Homogeneous Transformation and Kinematics of a Steering Tyre

Dai Quoc Vo[1(✉)], Nam Ky Le[1], Hormoz Marzbani[2], and Reza N. Jazar[2]

[1] Le Quy Don Technical University, Hanoi 10000, Vietnam
dai.voquoc@lqdtu.edu.vn
[2] RMIT University, Bundoora, VIC 3083, Australia

Abstract. When steering about the titled kingpin axis, the tyre-road contact point moves along the tyre perimeter. In the literature on the tyre kinematics, this displacement, however, has not been taken into consideration. This results in an inaccuracy in the steering tyre kinematics, especially when the steering angle is large. This paper presents a novel method, utilising homogeneous transformation, to develop the kinematics of a steering tyre with the chance of the tyre-road contact being taken into account. The results show that this novel kinematic model is more accurate than those in the literature. The steering tyre kinematics developed in this investigation is then compared to that one built in ADAMS software for validating purpose.

Keywords: Homogeneous transformation · Coordinate frame
Steering tyre kinematics · Road-tyre contact

1 Introduction

A kinematic model of the steering wheel is usually required in the studies of steering system dynamics and lateral vehicle dynamics [1–3]. In the literature, the kinematics of the road steering wheel was developed by explicitly or implicitly accepting the assumption that the ground-tyre contact is a fixed point on the tyre perimeter [3–5]. It was also assumed that ‘the tyre-to-ground contact point is the wheel radius below the wheel centre’ [2]. However, when a wheel is steering around a titled kingpin axis with a large steering angle, the two assumptions are no longer reasonable. The former assumption will be proven inaccurate in Sect. 4. The latter assumption is not true as the wheel gains camber when it is steering about the tilted axis. Therefore, a kinematic model of the steering tyre that can take this effect into account is essential, especially when it comes to large steering angles such as in lock-to-lock steering manoeuvre. In this current investigation, we utilise different coordinate frames to describe the motion of the steering wheel. The homogeneous transformation is then employed to transform coordinates between the frames by which the steering wheel kinematics is developed. The kinematics is then compared to that of the model built in multi-body software, ADAMS, for validation.

H. Fujita et al. (Eds.): ICERA 2018, LNNS 63, pp. 340–349, 2019.
https://doi.org/10.1007/978-3-030-04792-4_45

2 Coordinate Frames

In this paper, we accept the assumption that the wheel (hereafter, there is no difference between wheel and tyre) is a flat and rigid disk. Without loss of generality, we also assume the wheel is upright when it is not steering, so it has zero initial camber. In order to establish the kinematics of the steering wheel the following coordinate frames, shown in Fig. 1, are utilised.

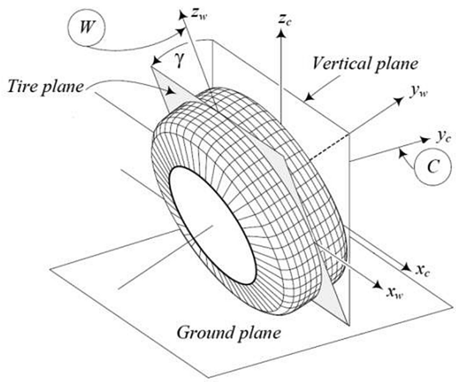

Fig. 1. The wheel coordinate frame *W* and the wheel-body coordinate frame *C*.

Wheel coordinate frame $W(x_w, y_w, z_w)$ is the frame that its origin *W* is attached to the wheel centre. Its x_w axis is initially parallel to the ground plane and directed forward; the z_w axis is vertical to the ground and upward, as long as the wheel is not steering. The y_w is spin axis and directed to the left (of the driver). The $(x_w W z_w)$ plane coincides with the tyre plane. The *W*-frame is fixed to the wheel. Therefore, it follows every motion of the wheel except the spin.

When the wheel is not steering, a wheel-body coordinate frame $C(x_c, y_c, z_c)$ that initially coincides with the wheel coordinate frame is attached to the car body. The wheel-body coordinate frame is motionless with the car body, so it does not follow any motion of the wheel when steering.

As defined, the orientations and locations of the two coordinate frames will not be affected by the spin motion of the wheel. Therefore, the spin motion is excluded from this kinematic analysis.

