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MIXED TYPE DUALITY FOR A CLASS OF MULTIPLE OBJECTIVE OPTIMIZATION PROBLEMS WITH AN INFINITE NUMBER OF CONSTRAINTS

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Dedicated to Professor Do Sang Kim on the occasion of his 65th birthday with respect

Abstract. This paper focuses on the study of optimality conditions and mixed type duality for a class of multiple objective optimization problems with an infinite number of constraints, denoted by (MOSIP). First, we explore a constraint qualification that is used to propose necessary optimality condition for problem (MOSIP). Then, a mixed type dual model is established, and weak duality under generalized convexity and strong duality with the assumption of the proposed constraint qualification are investigated.

1. Introduction

Multiobjective optimization has many applications in science, engineering, economics and logistics where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives [4, 5, 8, 14, 23]. Recently, optimization problems with an infinite number of constraints have been studied in many research papers; see [6, 16, 17, 20, 24, 26–28] and the references therein. This paper considers a class of multiple objective optimization problems with an infinite number of constraints which enjoys the following form:

$$\text{(MOSIP)} \qquad \text{Min}_{\mathbb{R}^m_+} f(x) \quad \text{subject to} \quad x \in F,$$

where $\mathbb{R}^m_+$ denotes the nonnegative orthant of $\mathbb{R}^m$, $f := (f_1, \ldots, f_m)$ is a vector function that each component $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in M := \{1, 2, \ldots, m\}$ are locally Lipschitz functions; moreover, $F := \{x \in C \colon g_t(x) \leqq 0, t \in T\}$ denotes the feasible set of problem (MOSIP), in which $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are locally Lipschitz with respect to x uniformly in t, T is an index set (compact but possibly infinite), and C is a closed (not necessarily convex) subset of $\mathbb{R}^n$.

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be Euclidean vector spaces referred to decision space and the image space. For $a, b \in \mathbb{R}^m$, we understand that

- $a \leq b \Leftrightarrow a_i \leqq b_i, \forall i = 1, \ldots, m$ but $a \neq b$; $a \nleq b$ is the negation of $a \leq b$.
- $a \leqq b$ and $a < b$ have the usual meaning in partial orderings.

Below, we recall the notion of (*weakly*) *efficient solution* to problem (MOSIP).

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Definition 1.1 ([8, Definition 2.24]). A point $\bar{x} \in F$ is said to be

(i) an *efficient solution* to problem (MOSIP) if there exists no $x \in F$ such that $f(x) \leq f(\bar{x})$;
(ii) a *weakly efficient solution* to problem (MOSIP) if there exists no $x \in F$ such that $f(x) < f(\bar{x})$.

Indeed, in the face of problem (MOSIP), there are many methods dealing with it; among them, scalarization method is shown to be an important one. The relevance of using scalarization methods to study problem (MOSIP) is that scalar problems can have more effective means of finding optimal solutions than vector problems. For a deeper study, the reader is referred to the books [3, 5, 8, 14] and the papers [9, 13, 21, 24] and the references therein. In our research, we are interested in the so-called *ϵ-constraint method*, which was minutely studied by Chankong and Haimes [5] and improved by Ehrgott and Ruzika [9]; moreover, it is worth mentioning that the ϵ-constraint method is proved to be an effective one to find efficient solutions of a class of multiobjective optimization problems with SOS-convex polynomials; see, [15, 18, 19] for example.

Mathematically, the ϵ-constraint method is based on a scalarization where one of the objective functions is minimized while all the other objective functions are bounded from above by means of additional constraints:

$$(\mathrm{P}_j(\epsilon)) \qquad \mathrm{Min}_{x\in F} f_j(x) \quad \text{subject to} \quad f_k(x) \leqq \epsilon_k,\ k \in M^j := M \setminus \{j\},$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m$, note that the component ϵ_j is unrelated for problem $(\mathrm{P}_j(\epsilon))$, the convention involving it here will be convenient. For each $j = 1, \ldots, m$, let $F_j(\epsilon) := \{x \in F : f_k(x) \leqq \epsilon_k,\ k \neq j\}$ be the feasible set of problem $(\mathrm{P}_j(\epsilon))$, which is assumed to be nonempty.

In what follows, we state the criteria of ϵ-constraint method (see [5, Theorem 4.1] or [8, Theorem 4.5]) that will play a key role in the present paper.

Theorem 1.2. *A point $\bar{x} \in F$ is an efficient solution to problem* (MOSIP) *if and only if it is an optimal solution to problem* $(\mathrm{P}_j(\epsilon))$ *for each $j \in M$, where $\epsilon_k = f_k(\bar{x})$ for $k \in M^j$.*

Remark 1.3. Observe that if $\bar{x}$ is an efficient solution to problem (MOSIP), then it is also an optimal solution to problem $(\mathrm{P}_j(\epsilon))$ for some $j \in M$; however the converse is not always true (actually, it was shown that if $\bar{x}$ is an optimal solution to problem $(\mathrm{P}_j(\epsilon))$ for some j, then $\bar{x}$ is a *weakly efficient solution* to problem (MOSIP) [8, Proposition 4.3]).

According to Remark 1.3, Piao *et al.* [24] studied necessary conditions for problems $(\mathrm{P}_j(\epsilon))$ and (MOSIP) under the assumption of some suitable constraint qualifications. In this paper, we will propose necessary optimality condition for problem (MOSIP) by using Theorem 1.2 rather than Remark 1.3.

