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Parallel extragradient algorithms for multiple set split equilibrium problems in Hilbert spaces

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Abstract In this paper, we introduce an extension of multiple set split variational inequality problem (Censor et al. Numer. Algor. **59**, 301–323 2012) to multiple set split equilibrium problem (MSSEP) and propose two new parallel extragradient algorithms for solving MSSEP when the equilibrium bifunctions are Lipschitz-type continuous and pseudo-monotone with respect to their solution sets. By using extragradient method combining with cutting techniques, we obtain algorithms for these problems without using any product space. Under certain conditions on parameters, the iteration sequences generated by the proposed algorithms are proved to be weakly and strongly convergent to a solution of MSSEP. An application to multiple set split variational inequality problems and a numerical example and preliminary computational results are also provided.

Keywords Multiple set split equilibrium problem · Pseudo-monotonicity · Extragradient method · Parallel algorithm · Weak and strong convergence

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1 Introduction

In a very interesting paper, Censor et al. [8] introduced the following split variational inequality problem (SVIP):

$$\begin{cases} \text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \ \geq 0, \forall y \in C \\ \text{and } u^* = Ax^* \in Q \text{ solves } \langle G(u^*), v - u^* \rangle \ \geq 0, \forall v \in Q, \end{cases}$$

where C and Q are nonempty, closed and convex subsets in real Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, $A : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator, $F : \mathbb{H}_1 \to \mathbb{H}_1$, $G : \mathbb{H}_2 \to \mathbb{H}_2$ are given operators. It was pointed out in [8] that many important problems arising from real-world problems can be formulated as (SVIP) such that: split minimization problem (SMP), split zeroes problem (SZP), and the split feasibility problem (SFP) which had been used for studying signal processing, medical image reconstruction, intensity-modulated radiation therapy, sensor networks, and data compression, see [3, 5, 6, 10, 12, 13, 15] and references quoted therein.

To find a solution of SVIP, Censor et al. [8] proposed to use two methods, the first one is to reformulate SVIP as a constrained variational inequality problem (CVIP) in a product space and solve CVIP when mappings F and G are monotone and Lipschitz continuous. The second one is to solve SVIP without using product space when F and G are inverse strongly monotone mappings and satisfying certain additional conditions.

Moudafi [29] (see also He [21]) introduced an extension of SVIP to split equilibrium problem (SEP) which can be stated as follow:

$$\begin{cases} \text{Find } x^* \in C \text{ such that } f(x^*, y) \ \geq 0, \forall y \in C \\ \text{and } u^* = Ax^* \in Q \text{ solves } g(u^*, v) \ \geq 0, \forall v \in Q, \end{cases}$$

where $f : \mathbb{H}_1 \times \mathbb{H}_1 \to \mathbb{R}$, and $g : \mathbb{H}_2 \times \mathbb{H}_2 \to \mathbb{R}$ are bifunctions such that $f(x, x) = g(u, u) = 0, \forall x \in C$, and $\forall u \in Q$, respectively.

For obtaining a solution of SEP without using product space, He [21] suggested to use the proximal method (see [31, 37]) and introduced an iterative method, which generated a sequence $\{x^k\}$ by

$$\begin{cases} x^0 \in C; \{\rho_k\} \subset (0, +\infty); \mu > 0, \\ f(y^k, y) + \dfrac{1}{\rho_k}\langle y - y^k, y^k - x^k \rangle \geq 0, \ \forall y \in C, \\ g(u^k, v) + \dfrac{1}{\rho_k}\langle v - u^k, u^k - Ay^k \rangle \geq 0, \ \forall v \in Q, \\ x^{k+1} = P_C(y^k + \mu A^*(u^k - Ay^k)), \ \forall k \geq 0, \end{cases}$$

where A^* is the adjoint operator of A.

Under suitable conditions on parameters, the author showed that $\{x^k\}$, $\{y^k\}$ converge weakly to a solution of SEP provided that f, g are monotone bifunctions on C and Q respectively. Since then, many solution methods for SEP and related problems when f is monotone or pseudo-monotone and g is monotone have been proposed, see, for example [16, 17, 19, 25–27, 34, 38].

Another extension of SVIP is multiple set split variational inequality problem (MSSVIP), which is formulated as follows (see [8, Section 6.1]):

$$\begin{cases} \text{Find } x^* \in C := \cap_{i=1}^N C_i \text{ such that } \langle F_i(x^*), y - x^* \rangle \geq 0, \quad \text{for all } \in C_i \\ \text{and for all } i = 1, 2, ..., N, \text{ and such that} \\ \text{the point } u^* = Ax^* \in Q := \cap_{j=1}^M Q_j \text{ solves } \langle G_j(u^*), v - u^* \rangle \geq 0, \quad \text{for all } v \in Q_j \\ \text{and for all } j = 1, 2, ..., M, \end{cases}$$

where, as before $A : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator, $F_i : \mathbb{H}_1 \to \mathbb{H}_1$, $i = 1, 2, ..., N$, and $G_j : \mathbb{H}_2 \to \mathbb{H}_2$, $j = 1, 2, ..., M$, are mappings and $C_i \subset \mathbb{H}_1$, for all $i = 1, 2, ..., N$; $Q_j \subset \mathbb{H}_2$, for all $j = 1, 2, ..., M$ are nonempty, closed, and convex subsets, respectively.