3 Steering Motions and Homogeneous Transformation

With the defined coordinate frames, the steering motion of the wheel is actually a rotation of the W-frame about the steering axis with respect to the C-frame. It is convenient to express the kinematics of the steering tyre in the C-frame, which is motionless with the car body. In order to do so, first we have to describe the steering axis in the C-frame. The steering axis is denoted by its unit vector and an arbitrary point on it.

Figure 2 illustrates the location and orientation of the kingpin axis. The kingpin axis has a caster angle ϕ with (y_c, z_c) plane and lean angle θ with (z_c, x_c) plane. They are measured about y_c and x_c axes, respectively. The steering axis is the intersection line of two planes: caster plane p_π and lean plane p_L. The caster plane is the plane that has an angle θ with the (z_c, x_c) plane whist the lean plane has an angle ϕ with (y_c, z_c) plane; and both of them contain steering axis. The two planes can be indicated by their normal unit vectors in the wheel-body coordinate frame, respectively:

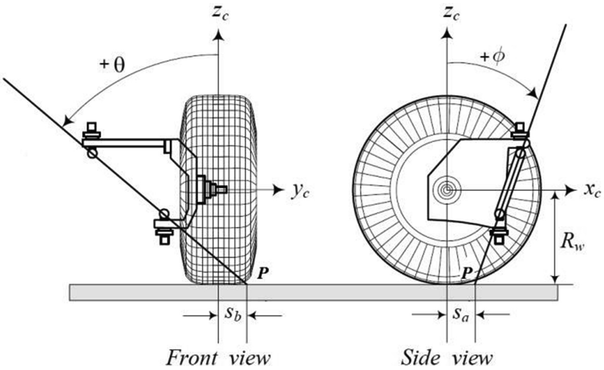

Fig. 2. The location and orientation of kingpin axis

$$\hat{n}_\pi = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}; \quad \hat{n}_L = \begin{bmatrix} -\cos\phi \\ 0 \\ \sin\phi \end{bmatrix} \tag{1}$$

The direction unit vector of the steering axis $\hat{u}$ can be determined as:

$$\hat{u} = {}^C\hat{u} = \frac{\hat{u}_\pi \times \hat{n}_L}{|\hat{n}_\pi \times \hat{n}_L|} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{\sqrt{\cos^2\phi + \cos^2\theta\sin^2\phi}} \begin{bmatrix} \cos\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta\cos\phi \end{bmatrix} \tag{2}$$

The point on the steering axis is the intersection P between it and the ground:

$$d_P = {}^C d_P = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} s_a \\ s_b \\ -R_w \end{bmatrix} \tag{3}$$

According to [6], the transformation matrix to map coordinates from the W-frame to the C-frame is:

$$ {}^{C}T_{W} = \begin{bmatrix} R_{\hat{u},\delta} & d_P - R_{\hat{u},\delta} d_P \\ 0 & 1 \end{bmatrix} \quad (4) $$

where $R_{u,\delta}$ is the transformation matrix between the two frames with the rotation axis passing though the origin; $d_P$is position vector of the point P on the rotation axis expressed in the C-frame.

$$ R_{\hat{u},\delta} = \begin{bmatrix} u_1^2 \operatorname{vers}\delta + \cos\delta & u_1u_2 \operatorname{vers}\delta - u_3\sin\delta & u_1u_3 \operatorname{vers}\delta + u_2\sin\delta \\ u_1u_2 \operatorname{vers}\delta + u_3\sin\delta & u_2^2 \operatorname{vers}\delta + \cos\delta & u_2u_3 \operatorname{vers}\delta - u_1\sin\delta \\ u_1u_3 \operatorname{vers}\delta - u_2\sin\delta & u_2u_3 \operatorname{vers}\delta + u_1\sin\delta & u_3^2 \operatorname{vers}\delta + \cos\delta \end{bmatrix} \quad (5) $$

Transforming coordinates from the wheel frame to the wheel-body frame is governed by:

$$ {}^{C}r = {}^{C}T_{W}\, {}^{W}r \quad (6) $$

where ${}^{C}r$ and ${}^{W}r$ are homogeneous coordinates of a point expressed in the C-frame and the W-frame, respectively [6].

$$ {}^{C}r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 1 \end{bmatrix}; {}^{W}r = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ 1 \end{bmatrix} \quad (7) $$

4 Kinematics of a Steering Tyre

Apply the homogeneous transformation in Eq. (6), we can calculate the coordinates of any point on the steering wheel expressed in the C-frame. By this way, the steering tyre kinematics is developed. Here, we derive some kinematic parameters using this transformation.