The rest of the paper is as follows: In Section 2, we state some preliminaries and basic concepts in nonsmooth analysis and linear space for semi-infinite programs. We present the main results in Section 3. In the first part, we explore a constraint qualification which is used to propose necessary optimality condition for problem (MOSIP), then we formulate necessary optimality conditions for problem (MOSIP) under the assumption of the proposed constraint qualification. In the second part, a mixed type dual model, which is a little bit different from the classical ones, for instance, see [1, 29], is established; weak duality under generalized convexity and strong duality with the assumption of the proposed constraint qualification are investigated, successively. Finally, we give our conclusions in Section 4.

2. Preliminaries

In this section, we overview briefly some notions of convex analysis and nonsmooth analysis widely used in the present paper; see [7, 25] for more details. Let $\mathbb{R}^n$ denote the Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be a *locally Lipschitz function* if for any $x \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of x such that

$$|\phi(y) - \phi(z)| \leqq K\|y - z\|, \quad \forall y, z \in N(x);$$

moreover, for a sequence $(\phi_t)_{t\in T}, \phi_t : \mathbb{R}^n \to \mathbb{R}$, where T is a compact index set, we say that $(\phi_t)_{t\in T}$ is locally Lipschitz with respect to x uniformly in t if there exists a neighborhood $N(x)$ and a constant $K > 0$ such that

$$|\phi_t(y) - \phi_t(z)| \leqq K\|y - z\|, \quad \forall y, z \in N(x) \text{ and } \forall t \in T.$$

2.1. **Nonsmooth analysis.** The *generalized directional derivative* of ϕ at x in the direction $d \in \mathbb{R}^n$ in the sense of Clarke [7, Chapter 2] is defined by

$$\phi^\circ(x; d) := \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{\phi(y + td) - \phi(y)}{t}$$

and the *Clarke subdifferential* of ϕ at x, denoted by $\partial^c\phi(x)$, is

$$\partial^c\phi(x) := \{u \in \mathbb{R}^n : \phi^\circ(x; d) \geqq \langle u, d\rangle, \ \forall d \in \mathbb{R}^n\}.$$

In particular, for $d \in \mathbb{R}^n$, if $\lim_{t\downarrow 0} \frac{\phi(x+td)-\phi(x)}{t}$ exists, then it is called the *directional derivative* of ϕ at x in the direction d and is denoted by $\phi'(x; d)$.

Definition 2.1 ([7, Definition 2.3.4])**.** The function ϕ is said to be *regular* at x, if $\phi'(x; d)$ exists and coincides with $\phi^\circ(x; d)$ for each $d \in \mathbb{R}^n$.

For a closed subset $D \subset \mathbb{R}^n$, the *tangent cone* to D at x is defined by

$$T_D(x) := \{h \in \mathbb{R}^n : d^\circ_D(x; h) = 0\},$$

where d_D denotes the distance function to D. The *normal cone* to D at x is defined by

$$N_D(x) := \{u \in \mathbb{R}^n : \langle u, h\rangle \leqq 0, \ \forall h \in T_D(x)\}.$$

If in addition D is convex, the normal cone to D coincides with the one in the sense of convex analysis:

$$N_D(x) = \{u \in \mathbb{R}^n : \langle u, y - x\rangle \leqq 0, \ \forall y \in D\}.$$

2.2. **Linear space in semi-infinite programs.** The following linear space is used for semi-infinite programming [10]:

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \colon \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) = \{t \in T \colon \lambda_t \neq 0\}$, is a finite subset of T. The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by:

$$\mathbb{R}^{(T)}_{+} = \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \colon \lambda_t \geqq 0, t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space, we understand that

$$\sum_{t\in T} \lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $g_t, t \in T$,

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

3. Main results

In this section, we present our main results: (i) we explore a constraint qualification that is used to propose necessary optimality condition for problem (MOSIP), and then under the assumption of the explored constraint qualification we establish necessary conditions for problem (MOSIP). Besides, we also study the sufficient optimality condition for problem (MOSIP) by means of introducing the concepts of (strictly) generalized convexity, which is different to the results in the paper [24]. (ii) We formulate a mixed type dual model due to the primal problem; weak and strong duality are then investigated.

3.1. **Optimality Conditions.** First of all, we consider the following scalar optimization problem in order to recall some results for a single objective optimization problem.

$$\text{(P)} \qquad \text{Min}_{x\in\mathbb{R}^n} f_0(x) \quad \text{subject to} \quad x \in F,$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, and $F := \{x \in C \colon g_t(x) \leqq 0, t \in T\}$ is the feasible set of problem (P), which is same to the one of problem (MOSIP).

Definition 3.1. Let $\bar{x} \in F$. We say that the following (CQ) *condition* for problem (P) is satisfied at $\bar{x}$, if

$$\text{(CQ)} \qquad \exists d \in T_C(\bar{x}) : g^c_t(\bar{x}; d) < 0, \text{ for all } t \in I(\bar{x}),$$

where $I(\bar{x}) := \{t \in T \colon g_t(\bar{x}) = 0\}$.

Remark 3.2. It is worth mentioning that (i) if T is finite, g_t, $t \in T$ are smooth functions and $C = \mathbb{R}^n$, the (CQ) condition coincides with the Mangasarian–Fromovitz constraint qualification [2], i.e., there is a direction d in $\mathbb{R}^n$ such that $\langle \nabla g_t(\bar{x}), d\rangle < 0$ for all $t \in I(\bar{x})$; (ii) if T is finite and $g_t, t \in T$ are locally Lipschitz functions, the (CQ) condition coincides with the condition in [11], which says that there is a direction d in $T_C(\bar{x})$ satisfying $g^c_t(\bar{x}; d) < 0$ for all $t \in I(\bar{x})$.