To solve MSSVIP, Censor et al. [8, Section 6.1] proposed to reformulate MSSVIP as a SVIP via certain product space and applying their first method to solve SVIP. The iteration sequence generated by their method was proved to converge weakly to a solution of MSSVIP when F_i, $i = 1, 2, ..., N$ and G_j, $j = 1, 2, ..., M$, are inverse strongly monotone mappings and satisfying some additional conditions.

In this paper, motivated by the results mentioned above, we introduce an extension of MSSVIP to multiple set split equilibrium problem (MSSEP) and propose two parallel extragradient methods for MSSEP. We make use of the extragradient algorithm for solving equilibrium problems in the corresponding spaces to design the weak convergence algorithm, we then combine this algorithm with the hybrid cutting technique (see [9, 41]) to get the strong convergence algorithm for MSSEP. By using the extragradient methods, we not only deal with the case MSSEP is pseudo-monotone but also solve MSSEP directly without using any product space.

The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce MSSEP and parallel extragradient algorithms for solving it. The last section presents a numerical example to demonstrate the proposed algorithms.

2 Preliminaries

We assume that $\mathbb{H}_1$ and $\mathbb{H}_2$ are real Hilbert spaces with an inner product and the associated norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively, whereas $\mathbb{H}$ refers to any of these spaces. We denote the strong convergence by '$\to$' and the weak convergence by '$\rightharpoonup$' in $\mathbb{H}$. Let C be a nonempty, closed and convex subset of $\mathbb{H}$. By P_C, we denote the metric projection operator onto C, that is

$$P_C(x) \in C : \|x - P_C(x)\| \leq \|x - y\|, \; \forall y \in C.$$

The following well known results will be used in the sequel.

Lemma 2.1 *Suppose that C is a nonempty, closed and convex subset in $\mathbb{H}$. Then P_C has following properties:*

(a) $P_C(x)$ *is singleton and well defined for every* x*;*
(b) $z = P_C(x)$ *if and only if* $\langle x - z, y - z \rangle \leq 0, \; \forall y \in C$*;*

(c) $\|y - P_C(x)\|^2 \le \|x - y\|^2 - \|x - P_C(x)\|^2, \ \forall y \in C$;
(d) $\|P_C(x) - P_C(y)\|^2 \le \langle P_C(x) - P_C(y), x - y\rangle, \ \forall x, y \in \mathbb{H}$;
(e) $\|P_C(x) - P_C(y)\|^2 \le \|x - y\|^2 - \|x - P_C(x) - y + P_C(y)\|^2, \ \forall x, y \in \mathbb{H}$.

Definition 2.1 [4, 28, 32] Let $\varphi : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction, and C be a nonempty, closed and convex subset of $\mathbb{H}$, $\emptyset \neq S \subset C$. Bifunction φ is said to be:

(a) monotone on C if

$$\varphi(x, y) + \varphi(y, x) \le 0, \ \forall x, y \in C;$$

(b) pseudo-monotone on C if

$$\forall x, y \in C, \varphi(x, y) \ge 0 \Longrightarrow \varphi(y, x) \le 0;$$

(c) pseudo-monotone on C with respect to S if

$$\forall x^* \in S, \ \forall y \in C, \varphi(x^*, y) \ge 0 \Longrightarrow \varphi(y, x^*) \le 0;$$

(d) Lipschitz-type continuous on C if there exist positive constants L_1 and L_2 such that

$$\varphi(x, y) + \varphi(y, z) \ge \varphi(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \ \forall x, y, z \in C.$$

From Definition 2.1, we have the followings

Remark 2.1 (i) it is clear that $(a) \Longrightarrow (b) \Longrightarrow (c), \forall S \subset C$.
(ii) If $\varphi(x, y) = \langle \Phi(x), y - x\rangle$, for a mapping $\Phi : \mathbb{H} \to \mathbb{H}$. Then the notions of monotonicity of bifunction φ collapse to the notions of monotonicity of mapping Φ, respectively. In addition, if mapping Φ is L-Lipschitz on C, i.e., $\|\Phi(x) - \Phi(y)\| \le L\|x - y\|, \forall x, y \in C$. Then, φ is also Lipschitz-type continuous on C (see [28, 42]), for example, with constants $L_1 = \frac{L}{2\epsilon}$, $L_2 = \frac{L\epsilon}{2}$, for any $\epsilon > 0$.
(iii) If φ_1 and φ_2 are Lipschitz-type continuous on C with constants L_1^1, L_2^1, and L_1^2, L_2^2, respectively. Then φ_1 and φ_2 are also Lipschitz-type continuous on C with the same constants L_1, L_2, for instance, take $L_1 = \max\{L_1^1, L_2^1\}$, $L_2 = \max\{L_1^2, L_2^2\}$.

Lemma 2.2 (Opial's condition) [33] *For any sequence $\{x^k\} \subset \mathbb{H}$ with $x^k \rightharpoonup x$, the inequality*

$$\liminf_{k \longrightarrow +\infty} \|x^k - x\| < \liminf_{k \longrightarrow +\infty} \|x^k - y\|$$

holds for each $y \in \mathbb{H}$ with $y \neq x$.