4.1 The Wheel Center

Because the coordinates of the centre of the wheel in the W-frame is:

$$ {}^{W}r_{W} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (8) $$

Its coordinates expressed in the wheel-body frame will be:

$$^{C}r_{W} = {}^{C}T_{W}{}^{W}r_{W} = \begin{bmatrix} x_{W} \\ y_{W} \\ z_{W} \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{u},\delta} & d_P - R_{\hat{u},\delta}d_P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_P - R_{\hat{u},\delta}d_P \\ 1 \end{bmatrix} \tag{9}$$

4.2 Wheel Camber

Camber is the angle γ that the tyre plane rotates about the x_t axis, from the vertical position. It is convenient to calculate camber through the angle ρ between the normal vector of the tyre plane and the normal vector of the ground plane (Fig. 3).

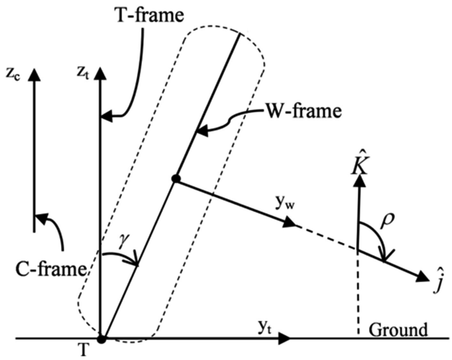

Fig. 3. The front view of the steered wheel.

If $\hat{I}$, $\hat{J}$,$\hat{K}$, are the unit vectors of the wheel-body coordinate frame; and, $\hat{i}$, $\hat{j}$, $\hat{k}$, are the unit vectors of the wheel coordinate frame; the angle ρ must be:

$$\rho = a\cos\left(\frac{{}^{C}\hat{j}\,{}^{C}\hat{K}}{\left|{}^{C}\hat{j}\,{}^{C}\hat{K}\right|}\right) \tag{10}$$

where

$$^{C}\hat{K} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

$$^{C}\hat{j} = {}^{C}T_{W}{}^{C}\hat{j} = \begin{bmatrix} R_{\hat{u},\delta} & d_P - R_{\hat{u},\delta}d_P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1u_2(1 - \cos\delta) - u_3\sin\delta \\ u_2^2(1 - \cos\delta) + \cos\delta \\ u_2u_3(1 - \cos\delta) + u_1\sin\delta \\ 0 \end{bmatrix} \tag{12}$$

Substituting ${}^{C}\hat{K}$ and ${}^{C}\hat{j}$ from (11) and (12) in (10), we have:

$$\rho = a\cos\left(\frac{{}^{C}\hat{j}\,{}^{C}\hat{K}}{\left|{}^{C}\hat{j}\,{}^{C}\hat{K}\right|}\right) = a\cos[u_2 u_3(1-\cos\delta) + u_1\sin\delta] \tag{13}$$

As camber is negative if its plane is turned about - x_t axis [7], it can be defined by the following equation:

$$\gamma = \frac{\pi}{2} - \rho \tag{14}$$

Substituting (2) in (13) and then (14) we have:

$$\gamma = \frac{\pi}{2} - a\cos\left[\frac{\cos\theta\sin\phi}{\sqrt{\cos^2\phi + \cos^2\theta\sin^2\phi}}\sin\delta - \frac{\cos^2\phi\sin\theta\cos\theta}{\cos^2\phi + \cos^2\theta\sin^2\phi}\operatorname{vers}\delta\right] \tag{15}$$

4.3 The Change of Ground-Contact Point Along the Tyre Perimeter

In this section, we show that the point on the tyre being in contact with the ground moves along its perimeter when steering, and that the movement is significant for large steering angles.