According to [27, Theorems 4.1 and 4.2] (where problem (P) works on a Banach space), we can derive the following lemma for the case of the related functions defined on $\mathbb{R}^n$, the proof was given in [24].

Lemma 3.3 ([24, 27])**.** *Let $\bar{x}$ be an optimal solution to problem* (P), *assume that the* (CQ) *condition holds at $\bar{x}$, then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f_0(\bar{x}) + \sum_{t\in T} \lambda_t \partial^c g_t(\bar{x}) + N_C(\bar{x}), \textit{ and } \quad g_t(\bar{x}) = 0, \ \forall t \in T(\lambda).$$

The following definition, which is a modified version of the (CQ) condition for problem (P) (see Definition 3.1), is associated to problem ($\mathrm{P}_j(\epsilon)$). Recall that the feasible set of problem ($\mathrm{P}_j(\epsilon)$) is denoted by $F_j(\epsilon)$, observe that we have $F_j(\epsilon) \subset F$ since $F_j(\epsilon) = F \cap \{x \in \mathbb{R}^n \colon f_k(x) \leqq \epsilon_k, \ k \neq j\}$.

Definition 3.4 ([17])**.** Let $\bar{x} \in \mathbb{R}^n$, we say that the following (MCQ_j) *condition* for problem ($\mathrm{P}_j(\epsilon)$) is satisfied at $\bar{x}$, if

$$(\mathrm{MCQ}_j) \qquad \exists d \in T_C(\bar{x}) : \begin{cases} g_t^c(\bar{x}; d) < 0, \text{ for all } t \in I(\bar{x}), \\ f_k^c(\bar{x}; d) < 0, \text{ for all } k \in H_j(\bar{x}). \end{cases}$$

where $I(\bar{x}) = \{t \in T \colon g_t(\bar{x}) = 0\}$, $H_j(\bar{x}) = \{k \in M^j \colon f_k(\bar{x}) = \epsilon_k\}$, $I(\bar{x}) \cap H_j(\bar{x}) = \emptyset$ and $\bar{I}(\bar{x}) = I(\bar{x}) \cup H_j(\bar{x})$.

Now, we give a KKT type optimality condition for an optimal solution to problem ($\mathrm{P}_j(\epsilon)$) for each $j \in M$ under the fulfilment of the (MCQ_j) condition for each corresponding j.

Theorem 3.5. *For each given $j \in M$, let $\bar{x} \in F_j(\epsilon)$ be an optimal solution to problem* ($\mathrm{P}_j(\epsilon)$) *and assume that the* (MCQ_j) *condition holds at $\bar{x}$. Then there exist $\alpha_k \geqq 0, k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f_j(\bar{x}) + \sum_{k\in M^j} \alpha_k \partial^c f_k(\bar{x}) + \sum_{t\in T} \lambda_t \partial^c g_t(\bar{x}) + N_C(\bar{x}), \tag{3.1}$$
$$g_t(\bar{x}) = 0, \ \forall\, t \in T(\lambda).$$

Proof. For each given $j \in M$, let $\bar{x}$ be an optimal solution to problem ($\mathrm{P}_j(\epsilon)$), then $f_j(\bar{x}) \leqq f_j(x), \ \forall\, x \in F_j(\epsilon)$. Without loss of generality, we assume that $T \cap M^j = \emptyset$, set

$$G_t(\cdot) = \begin{cases} f_t(\cdot) - \epsilon_t, & t \in M^j, \\ g_t(\cdot), & t \in T, \end{cases} \quad \text{and } \bar{T} = T \cup M^j.$$

Since the (MCQ_j) condition holds at $\bar{x}$ for each given $j \in M$, by applying Lemma 3.3 to the following scalar problem:

$$(\mathrm{EP}_j(\epsilon)) \qquad \min \ f_j(x) \quad \text{subject to } \ x \in C, \ G_t(x) \leqq 0, \ t \in \bar{T},$$

there exist $\bar{\lambda} \in \mathbb{R}_+^{(\bar{T})}$ such that

$$0 \in \partial^c f_j(\bar{x}) + \sum_{t\in \bar{T}} \bar{\lambda}_t \partial^c G_t(\bar{x}) + N_C(\bar{x}), \ G_t(\bar{x}) = 0, \ \forall t \in \bar{T}(\bar{\lambda}).$$

Hence $0 \in \partial^c f_j(\bar{x}) + \sum_{k\in M^j} \alpha_k \partial^c f_k(\bar{x}) + \sum_{t\in T} \lambda_t \partial^c g_t(\bar{x}) + N_C(\bar{x})$, where $\bar{\lambda}_t$ is replaced by α_k if $t \in M^j$ and by λ_t if $t \in T$ as well as $g_t(\bar{x}) = 0, \forall t \in T(\lambda)$. □

Remark 3.6. Note that problem ($\mathrm{EP}_j(\epsilon)$) is equivalent to problem ($\mathrm{P}_j(\epsilon)$).

Below, with the aid of Theorems 1.2 and 3.5, we propose a KKT type necessary optimality condition for an efficient solution to problem (MOSIP).