3 Main results

Now, given a bounded linear operator $A : \mathbb{H}_1 \to \mathbb{H}_2$, nonempty, closed, and convex subsets $C_i \subset \mathbb{H}_1$, $Q_j \subset \mathbb{H}_2$, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, and bifunctions

$f_i : \mathbb{H}_1 \times \mathbb{H}_1 \to \mathbb{R}$, and $g_j : \mathbb{H}_2 \times \mathbb{H}_2 \to \mathbb{R}$ such that $f_i(x, x) = g_j(u, u) = 0$, for every $x \in C_i$, $u \in Q_j$, and for every $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, respectively, the multiple set split equilibrium problem (MSSEP) is stated as follows:

$$\begin{cases} \text{Find } x^* \in C := \cap_{i=1}^{N} C_i \text{ such that } f_i(x^*, y) \geq 0, \quad \text{for all } y \in C_i \\ \text{and for all } i = 1, 2, ..., N, \text{ and such that} \\ \text{the point } u^* = Ax^* \in Q := \cap_{j=1}^{M} Q_j \text{ solves } g_j(u^*, v) \geq 0, \quad \text{for all } v \in Q_j \\ \text{and for all } j = 1, 2, ..., M. \end{cases}$$

Since variational inequality problem could be consider as a special case of equilibrium problem [4, 32], MSSVIP is a particular case of MSSEP, for instance, take $f_i(x, y) = \langle F_i(x), y - x\rangle$, $i = 1, 2, ..., N$, $g_j(u, v) = \langle G_j(u), v - u\rangle$, $j = 1, 2, ..., M$. Similarly, the split common fixed point problem [11, 23, 30] and the split common null point problem [7, 40] can be formulated as MSSEP. Further more, if $Q_j = \mathbb{H}_2$ and $g_j(u, v) = 0$, $\forall u, v \in \mathbb{H}_2$, and for all $j = 1, 2, ..., M$. Then MSSEP becomes the problem of finding a common element of the set of solutions of equilibrium problems, see, for example [35, 39] and references quoted therein.

By $\mathrm{Sol}(C_i, f_i)$, we denote the solution set of the equilibrium problem $\mathrm{EP}(C_i, f_i)$, i.e.,

$$\mathrm{Sol}(C_i, f_i) = \{\bar{x} \in C_i \text{ such that } f_i(\bar{x}, y) \geq 0, \ \forall y \in C_i\},$$

for all $i = 1, 2, ..., N$, and $\mathrm{Sol}(Q_j, g_j)$ stands for the solution set of the equilibrium problem $\mathrm{EP}(Q_j, g_j)$, for all $j = 1, ..., M$.

Now, let K be a nonempty, closed and convex subset of $\mathbb{H}$ and $\varphi : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction such that $\varphi(x, x) = 0$, $\forall x \in$ K. In order to find a solution of MSSEP, we make use of the following blanket assumptions:

Assumptions $\mathcal{A}$

- (A_1) φ is pseudo monotone on K with respect to Sol(K, φ);
- (A_2) $\varphi(x, \cdot)$ is convex, subdifferentiable on K, for all $x \in$ K;
- (A_3) φ is weakly continuous on K×K in the sense that if $x, y \in$ K and $\{x^k\}, \{y^k\} \subset$ K converge weakly to x and y, respectively, then $\varphi(x^k, y^k) \to \varphi(x, y)$ as $k \to +\infty$;
- (A_4) φ is Lipschitz-type continuous on K with constants $L_1 > 0$ and $L_2 > 0$.

The extragradient algorithm was first proposed by Korpelevich [24] (see also [20]) for finding saddle points and other problems; recently, many authors have succeeded in applying this algorithm for solving equilibrium problems and other related problems [18, 42]. One advantage of the extragradient algorithm for solving equilibrium problems is that it can be apply not only for the pseudo-monotone equilibrium problem cases but also at each iteration, we only have to solve two strongly convex programs. We now present a parallel extragradient algorithm for MSSEP.

Algorithm 1 (*Parallel extragradient algorithm for MSSEP*)

Initialization. Pick $x^0 \in C = \cap_{i=1}^N C_i$, choose constants $0 < \underline{\rho} \leq \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $0 < \underline{\alpha} \leq \bar{\alpha} \leq 1$; for each $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, choose parameters $\{\rho_k^i\}, \{r_k^j\} \subset [\underline{\rho}, \bar{\rho}]$; $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha}, \bar{\alpha}]$, $\sum_{i=1}^N \alpha_k^i = \sum_{j=1}^M \beta_k^j = 1$; $\mu \in (0, \frac{1}{\|A\|})$.

Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:

Step 1. Solve N strongly convex programs in parallel

$$\begin{cases} y_i^k = \arg\min\{f_i(x^k, y) + \frac{1}{2\rho_k^i}\|y - x^k\|^2 : \; y \in C_i\} \\ z_i^k = \arg\min\{f_i(y_i^k, y) + \frac{1}{2\rho_k^i}\|y - x^k\|^2 : \; y \in C_i\}, \end{cases} \quad i = 1, 2, ..., N.$$

Step 2. Set $\bar{z}^k = \sum_{i=1}^N \alpha_k^i z_i^k$, and compute $\hat{v}^k = A\bar{z}^k$.