Here, T_0 and T respectively denote the ground-contact points (on the tyre perimeter) associated with zero and δ steering angles. The homogeneous representation of T_0 in the W-frame is:

$$ {}^{W}r_{T_0} = \begin{bmatrix} 0 \\ 0 \\ -R_w \\ 1 \end{bmatrix} \tag{16}$$

The expression in the C-frame is written as:

$$ {}^{C}r_{T_0} = \begin{bmatrix} {}^{C}x_{T_0} \\ {}^{C}y_{T_0} \\ {}^{C}z_{T_0} \\ 1 \end{bmatrix} = {}^{C}T_W\,{}^{W}r_{T_0} \tag{17}$$

Substituting (4) and (16) in (17) yields:

$$\begin{aligned} {}^{C}z_{T_0} = &-\frac{1}{2}\frac{s_a\cos^2\theta\sin 2\phi - s_b\sin 2\theta\cos^2\phi}{\cos^2\phi + \cos^2\theta\sin^2\phi}\operatorname{vers}\delta \\ &- \frac{s_a\cos\phi\sin\theta + s_b\cos\theta\sin\phi}{\sqrt{\cos^2\phi + \cos^2\theta\sin^2\phi}}\sin\delta - R_w \end{aligned} \tag{18}$$

${}^{C}z_{T_0}$ shows the vertical distance between point T_0 on the tyre and point C fixed to the car body when the wheel is steering. It can be verified that when caster and lean angles are zero, T_0 is the tyre radius below C.

According to [8], the z-coordinate of the point T in the C-frame after being steered is:

$$ {}^{C}z_T = {}^{C}z_W - R_w \cos\gamma \tag{19} $$

where ${}^{C}z_W$ is determined by (9).

We can determine the angle β that the ground-contact point on the tyre makes when it moves along the tyre perimeter (Fig. 4).

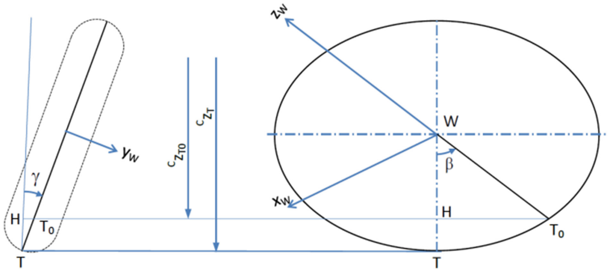

Fig. 4. The angular motion of the ground-contact point on the steering tyre perimeter.

$$ \cos\beta = \frac{WT - HT}{WT_0} = \frac{R_w - HT}{R_w} = 1 - \frac{HT}{R_w} = 1 - \frac{{}^{C}z_{T_0} - {}^{C}z_T}{R_w \cos\gamma} \tag{20} $$

Substituting (18) and (19) in (20) yields:

$$ \cos\beta = \frac{\cos\delta}{\cos\gamma} + \frac{\cos^2\theta\cos^2\phi}{(\cos^2\phi + \cos^2\theta\sin^2\phi)}(1 - \cos\delta) \tag{21} $$

As can be seen by (21), β depends only on the orientation of the steering axis and the steering angle. Furthermore, with a non-zero practical steering angle ($90° < \delta < 90°$), and an inclined steering pivot ($\phi, \theta \neq 0$), β is always non-zero.

The angular motion that the ground-contact point makes along the tyre perimeter, for an exemplary configuration ($\phi = -5^0$, $\theta = 13^0$), is illustrated in Fig. 5. The dependence of the motion on the caster and lean angles, for a large steering angle (*−45°*), is also visualised in Fig. 6.

It can be clearly seen from the graphs that the displacement of the ground-contact point along the tyre perimeter is considerable for large steering angles (around 7° for 30° of steering angle). Therefore, when determining kingpin axis moment, especially at low speed with large steering angles, this effect should not be neglected.

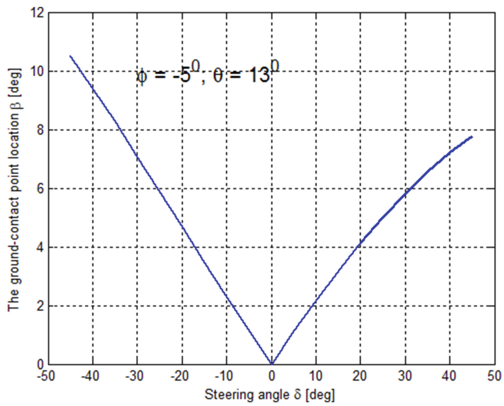

Fig. 5. The angular motion of the ground-contact point as a function of δ.