Theorem 3.7 (Necessary Condition)**.** *Let $\bar{x} \in F$ be an efficient solution to problem* (MOSIP)*, and assume that the* (MCQ_j) *condition holds at $\bar{x}$ for each $j \in M$, then there exist $\tau \in \mathbb{R}^m_+$ with $\sum_{j\in M}\tau_j = 1$ and $\nu \in \mathbb{R}^{(T)}_+$ such that*

$$0 \in \sum_{j\in M}\tau_j\partial^c f_j(\bar{x}) + \sum_{t\in T}\nu_t\partial^c g_t(\bar{x}) + N_C(\bar{x}), \tag{3.2}$$
$$g_t(\bar{x}) = 0, \ \forall t \in T(\lambda).$$

Proof. Since $\bar{x}$ is an efficient solution to problem (MOSIP), then along with the criteria of the ϵ-constraint method (Theorem 1.2), we have $\bar{x}$ is an optimal solution to problem ($\mathrm{P}_j(\epsilon)$) for each $j \in M$, and $\bar{x} \in F_j(\epsilon)$ since we take $\epsilon_k = f_k(\bar{x})$ in Theorem 1.2.

On the other hand, by Theorem 3.5, there exist $\alpha_k \geqq 0, k \in M^j$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that (3.1) holds as well as $g_t(\bar{x}) = 0, \ \forall t \in T(\lambda)$. This implies that

$$0 \in \sum_{j\in M}\tau_j\partial^c f_j(\bar{x}) + \sum_{t\in T}\nu_t\partial^c g_t(\bar{x}) + N_C(\bar{x}),$$

where $\tau_j = \dfrac{1}{1+\sum_{k\in M^j}\alpha_k}$, $\tau_k = \dfrac{\alpha_k}{1+\sum_{k\in M^j}\alpha_k}$, $k \in M^j$, and $\nu_t = \dfrac{\lambda_t}{1+\sum_{k\in M^j}\alpha_k}$, $t \in T$.

It is easy to check that $\sum_{j\in M}\tau_j = 1$ and $\dfrac{1}{1+\sum_{k\in M^j}\alpha_k}N_C(\bar{x}) \subset N_C(\bar{x})$. □

It should be noted here that the (MCQ_j) *condition* is essential since the constraint qualification in Theorem 3.7 is not for problem (MOSIP) but for problem ($\mathrm{P}_j(\epsilon)$); in other words, we proposed the KKT type necessary optimality condition for problem (MOSIP) via the one for problem ($\mathrm{P}_j(\epsilon)$).

Now, we design an example to illustrate the obtained two theorems.

Example 3.8. Consider the following nonsmooth multiobjective semi-infinite problem $(\mathrm{MOSIP})_1$:

$$(\mathrm{MOSIP})_1 \quad \begin{array}{ll} \text{Minimize} & f(x) := (f_1(x), f_2(x)) \\ \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & x \in C, \end{array}$$

where $f_1(x) = |x-2|$, $f_2(x) = (x-1)^2$ and $g_t(x) = tx^2 - 2x$, $t \in [0,1]$, $C := [0,3]$. It is easy to verify that the feasible set of $(\mathrm{MOSIP})_1$ is $[0,2]$ and its efficient solution set is $[1,2]$.

Now take $\bar{x} = 2$ to be the optimal solution of both ($\mathrm{P}_1(\epsilon)$) and ($\mathrm{P}_2(\epsilon)$) by taking $\epsilon_k = f_k(\bar{x})$ as the statement in Theorem 1.2. Observe that both the (MCQ_1) condition and (MCQ_2) condition hold. First, let us deal with $j = 1 \in \{1,2\}$, that is,

$$(\mathrm{P}_1(\epsilon)) \quad \min \quad f_1(x)$$

$$\text{s.t.} \quad g_t(x) \leqq 0,\ f_2(x) \leqq \epsilon_2,\ x \in C,$$

and we have $0 \in [-1, 1] + 2\alpha_2 + 2\lambda_1 + \{0\}$, so we may get $\alpha_2 = \frac{1}{4}$ and $\lambda_1 = \frac{1}{4}$ for $t = 1$, $\lambda_t = 0$ for all $t \in [0, 1)$ such that Theorem 3.5 holds for $j = 1$. Second, for $j = 2 \in \{1, 2\}$, that is,

$$\begin{aligned} (\mathrm{P}_2(\epsilon)) \quad & \min \quad f_2(x) \\ & \text{s.t.} \quad g_t(x) \leqq 0,\ f_1(x) \leqq \epsilon_1,\ x \in C, \end{aligned}$$

one has easily that $0 \in 2 + \alpha_1[-1, 1] + 2\lambda_1$ and we may get $\alpha_1 = 3$ and $\lambda_1 = \frac{1}{4}$ for $t = 1$, $\lambda_t = 0$ for all $t \in [0, 1)$ such that Theorem 3.5 holds for $j = 2$. Up to now, we declare that Theorem 3.5 holds.

We turn to check Theorem 3.7. Let $\bar{x} = 2$ be the efficient solution of $(\mathrm{MOSIP})_1$, then $\bar{x} = 2$ solves $(\mathrm{P_j}(\epsilon))$ for each $j \in \{1, 2\}$, as we discussed above it is easy to obtain that there exist $\tau = (\frac{4}{5}, \frac{1}{5})$ and $\nu_1 = \frac{1}{5}$ for $t = 1$ and $\nu_t = 0$ for all $t \in [0, 1)$ such that Theorem 3.7 holds. (or there exist $\tau = (\frac{1}{4}, \frac{3}{4})$ and $\nu_1 = \frac{1}{5}$ for $t = 1$ and $\nu_t = 0$ for all $t \in [0, 1)$ such that Theorem 3.7 holds.)