Step 3. Solve M the following strongly convex programs in parallel

$$\begin{cases} v_j^k = \arg\min\{g_j(\hat{v}^k, v) + \frac{1}{2r_k^j}\|v - \hat{v}^k\|^2 : v \in Q_j\} \\ u_j^k = \arg\min\{g_j(v_j^k, v) + \frac{1}{2r_k^j}\|v - \hat{v}^k\|^2 : v \in Q_j\}, \end{cases} \quad j = 1, 2, ..., M.$$

Step 4. Compute $\bar{u}^k = \sum_{j=1}^M \beta_k^j u_j^k$.

Step 5. Take $x^{k+1} = P_C(\bar{z}^k + \mu A^*(\bar{u}^k - \hat{v}^k))$, and go to **Iteration** k with k is replaced by $k + 1$.

Remark 3.1 (i) At iteration k, if $y_i^k = x^k$, then x^k is a solution of EP(C_i, f_i). Similarly, if $v_j^k = \hat{v}^k$, then $\hat{v}^k$ is a solution of EP(Q_j, g_j).

(ii) At each iteration k, parameters $\{\alpha_k^i\}$ can be chosen basing on the relative position of z_i^k and x^k, $i = 1, ..., N$. Similarly, parameters $\{\beta_k^j\}$, can be chosen by using the distance between u_j^k and $\hat{v}^k$, $j = 1, 2, ..., M$ (see [1, 22]).

(iii) We may assume without loss of generality that bifunctions f_i, $i = 1, 2, ..., N$ and g_j, $j = 1, 2, ..., M$, satisfying assumptions (A_4) with the same Lipschitz-type constants L_1 and L_2.

Before going to prove the convergence of Algorithm 1, let us recall the following result which was in [2, 42]

Lemma 3.1 ([2, 42]) *Suppose that f_i, $i = 1, 2, ..., N$, satisfy assumptions (A_1), (A_2), (A_4) such that $\cap_{i=1}^N Sol(C_i, f_i)$ is nonempty. Then, for each $i = 1, 2, ..., N$, we have:*

(i) $\rho_k^i[f_i(x^k, y) - f_i(x^k, y_i^k)] \geq \langle y_i^k - x^k, y_i^k - y\rangle, \; \forall y \in C_i$.

(ii) $\|z_i^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - 2\rho_k^i L_1)\|x^k - y_i^k\|^2 - (1 - 2\rho_k^i L_2)\|y_i^k - z_i^k\|^2, \; \forall x^* \in \cap_{i=1}^N Sol(C_i, f_i), \; \forall k$.

We are now in a position to prove the convergence of Algorithm 1.

Theorem 3.1 *Let bifunctions f_i, g_j satisfy assumptions $\mathcal{A}$, on C_i and Q_j, respectively, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded* *linear operator with its adjoint A^*. If $\Omega = \{x^* \in \bigcap_{i=1}^N Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)\} \neq \emptyset$, then the sequences $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ converge weakly to an element $x^* \in \Omega$ and $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ converge weakly to $Ap \in \bigcap_{j=1}^M Sol(Q_j, g_j)$.*

Proof Let $p \in \Omega$, so $p \in \bigcap_{i=1}^N \mathrm{Sol}(C_i, f_i)$ and $Ap \in \bigcap_{j=1}^M \mathrm{Sol}(Q_j, g_j)$. For each $i = 1, 2, ..., N$ by Lemma 3.1, we have

$$\|z_i^k - p\|^2 \leq \|x^k - p\|^2 - (1 - 2\rho_k^i L_1)\|x^k - y_i^k\|^2 - (1 - 2\rho_k^i L_2)\|y_i^k - z_i^k\|^2$$

Combining with step 2, one has

$$\begin{aligned} \|\bar{z}^k - p\|^2 &= \|\sum_{i=1}^N \alpha_k^i z_i^k - p\|^2 = \|\sum_{i=1}^N \alpha_k^i (z_i^k - p)\|^2 \\ &= \sum_{i=1}^N \alpha_k^i \|z_i^k - p\|^2 - \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \alpha_k^i \alpha_k^j \|z_i^k - z_j^k\|^2 \\ &\leq \|x^k - p\|^2 - \sum_{i=1}^n \alpha_k^i (1 - 2\rho_k^i L_1)\|x^k - y_i^k\|^2 \\ &\quad - \sum_{i=1}^n \alpha_k^i (1 - 2\rho_k^i L_2)\|y_i^k - z_i^k\|^2. \end{aligned} \tag{3.1}$$

Similarly, assertion (ii) in Lemma 3.1 implies that

$$\|u_j^k - Ap\|^2 \leq \|A\bar{z}^k - Ap\|^2 - (1 - 2r_k^j L_1)\|A\bar{z}^k - v_j^k\|^2 - (1 - 2r_k^j L_2)\|v_j^k - u_j^k\|^2, \forall j = 1, 2, ..., M.$$