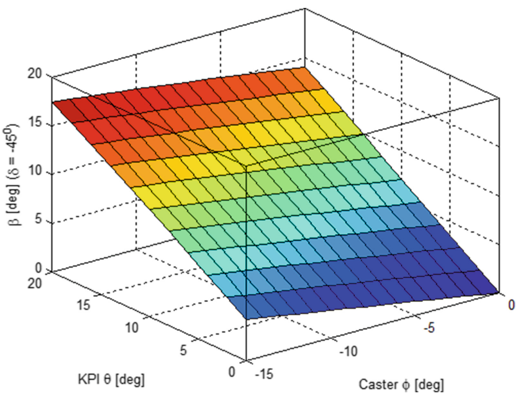

Fig. 6. The angular motion of the ground-contact point as a function of ϕ and θ.

5 Validation

To validate the tyre kinematics, a model of the rigid steerable wheel was built using multi-body software ADAMS. Parameters derived from the homogeneous transformation were compared with those from the multi-body model.

Figure 7 illustrates how much the camber is generated using the homogeneous transformation method and ADAMS model for different orientations. We also compare the results with those developed by Alberding [5] and by Dixon [4]. For the sake of mathematical exactness, the comparison was made over the steering angle range of 0^0 to 360^0 which is even much more than practical steering angles.

It can clearly be seen that, there is no difference between the generated camber values derived from the homogeneous transformation method and the multi-body model for different steering pivot's orientations. In contrast, the gained camber using Alberding-formula [5] and Dixon-formula [4] only shows a good agreement with that

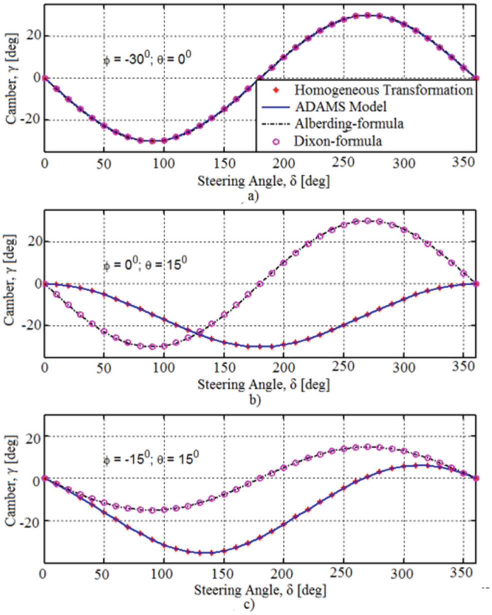

Fig. 7. A comparison of generated camber between different models

of the multi-body model when there is no lean angle or the steering angle is very small; when there is a lean angle and the steering angle is relatively large the camber values significantly differ from the multi-body data. The comparison convinces that the kinematics of the steered wheel developed here using the homogeneous transformation is mathematically accurate. Therefore, it is applicable to the case of large steering angles.

6 Conclusion

The homogeneous transformation, utilized in this investigation, is an effective method to develop a more accurate kinematic model of a steering tyre than those in the literature. In order to do so, a number of coordinate systems is defined first. The homogeneous transformation is then applied to map the coordinates between the frames. The results show that this model can take the change of the road-tyre contact point along its perimeter into account, which is more accurate than the existing models. Thereby, this kinematic model of the steering tyre can be applicable to the case of any steering angles, and kingpin orientation angles.

References

1. Pfeffer, P.E., Harrer, M.: On-centre steering wheel torque characteristics during steady state cornering. In: SAE International (2008)
2. Sharp, R.S., Granger, R.: On car steering torques at parking speeds. Proc. Inst. Mech. Eng. **217**(2), 87 (2003)
3. Cho, Y.: Vehicle steering returnability with maximum steering wheel angle at low speeds. Int. J. Automot. Technol. **10**(4), 431–439 (2009)
4. Dixon, J.: Suspension Geometry and Computation. Wiley, Hoboken (2009)
5. Alberding, M.B., et al.: Variable caster steering. IEEE Trans. Veh. Technol. **63**(4), 1513–1529 (2014)
6. Jazar, R.N.: Theory of Applied Robotics: Kinematics, Dynamics, and Control, 2nd edn. Springer, Boston (2010)
7. Jazar, R.N.: Vehicle Dynamics. Springer, New York (2014)
8. Vo, D.Q., Jazar, R.N., Fard, M.: A comparison between caster and lean angle in generating variable camber. In: SAE International (2015)