Before we discuss the sufficient conditions for (weakly) efficient solutions of problem (MOSIP), we would recall the concepts of (strict) generalized convexity, which is defined in [28, Definition 3.1] by means of Clarke subdifferential, for a family of locally Lipschitz functions; and furthermore the concepts are inspired by [6].

Definition 3.9. Let $f := (f_1, \dots, f_m)$ and $g_T := (g_t)_{t \in T}$.

(i) We say that (f, g_T) is *generalized convex* on C at $\bar{x}$, if for any $x \in C$, $z^* \in \partial^c f_j(\bar{x}), j = 1, \dots, m$ and $x_t^* \in \partial^c g_t(\bar{x}), t \in T$, there exists $\omega \in T_C(\bar{x})$ satisfying

$$\begin{aligned} f_j(x) - f_j(\bar{x}) &\geqq \langle z_j^*, \omega \rangle,\ j = 1, \dots, m, \\ g_t(x) - g_t(\bar{x}) &\geqq \langle x_t^*, \omega \rangle,\ t \in T. \end{aligned}$$

(ii) We say that (f, g_T) is *strictly generalized convex* on C at $\bar{x}$, if for any $x \in C \backslash \{\bar{x}\}$, $z^* \in \partial^c f_j(\bar{x}), j = 1, \dots, m$ and $x_t^* \in \partial^c g_t(\bar{x}), t \in T$, there exists $\omega \in T_C(\bar{x})$ satisfying

$$\begin{aligned} f_j(x) - f_j(\bar{x}) &> \langle z_j^*, \omega \rangle,\ j = 1, \dots, m, \\ g_t(x) - g_t(\bar{x}) &\geqq \langle x_t^*, \omega \rangle,\ t \in T. \end{aligned}$$

Remark 3.10. Observe that, if C is convex and $f_i, i \in M$, and $g_t, t \in T$ are convex (resp., strictly convex), then (f, g_T) is generalized convex (resp., strictly generalized convex) on C at any $\bar{x} \in C$ with $\omega := x - \bar{x}$ for each $x \in C$. Furthermore, as the authors [28] pointed that by a similar argument in [6, Example 3.2], we can show that the class of generalized convex functions is properly larger than the one of convex functions.

Theorem 3.11 (Sufficient Condition)**.** *Let $\bar{x} \in F$ satisfy* (3.2).

(i) *If (f, g_T) is generalized convex on C at $\bar{x}$, then $\bar{x}$ is a weakly efficient solution.*

(ii) *If (f, g_T) is strictly generalized convex on C at $\bar{x}$, then $\bar{x}$ is an efficient solution.*

Proof. Since $\bar{x} \in F$ satisfies (3.2), there exist $\tau \in \mathbb{R}^m_+$ with $\sum_{j\in M} \tau_j = 1$, $\nu \in \mathbb{R}^{(T)}_+$ and $z_j^* \in \partial^c f_j(\bar{x}),\ j = 1, \ldots, m,\ x_t^* \in \partial^c g_t(\bar{x}),\ t \in T$ such that

$$-\Big(\sum_{j\in M} \tau_j z_j^* + \sum_{t\in T} \nu_t x_t^*\Big) \in N_C(\bar{x}). \tag{3.3}$$

We first show (i). Assume to the contrary that $\bar{x}$ is not a weakly efficient solution, which tells us that there exists $\hat{x} \in F$ such that

$$f(\bar{x}) - f(\hat{x}) \in \operatorname{int} \mathbb{R}^m_+. \tag{3.4}$$

By relationship between tangent cone and normal cone, and the generalized convexity of (f, g_T), we deduce from (3.3) that, for such $\hat{x}$, there exists $\omega \in T_C(\bar{x})$ such that

$$0 \leqq \sum_{j\in M} \tau_j \langle z_j^*, \omega\rangle + \sum_{t\in T} \nu_t \langle x_t^*, \omega\rangle \leqq \sum_{j\in M} \tau_j [f_j(\hat{x}) - f_j(\bar{x})] + \sum_{t\in T} \nu_t [g_t(\hat{x}) - g_t(\bar{x})].$$

Hence,

$$\sum_{j\in M} \tau_j f_j(\bar{x}) + \sum_{t\in T} \nu_t g_t(\bar{x}) \leqq \sum_{j\in M} \tau_j f_j(\hat{x}) + \sum_{t\in T} \nu_t g_t(\hat{x}),$$

combining this with the facts that $\nu_t g_t(\bar{x}) = 0$, and $\nu_t g_t(\hat{x}) \leqq 0$ for all $t \in T$, we conclude that

$$\begin{aligned}\sum_{j\in M} \tau_j f_j(\bar{x}) &= \sum_{j\in M} \tau_j f_j(\bar{x}) + \sum_{t\in T} \nu_t g_t(\bar{x}) \\ &\leqq \sum_{j\in M} \tau_j f_j(\hat{x}) + \sum_{t\in T} \nu_t g_t(\hat{x}) \\ &\leqq \sum_{j\in M} \tau_j f_j(\hat{x}).\end{aligned}$$

This entails that there exists $j_0 \in M$ such that $f_{j_0}(\bar{x}) \leqq f_{j_0}(\hat{x})$ due to $\tau \in \mathbb{R}^m_+ \backslash \{0\}$. This, along with (3.4), gives a contradiction.