Thus,

$$\begin{aligned} \|\bar{u}^k - Ap\|^2 &= \|\sum_{j=1}^M \beta_k^j (u_j^k - Ap)\|^2 \\ &= \sum_{j=1}^M \beta_k^j \|u_j^k - Ap\|^2 - \frac{1}{2}\sum_{i=1}^M \sum_{j=1}^M \beta_k^i \beta_k^j \|u_i^k - u_j^k\|^2 \\ &\leq \|A\bar{z}^k - Ap\|^2 - \sum_{j=1}^M \beta_k^j (1 - 2r_k^j L_1)\|A\bar{z}^k - v_j^k\|^2 \\ &\quad - \sum_{j=1}^M \beta_k^j (1 - 2r_k^j L_2)\|v_j^k - u_j^k\|^2. \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned} 2\langle A(\bar{z}^k - p), \bar{u}^k - A\bar{z}^k\rangle &= 2\langle A(\bar{z}^k - p) + \bar{u}^k - A\bar{z}^k - (\bar{u}^k - A\bar{z}^k), \bar{u}^k - A\bar{z}^k\rangle \\ &= 2\langle \bar{u}^k - Ap, \bar{u}^k - A\bar{z}^k\rangle - 2\|\bar{u}^k - A\bar{z}^k\|^2 \\ &= \|\bar{u}^k - Ap\|^2 + \|\bar{u}^k - A\bar{z}^k\|^2 - \|A\bar{z}^k - Ap\|^2 \\ &\quad - 2\|\bar{u}^k - A\bar{z}^k\|^2 \\ &= \|\bar{u}^k - Ap\|^2 - \|\bar{u}^k - A\bar{z}^k\|^2 - \|A\bar{z}^k - Ap\|^2. \end{aligned} \tag{3.3}$$

By definition of x^{k+1}, we have

$$\begin{aligned}\|x^{k+1}-p\|^2 &= \|P_C(\bar{z}^k+\mu A^*(\bar{u}^k-A\bar{z}^k))-P_C(p)\|^2\\ &\le \|(\bar{z}^k-p)+\mu A^*(\bar{u}^k-A\bar{z}^k)\|^2\\ &= \|\bar{z}^k-p\|^2+\|\mu A^*(\bar{u}^k-A\bar{z}^k)\|^2+2\mu\langle \bar{z}^k-p, A^*(\bar{u}^k-A\bar{z}^k)\rangle\\ &\le \|\bar{z}^k-p\|^2+\mu^2\|A^*\|^2\|\bar{u}^k-A\bar{z}^k\|^2+2\mu\langle \bar{z}^k-p, A^*(\bar{u}^k-A\bar{z}^k)\rangle\\ &= \|\bar{z}^k-p\|^2+\mu^2\|A\|^2\|\bar{u}^k-A\bar{z}^k\|^2+2\mu\langle A(\bar{z}^k-p), (\bar{u}^k-A\bar{z}^k)\rangle.\end{aligned}$$

Using (3.3), we get

$$\begin{aligned}\|x^{k+1}-p\|^2 \le\ & \|\bar{z}^k-p\|^2+\mu^2\|A\|^2\|\bar{u}^k-A\bar{z}^k\|^2\\ &+\mu\big[\|\bar{u}^k-Ap\|^2-\|\bar{u}^k-A\bar{z}^k\|^2-\|A\bar{z}^k-Ap\|^2\big]\\ =\ & \|\bar{z}^k-p\|^2-\mu(1-\mu\|A\|^2)\|\bar{u}^k-A\bar{z}^k\|^2\\ &+\mu\big[\|\bar{u}^k-Ap\|^2-\|A\bar{z}^k-Ap\|^2\big].\end{aligned} \tag{3.4}$$

In combination with (3.1) and (3.2), (3.4) becomes

$$\begin{aligned}\|x^{k+1}-p\|^2 \le \|x^k-p\|^2-\sum_{i=1}^{N}\alpha_k^i\big[(1-2\rho_k^i L_1)\|x^k-y_i^k\|^2-(1-2\rho_k^i L_2)\|y_i^k-z_i^k\|^2\big]\\ -\mu(1-\mu\|A\|^2)\|\bar{u}^k-A\bar{z}^k\|^2\\ -\mu\sum_{j=1}^{M}\big[\beta_k^j(1-2r_k^j L_1)\|A\bar{z}^k-v_j^k\|^2-(1-2r_k^j L_2)\|v_j^k-u_j^k\|^2\big].\end{aligned} \tag{3.5}$$

Since $0<\mu<\dfrac{1}{\|A\|^2}$, $\{\rho_k^i\}, \{r_k^j\} \subset [\underline{\rho}, \bar{\rho}] \subset (0, \min\{\frac{1}{2L_1}, \frac{1}{2L_1}\})$, $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha}, \bar{\alpha}] \subset (0, 1]$. Equation (3.5) implies that $\{\|x^k-p\|^2\}$ is a nonincreasingly sequence.

