$\varphi(t,x) = t^{\alpha-1}E_{\alpha,\alpha}(-Lt^{\alpha})\,x + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-L(t-\tau)^{\alpha})\,g(\tau,\varphi(\tau,x))\,d\tau.$$

Lemma 1. *For any $x < y$, we have $\varphi(t,x) < \varphi(t,y)$ for all $t \in (0,T]$.*

Proof. Suppose the contrary, i.e., the set

$$\{t \in (0,T] : \varphi(t,x) \geq \varphi(t,y)\}$$

is not empty. Let

$$\kappa := \inf\{t \in (0,T] : \varphi(t,x) \geq \varphi(t,y)\}.$$

By the definition of the intial condition, we have $\kappa > 0$ and by the continuity property of solutions, we have

$$\varphi(\kappa,x) = \varphi(\kappa,y) \quad \text{and} \quad \varphi(t,x) < \varphi(t,y) \tag{6}$$

for all $0 < t < \kappa$. By Remark 1(ii), we have

$$\varphi(\kappa,y) - \varphi(\kappa,x) = \kappa^{\alpha-1}E_{\alpha,\alpha}(-L\kappa^{\alpha})(y-x) + \int_0^{\kappa}(\kappa-\tau)^{\alpha-1}E_{\alpha,\alpha}(-L(\kappa-\tau)^{\alpha})\,(g(\tau,\varphi(\tau,y)) - g(\tau,\varphi(\tau,x)))\ d\tau.$$

Using Remark 1(i) and (6), we obtain that

$$g(\tau, \varphi(\tau, y)) \geq g(\tau, \varphi(\tau, x))$$

for almost all $\tau \in [0, \kappa]$. Therefore, by positivity of the function $E_{\alpha,\alpha}(z)$ for $z \in \mathbb{R}$ we have

$$\varphi(\kappa, y) - \varphi(\kappa, x) \geq \kappa^{\alpha-1} E_{\alpha,\alpha}(-L\kappa^{\alpha})(y - x) > 0$$

which contradicts to (6). The proof is complete. □

Now we are in a position to prove our main theorem of this paper.

Proof of Theorem 1. Without loss of generality, we assume that $x < y$. Then by Lemma 1, we have $\varphi(t, x) < \varphi(t, y)$ for any $t \in [0, T]$. Let $t \in [0, T]$ be arbitrary. Using Remark 1(ii), we obtain that

$$\begin{aligned} \varphi(t, y) - \varphi(t, x) &= t^{\alpha-1} E_{\alpha,\alpha}(-Lt^{\alpha})(y - x) + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-L(t - \tau)^{\alpha})(g(\tau, \varphi(\tau, y)) - g(\tau, \varphi(\tau, x)))d\tau \\ &\geq t^{\alpha-1} E_{\alpha,\alpha}(-Lt^{\alpha})(y - x), \end{aligned}$$

which completes the proof. □

Conclusions

Our main interest in this paper is to study the separation between two solutions of a scalar Riemann-Liouville fractional differential equation. By proving non-intersection of two distinct solutions (Lemma 1), a lower bound on the separation of two distinct solutions was established (Theorem 1). Based on this estimate, we showed that the Lyapunov exponent of an arbitrary non-trivial solution of a linear bounded scalar Riemann-Liouville equation is always non-negative (Corollary 1).

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