Now, we prove (ii). Suppose for contradiction that $\bar{x}$ is not an efficient solution. This means that there exists $\hat{x} \in F$ such that

$$f(\bar{x}) - f(\hat{x}) \in \mathbb{R}^m_+ \backslash \{0\}. \tag{3.5}$$

Again, by relationship between tangent cone and normal cone, and the strictly generalized convexity of (f, g_T), we deduce from (3.3) that, for such $\hat{x}$, there exists $\omega \in T_C(\bar{x})$ such that

$$0 \leqq \sum_{j\in M} \tau_j \langle z_j^*, \omega\rangle + \sum_{t\in T} \nu_t \langle x_t^*, \omega\rangle < \sum_{j\in M} \tau_j [f_j(\hat{x}) - f_j(\bar{x})] + \sum_{t\in T} \nu_t [g_t(\hat{x}) - g_t(\bar{x})].$$

Thus,

$$\sum_{j\in M} \tau_j f_j(\bar{x}) + \sum_{t\in T} \nu_t g_t(\bar{x}) < \sum_{j\in M} \tau_j f_j(\hat{x}) + \sum_{t\in T} \nu_t g_t(\hat{x}).$$

Observe that $\nu_t g_t(\bar{x}) = 0$, and $\nu_t g_t(\hat{x}) \leqq 0$ for all $t \in T$. Therefore, we have

$$\sum_{j\in M} \tau_j f_j(\bar{x}) = \sum_{j\in M} \tau_j f_j(\bar{x}) + \sum_{t\in T} \nu_t g_t(\bar{x})$$

$$< \sum_{j\in M} \tau_j f_j(\hat{x}) + \sum_{t\in T} \nu_t g_t(\hat{x})$$
$$\leqq \sum_{j\in M} \tau_j f_j(\hat{x}).$$

This implies that there exists $j_0 \in M$ such that $f_{j_0}(\bar{x}) < f_{j_0}(\hat{x})$. This, along with (3.5), gives a contradiction. □

3.2. **Duality Theorems.** In this part, we introduce a mixed type dual problem (MD), which combines the type of Wolfe [29] with the type of Mond–Weir [22] dual problems. Then, we establish weak and strong duality theorems between the corresponding ones.

$$\begin{aligned}
\text{(MD)}\quad \text{Maximize}\quad & f(y) + \sum_{t\in T} \lambda_t g_t(y) e \\
\text{subject to}\quad & 0 \in \sum_{j\in M} \tau_j \partial^c f_j(y) + \sum_{t\in T} (\lambda_t + \mu_t) \partial^c g_t(y) + N_C(y), \\
& \mu_t g_t(y) \geqq 0, t \in T, \\
& \tau \in \mathbb{R}^m_+, \tau^T e = 1, e = (1, \ldots, 1) \in \mathbb{R}^m, \\
& (y, \tau, \lambda, \mu) \in C \times \mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+ \times \mathbb{R}^{(T)}_+.
\end{aligned}$$

Let us denote by G the feasible set of (MD). It is worth noting that the problem model (MD) is a vector version of the one due to [26]. It is obvious that, for $\mu = 0$, the problem (MD) is equivalent to the dual of problem (MOSIP) in the sense of Wolfe [29], denoted by (MD)_W:

$$\begin{aligned}
\text{(MD)}_\text{W}\quad \text{Maximize}\quad & f(y) + \sum_{t\in T} \lambda_t g_t(y) e \\
\text{subject to}\quad & 0 \in \sum_{j\in M} \tau_j \partial^c f_j(y) + \sum_{t\in T} \lambda_t \partial^c g_t(y) + N_C(y), \\
& \tau \in \mathbb{R}^m_+, \tau^T e = 1, e = (1, \ldots, 1) \in \mathbb{R}^m, \\
& (y, \tau, \lambda) \in C \times \mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+.
\end{aligned}$$

For $\lambda = 0$, the problem (MD) is equivalent to the dual of problem (MOSIP) in the sense of Mond–Weir [22], denoted by (MD)_M:

$$\begin{aligned}
\text{(MD)}_\text{M}\quad \text{Maximize}\quad & f(y) \\
\text{subject to}\quad & 0 \in \sum_{j\in M} \tau_j \partial^c f_j(y) + \sum_{t\in T} \mu_t \partial^c g_t(y) + N_C(y), \\
& \mu_t g_t(y) \geqq 0, t \in T, \\
& \tau \in \mathbb{R}^m_+, \tau^T e = 1, e = (1, \ldots, 1) \in \mathbb{R}^m, \\
& (y, \tau, \mu) \in C \times \mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+.
\end{aligned}$$

The following theorem presents weak duality relations between the primal problem (MOSIP) and the dual problem (MD).