So

$$\lim_{k\to\infty}\|x^k-p\|^2=a(p). \tag{3.6}$$

From (3.5) and (3.6), we get

$$\lim_{k\to+\infty}\|x^k-y_i^k\|=\lim_{k\to+\infty}\|y_i^k-z_i^k\|=0,\ \forall i=1,2,...,N. \tag{3.7}$$

$$\lim_{k\to+\infty}\|A\bar{z}^k-v_j^k\|=\lim_{k\to+\infty}\|v_j^k-u_j^k\|=0,\ \forall j=1,2,...,M. \tag{3.8}$$

Because $\lim_{k\to+\infty}\|x^k-p\|=a(p)$, $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_m}\}$ of $\{x^k\}$ such that x^{k_m} converges weakly to some x^* as $m\to+\infty$. Remember that $x^k\in C=\cap_{i=1}^N C_i$, $\forall k$ and C_i are closed and convex sets for all $i=1,2,...,N$. We have that C is also closed and convex, so C is weakly closed and therefore $x^*\in C$, i.e., $x^*\in C_i$ for all $i=1,2,...,N$. From (3.7), we also get that $\{y_i^{k_m}\}$, $\{z_i^{k_m}\}$ converge weakly to x^*, for all $i=1,2,...,N$. Hence, $\{\bar{z}^{k_m}\}$ converges weakly to x^*, consequently $\{A\bar{z}^{k_m}\}$ converges weakly to Ax^*. Together with (3.8), we obtain that $\{v_j^{k_m}\}$, $\{u_j^{k_m}\}$ converge weakly to Ax^* for all $j=1,2,...,M$ as $m\to+\infty$. Since $\{v_j^k\}\subset Q_j$, Q_j is closed and convex, so $Ax^*\in Q_j, \forall j$.

For each $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$, assertion (i) in Lemma 3.1 implies that

$$\rho^i_{k_m}[f_i(x^{k_m}, y) - f_i(x^{k_m}, y_i^{k_m})] \geq \langle y_i^{k_m} - x^{k_m}, y_i^{k_m} - y\rangle, \ \forall y \in C_i, \ \text{and}$$

$$r^j_{k_m}[g_j(A\bar{z}^{k_m}, v) - g_j(A\bar{z}^{k_m}, v_j^{k_m})] \geq \langle v_j^{k_m} - A\bar{z}^{k_m}, v_j^{k_m} - v\rangle, \ \forall v \in Q_j.$$

Since

$$\langle y_i^{k_m} - x^{k_m}, y_i^{k_m} - y\rangle \geq -\|y_i^{k_m} - x^{k_m}\|\|y_i^{k_m} - y\| \text{ and}$$

$$\langle v_j^{k_m} - A\bar{z}^{k_m}, v_j^{k_m} - v\rangle \geq -\|v_j^{k_m} - A\bar{z}^{k_m}\|\|v_j^{k_m} - v\|,$$

it follows from the last inequalities that

$$\rho^i_{k_m}\left[f_i(x^{k_m}, y) - f_i(x^{k_m}, y_i^{k_m})\right] \geq -\|y_i^{k_m} - x^{k_m}\|\|y_i^{k_m} - y\|, \ \text{and}$$

$$r^j_{k_m}\left[g_j(A\bar{z}^{k_m}, v) - g_j(A\bar{z}^{k_m}, v_j^{k_m})\right] \geq -\|v_j^{k_m} - A\bar{z}^{k_m}\|\|v_j^{k_m} - v\|.$$

Hence,

$$f_i(x^{k_m}, y) - f_i(x^{k_m}, y_i^{k_m}) + \frac{1}{\rho^i_{k_m}}\|y_i^{k_m} - x^{k_m}\|\|y_i^{k_m} - y\| \geq 0, \ \text{and}$$

$$g_j(A\bar{z}^{k_m}, v) - g_j(A\bar{z}^{k_m}, v_j^{k_m}) + \frac{1}{r^j_{k_m}}\|v_j^{k_m} - A\bar{z}^{k_m}\|\|v_j^{k_m} - v\| \geq 0.$$

Letting $m \to +\infty$, in combination with (3.7), (3.8) and the continuity of f_i, g_j, yield

$$f_i(x^*, y) - f_i(x^*, x^*) \geq 0, \ \forall y \in C_i, \ \forall i = 1, 2, ..., N, \ \text{and}$$

$$g_j(Ax^*, v) - g_j(Ax^*, Ax^*) \geq 0, \ \forall v \in Q_j, \ \forall j = 1, 2, ..., M,$$

which means that $x^* \in \mathrm{Sol}(C_i, f_i)$ for all $i = 1, 2, ..., N$ and $Ax^* \in \mathrm{Sol}(Q_j, g_j)$ for all $j = 1, 2, ..., M$, or $x^* \in \Omega$.

Finally, we prove $\{x^k\}$ converges weakly to x^*. Indeed, if there exists a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that $x^{k_n} \rightharpoonup \tilde{x}$ with $\tilde{x} \neq x^*$, then we have $\tilde{x} \in \Omega$. By Opial's condition, yields

$$\begin{aligned}\liminf_{m\to+\infty}\|x^{k_m} - x^*\| &< \liminf_{m\to+\infty}\|x^{k_m} - \tilde{x}\| \\ &= \liminf_{n\to+\infty}\|x^{k_n} - \tilde{x}\| \\ &< \liminf_{n\to+\infty}\|x^{k_n} - x^*\| \\ &= \liminf_{m\to+\infty}\|x^{k_m} - x^*\|.\end{aligned}$$

This is a contradiction. Hence, $\{x^k\}$ converges weakly to x^*.