Theorem 3.12 (Weak Duality)**.** *Let x and (y,τ,λ,μ) be the feasible solutions of problems* (MOSIP) *and* (MD), *respectively.*

(i) *If (f, g_T) is generalized convex on C at y, then $f(x) \nless f(y)+\sum_{t\in T}\lambda_t g_t(y)e$.*
(ii) *If (f, g_T) is strictly generalized convex on C at y, then $f(x) \nleq f(y) + \sum_{t\in T}\lambda_t g_t(y)e$.*

Proof. Since $(y,\tau,\lambda,\mu)\in G$, there exist $\tau\in\mathbb{R}^m_+$ with $\sum_{j\in M}\tau_j = 1$, $\lambda,\mu\in\mathbb{R}^{(T)}_+$ and $z^*_j\in\partial^c f_j(\bar{x}),\ j=1,\dots,m,\ x^*_t\in\partial^c g_t(\bar{x}),\ t\in T$ such that

$$-\Big(\sum_{j\in M}\tau_j z^*_j + \sum_{t\in T}(\lambda_t+\mu_t)x^*_t\Big)\in N_C(\bar{x}). \tag{3.6}$$

(i) Assume to the contrary that

$$f(x) < f(y) + \sum_{t\in T}\lambda_t g_t(y)e.$$

Hence, $\langle \tau, f(x) - [f(y) + \sum_{t\in T}\lambda_t g_t(y)e]\rangle < 0$, which is equivalent to the following inequality:

$$\sum_{j\in M}\tau_j[f_j(x)-f_j(y)] - \sum_{t\in T}\lambda_t g_t(y) < 0. \tag{3.7}$$

By relationship between tangent cone and normal cone, and the generalized convexity of (f, g_T) on C at y, we deduce from (3.6) that, for such x, there exists $\omega\in T_C(\bar{x})$ such that

$$\begin{aligned} 0 &\leqq \sum_{j\in M}\tau_j\langle z^*_j,\omega\rangle + \sum_{t\in T}\lambda_t\langle x^*_t,\omega\rangle \\ &\leqq \sum_{j\in M}\tau_j[f_j(x)-f_j(y)] + \sum_{t\in T}\lambda_t[g_t(x)-g_t(y)]. \end{aligned} \tag{3.8}$$

It follows by $x\in F$ that $\sum_{t\in T}\lambda_t g_t(\bar{x})\leqq 0$. Thus, (3.8) implies that

$$0\leqq \sum_{j\in M}\tau_j[f_j(x)-f_j(y)] - \sum_{t\in T}\lambda_t g_t(y).$$

It comes a contradiction in the view of (3.6) and (3.8).

We would leave the proof of (ii) to the reader. □

The forthcoming theorem describes strong duality relations between the primal problem (MOSIP) and the dual problem (MD).

Theorem 3.13 (Strong Duality)**.** *Let $\bar{x}$ be a weakly efficient solution to* (MOSIP) *such that the* (MCQ_j) *condition is satisfied at $\bar{x}$ for some $j\in M$. Then there exists $(\bar{\tau},\bar{\lambda},\bar{\mu})\in\mathbb{R}^m_+\times\mathbb{R}^{(T)}_+\times\mathbb{R}^{(T)}_+$ such that $(\bar{x},\bar{\tau},\bar{\lambda},\bar{\mu})$ is a feasible solution of* (MD) *and $f(\bar{x}) = f(\bar{x}) + \sum_{t\in T}\bar{\lambda}_t g_t(\bar{x})e$. Furthermore,*

(i) *If (f, g_T) is generalized convex on C at any y, then $(\bar{x},\bar{\tau},\bar{\lambda},\bar{\mu})$ is a weakly efficient solution of* (MD).
(ii) *If (f, g_T) is strictly generalized convex on C at any y, then $(\bar{x},\bar{\tau},\bar{\lambda},\bar{\mu})$ is an efficient solution of* (MD).

Proof. Thanks to Theorem 3.7, there exist $\tau_j \in \mathbb{R}^m_+$ with $\sum_{j\in M} \tau_j = 1$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that

$$\begin{aligned} 0 &\in \sum_{j\in M} \tau_j \partial^c f_j(\bar{x}) + \sum_{t\in T} \lambda_t \partial^c g_t(\bar{x}) + N_C(\bar{x}) \\ &\subset \sum_{j\in M} \tau_j \partial^c f_j(\bar{x}) + \sum_{t\in T} (\lambda_t + \mu_t) \partial^c g_t(\bar{x}) + N_C(\bar{x}), \end{aligned} \tag{3.9}$$

and $(\lambda_t + \mu_t) g_t(\bar{x}) = 0$. Putting

$$\bar{\tau}_j := \frac{\tau_j}{\sum_{j\in M} \tau_j},\ j = 1, \dots, m, \quad \bar{\lambda}_t := \frac{\lambda_t}{\sum_{j\in M} \tau_j},\ \bar{\mu}_t := \frac{\mu_t}{\sum_{j\in M} \tau_j},\ t \in T,$$

then we have $\bar{\tau}_j \in \mathbb{R}^m_+$ with $\sum_{j\in M} \bar{\tau}_j = 1$, and $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{(T)}_+$, $\bar{\mu} := (\bar{\mu}_t)_{t\in T} \in \mathbb{R}^{(T)}_+$. Moreover, the inclusion (3.9) is still valid when τ_j's, λ_t's and μ_t's are replaced by $\bar{\tau}_j$'s, $\bar{\lambda}_t$'s and $\bar{\mu}_t$'s, respectively. Thus, $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (MD). In addition, since $\lambda_t g_t(\bar{x}) = 0$, This implies that $\bar{\lambda}_t g_t(\bar{x}) = 0$, and therefore,

$$f(\bar{x}) = f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) e.$$

(i) Since (f, g_T) is generalized convex on C at any y, according to (i) of Theorem 3.12, we have

$$f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) e = f(\bar{x}) \not< f(y) + \sum_{t\in T} \lambda_t g_t(y) e,$$

for any $(y, \tau, \lambda, \mu) \in G$. Thus, $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (MD).

(ii) Since (f, g_T) is strictly generalized convex on C at any y, according to (ii) of Theorem 3.12, we obtain

$$f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) e \not\leq f(y) + \sum_{t\in T} \lambda_t g_t(y) e,$$

for any $(y, \tau, \lambda, \mu) \in G$. Hence $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (MD). □

4. Conclusions

In this paper, we studied a class of multiple objective optimization problems with an infinite number of constraints by the so-called *ϵ-constraint method.* We explored a constraint qualification in order to propose necessary optimality condition for problem (MOSIP). Moreover, we investigated weak and strong duality theorems after a mixed type dual model was established. On the other hand, very recently, Lee and Lee [20] studied robust semi-infinite multiobjective optimization problems by another scalarization method (weight-sum method), but how to obtain efficient solutions of robust semi-infinite multiobjective optimization problems by ϵ-constraint method is also an interesting and important issue. This would be examined in a forthcoming study.

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References

[1] K. D. Bae, D. S. Kim and L. G. Jiao, *Mixed duality for a class of nondifferentiable multiobjective programming problems*, J. Nonlinear Convex Anal. **16** (2015), 255–263.

[2] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, CMS Books in Mathematics, Springer-Verlag, New York, 2006.

[3] R. I. Bot, S-M. Grad and G. Wanka, *Duality in Vector Optimization*, Springer-Verlag, Berlin, 2009.

[4] M. Caramia and P. Dell ' Olmo, *Multi-objective Management in Freight Logistics*, Springer-Verlag, London, 2008.

[5] V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, Amsterdam: North-Holland, 1983.

[6] T. D. Chuong and D. S. Kim, *Nonsmooth semi-infinite multiobjective optimization problems*, J. Optim. Theory Appl.,**160** (2014), 748–762.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[8] M. Ehrgott, *Multicretiria Optimiztaion*, Second Edition, Springer Verlag, Sept., 2005.

[9] M. Ehrgott and S. Ruzika, *Improved ϵ-constraint method for multiobjective progamming*, J. Optim. Theory Appl. **138** (2008), 375–396.

[10] M. A. Goberna and M. A. López, *Linear Semi-Infinite Optimization*, J. Wiley & Sons, Chichester 1998.

[11] J. B. Hiriart-Urruty, *On optimality conditions in nondifferentiable programming*, Math. Program. **14** (1978), 73–86.

[12] J. B. Hiriart-Urruty, *New concepts in nondifferentiable programming*, Bull. Soc. Math. France Mém. **60** (1979), 57–85.

[13] J. Jahn, *Scalarization in vector optimization*, Math. Program. **29** (1984), 203–218.

[14] J. Jahn, *Vector Optimization: Theory, Applications and Extensions*, Second Edition, Springer Verlag, Berlin 2011.

[15] L. G. Jiao and J. H. Lee, *Finding efficient solutions in robust multiple objective optimization with SOS-convex polynomial data*, Ann. Oper. Res. (2019). https://doi.org/10.1007/s10479-019-03216-z

[16] D. S. Kim and T. Q. Son, *Characterizations of solution sets of a class of nonconvex semi-infinite programming problems*, J. Nonlinear Convex Anal. **12** (2011), 429–440.

[17] D. S. Kim and T. Q. Son, *An approach to ϵ-duality theorems for nonconvex semi-infinite multiobjective optimization problems*, Taiwanese J. Math. **22** (2018), 1261–1287.

[18] J. H. Lee and L. G. Jiao, *Finding efficient solutions for multicriteria optimization problems with SOS-convex polynomials*, Taiwanese J. Math. **23** (2019), 1535–1550.

[19] J. H. Lee and L. G. Jiao, *Solving fractional multicriteria optimization problems with sum of squares convex polynomial data*, J. Optim. Theory Appl. **176**

(2018), 428–455.

[20] J. H. Lee and G. M. Lee, *On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems* , Ann. Oper. Res. **269** (2018), 419–438.

[21] D. T. Luc, *Scalarization of vector optimization problems*, J. Optim. Theory Appl. **55** (1987), 85–102.

[22] B. Mond and T. Weir, *Generalized convavity and duality*, in: Generalized Concavity in Optimization and Economics, S. Schaible and W.T. Ziemba (eds.), Academic Press, New York 1981, pp. 263–279.

[23] H. Nakayama, Y. Yun and M. Yoon, *Sequential Approximate Multiobjective Optimization Using Computational Intelligence*, Springer-Verlag, Berlin, 2009.

[24] G.-R. Piao, L. G. Jiao and D. S. Kim, *Optimality conditions in nonconvex semi-infinite multiobjective optimization problems*, J. Nonlinear Convex Anal. **17** (2016), 167–175.

[25] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[26] T. Q. Son and D. S. Kim, *A new approach to characterize the solution set of a pseudoconvex programming problem*, J. Comput. Appl. Math. **261** (2014), 333–340.

[27] T. Q. Son, J. J. Strodiot and V. H. Nguyen, *ϵ-Optimality and ϵ-Lagrangian duality for a nonconvex programming problem with an infinite number of constraints*, J. Optim. Theory Appl. **141** (2009), 389–409.

[28] T. Q. Son, N. V. Tuyen, C.-F. Wen, *Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints.* arXiv:1808.10100, 2018.

[29] P. Wolfe, *A duality theorem for non-linear programming*, Quart. Appl. Math. **19** (1961), 621–635.

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