Together with (3.7), we also get $y_i^k \rightharpoonup x^*$ and $z_i^k \rightharpoonup x^*$, for all $i = 1, 2, ..., N$. Therefore, $\bar{z}^k \rightharpoonup x^*$, and $A\bar{z}^k \rightharpoonup Ax^*$. Combining with (3.8), it is immediate that $\{v_j^k\}$, $\{u_j^k\}$ converge weakly to Ax^* for all $j = 1, 2, ..., M$. □

When $N = M = 1$, then $C_1 = C$ and $Q_1 = Q$, we get the following corollary.

Corollary 3.1 *Suppose that f, g are bifunctions satisfying assumptions $\mathcal{A}$ on C and Q respectively, suppose further that $A : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator with its adjoint A^*. Take $x^0 \in C$; $\{\rho_k\}$, $\{r_k\} \subset [\underline{\rho},\ \bar{\rho}]$, for some $\underline{\rho}$, $\bar{\rho}$ such that $0 < \underline{\rho} \le \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$; $0 < \mu < \dfrac{1}{\|A\|^2}$, and consider the sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, and $\{v^k\}$, $\{u^k\}$ defined by*

$$\begin{cases} y^k = \arg\min\left\{\rho_k f(x^k, y) + \dfrac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \arg\min\left\{\rho_k f(y^k, y) + \dfrac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ v^k = \arg\min\left\{r_k g(Az^k, v) + \dfrac{1}{2}\|v - Az^k\|^2 : v \in Q\right\}, \\ u^k = \arg\min\left\{r_k g(Av^k, y) + \dfrac{1}{2}\|v - Az^k\|^2 : v \in Q\right\}, \\ x^{k+1} = P_C(z^k + \mu A^*(u^k - Az^k)). \end{cases}$$

If $\Omega = \{x^* \in Sol(C, f) : Ax^* \in Sol(Q, g)\} \neq \emptyset$, then the sequences $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ converge weakly to an element $x^* \in \Omega$, and $\{v^k\}$, $\{u^k\}$ converge weakly to $Ax^* \in Sol(Q, g)$.

An important case of MSSEP is multiple set split variational inequality problem MSSVIP. In this case, we have the following corollary which not only solves MSSVIP without using any product space but also deals with the pseudo-monotone cases.

Corollary 3.2 *Suppose that F_i, G_j are mappings satisfying assumptions $\mathcal{A}$ on C_i and Q_j, respectively, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, and $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with its adjoint A^*. Take $x^0 \in C = \cap_{i=1}^N C_i$; $\{\rho_k^i\}$, $\{r_k^j\} \subset [\underline{\rho},\ \bar{\rho}]$, for some $\underline{\rho}$, $\bar{\rho}$ such that $0 < \underline{\rho} \le \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$; $\{\alpha_k^i\}$, $\{\beta_k^j\} \subset [\underline{\alpha},\ \bar{\alpha}] \subset (0, 1]$, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, and $\sum_{i=1}^N \alpha_k^i = \sum_{j=1}^M \beta_k^j = 1$; $0 < \mu < \dfrac{1}{\|A\|^2}$. Consider the sequences $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ and $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ defined by*

$$\begin{cases} \begin{cases} y_i^k = P_{C_i}(x^k - \rho_k^i F_i(x^k)) \\ z_i^k = P_{C_i}(x^k - \rho_k^i F_i(y_i^k)), \end{cases} \quad i = 1, 2, ..., N. \\ \bar{z}^k = \sum_{i=1}^N \alpha_k^i z_i^k, \\ \hat{v}^k = A\bar{z}^k, \\ \begin{cases} v_j^k = P_{Q_j}(\hat{v}^k - r_k^j G_j(\hat{v}^k)) \\ u_j^k = P_{Q_j}(\hat{v}^k - r_k^j G_j(v_j^k)), \end{cases} \quad j = 1, 2, ..., M. \\ \bar{u}^k = \sum_{j=1}^M \beta_k^j u_j^k, \\ x^{k+1} = P_C(\bar{z}^k + \mu A^*(\bar{u}^k - \hat{v}^k)). \end{cases}$$

If $\Omega = \{x^* \in \bigcap_{i=1}^N Sol(C_i, F_i) : Ax^* \in \bigcap_{j=1}^M Sol(Q_j, G_j)\} \neq \emptyset$, *then the sequences* $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ *converge weakly to an element* $x^* \in \Omega$, *and* $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ *converge weakly to* $Ax^* \in \cap_{j=1}^M Sol(Q_j, G_j)$.

In some cases, we wish to get the strong convergence algorithms to solve our problems. To do so, we combine Algorithm 1 with hybrid method [9, 41] to propose the following parallel hybrid extragradient algorithm for MSSEP.

Algorithm 2 (*Parallel Hybrid Extragradient Algorithm for MSSEP*)

Initialization. Pick $x^0 = x^g \in C = \cap_{i=1}^N C_i$, choose constants $0 < \underline{\rho} \leq \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $0 < \underline{\alpha} \leq \bar{\alpha} \leq 1$; for each $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, choose parameters $\{\rho_k^i\}, \{r_k^j\} \subset [\underline{\rho},\ \bar{\rho}]$; $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha},\ \bar{\alpha}]$, $\sum_{i=1}^N \alpha_k^i = \sum_{j=1}^M \beta_k^j = 1$; $\mu \in (0,\ \frac{1}{\|A\|})$; $B_0 = C$.
Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:

Step 1. Solve N strongly convex programs in parallel

$$\begin{cases} y_i^k = \arg\min\{f_i(x^k, y) + \frac{1}{2\rho_k^i}\|y - x^k\|^2 :\ y \in C_i\} \\ z_i^k = \arg\min\{f_i(y_i^k, y) + \frac{1}{2\rho_k^i}\|y - x^k\|^2 :\ y \in C_i\}, \end{cases} \quad i = 1, 2, ..., N.$$

Step 2. Set $\bar{z}^k = \sum_{i=1}^N \alpha_k^i z_i^k$, and compute $\hat{v}^k = A\bar{z}^k$.
Step 3. Solve M the following strongly convex programs in parallel

$$\begin{cases} v_j^k = \arg\min\{g_j(\hat{v}^k, v) + \frac{1}{2r_k^j}\|v - \hat{v}^k\|^2 : v \in Q_j\} \\ u_j^k = \arg\min\{g_j(v_j^k, v) + \frac{1}{2r_k^j}\|v - \hat{v}^k\|^2 : v \in Q_j\}, \end{cases} \quad j = 1, 2, ..., M.$$

Step 4. Compute $\bar{u}^k = \sum_{j=1}^M \beta_k^j u_j^k$.
Step 5. Take $t^k = P_C(\bar{z}^k + \mu A^*(\bar{u}^k - \hat{v}^k))$.
Step 6. Define $B_{k+1} = \{x \in B_k : \|x - t^k\| \leq \|x - \bar{z}^k\| \leq \|x - x^k\|\}$, compute $x^{k+1} = P_{B_{k+1}}(x^g)$, and go to **Iteration** k with k is replaced by $k + 1$.

Remark 3.2 By setting $H_k^1 = \{x \in \mathbb{H}_1 : \|x - t^k\| \leq \|x - \bar{z}^k\|\}$, we have

$$H_k^1 = \{x \in \mathbb{H}_1 : \langle \bar{z}^k - t^k, x\rangle \leq \frac{1}{2}(\|\bar{z}^k\|^2 - \|t^k\|^2)\},$$

hence H_k^1 is a halfspace. Similarly,

$$\begin{aligned} H_k^2 &= \{x \in \mathbb{H}_1 : \|x - \bar{z}^k\| \leq \|x - x^k\|\} \\ &= \{x \in \mathbb{H}_1 : \langle x^k - \bar{z}^k, x\rangle \leq \frac{1}{2}(\|x^k\|^2 - \|z^k\|^2)\}, \end{aligned}$$

is also a halfspace. Since $B_{k+1} = B_k \cap H_k^1 \cap H_k^2$, if $\mathbb{H}_1$ is the Euclidean space $\mathbb{R}^n$ and B_0 is a polyhedron, then by induction we get that B_k are polyhedra for all k.

Therefore, the computation of x^{k+1} in Algorithm 2 is equivalent to find the projection of x^g onto the polyhedron B_{k+1} which can be computed efficiently using, for example, strongly convex quadratic programming methods.

The following theorem give us the strong convergence of Algorithm 2.

Theorem 3.2 *Let bifunctions f_i, g_j satisfy assumptions $\mathcal{A}$ on C_i and Q_j, respectively, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with its adjoint A^*. If $\Omega = \{x^* \in \bigcap_{i=1}^N Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)\} \neq \emptyset$, then the sequences $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ converge strongly to an element $x^* \in \Omega$ and $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ converge strongly to $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$.*

Proof Firstly, we observe that B_k is a nonempty closed convex set for all $k \in \mathbb{N}^*$. In fact, let $p \in \Omega$, it follows from (3.4), (3.2), (3.1) that

$$\begin{aligned}
\|t^k - p\|^2 &\leq \|\bar{z}^k - p\|^2 - \mu(1-\mu\|A\|^2)\|\bar{u}^k - A\bar{z}^k\|^2 + \mu\big[\|\bar{u}^k - Ap\|^2 - \|A\bar{z}^k - Ap\|^2\big] \\
&\leq \|\bar{z}^k - p\|^2 - \mu(1-\mu\|A\|^2)\|\bar{u}^k - A\bar{z}^k\|^2 \\
&\quad - \mu\Big[\sum_{j=1}^M \beta_k^j(1-2r_k^j L_1)\|A\bar{z}^k - v_j^k\|^2 + \sum_{j=1}^M \beta_k^j(1-2r_k^j L_2)\|v_j^k - u_j^k\|^2\Big] \\
&\leq \|x^k - p\|^2 - \mu(1-\mu\|A\|^2)\|\bar{u}^k - A\bar{z}^k\|^2 \\
&\quad - \sum_{i=1}^N \alpha_k^i(1-2\rho_k^i L_1)\|x^k - y_i^k\|^2 - (1-2\rho_k^i L_2)\|y_i^k - z_i^k\|^2\big] \\
&\quad - \mu\sum_{j=1}^M \big[\beta_k^j(1-2r_k^j L_1)\|A\bar{z}^k - v_j^k\|^2 - (1-2r_k^j L_2)\|v_j^k - u_j^k\|^2\big].
\end{aligned} \tag{3.9}$$

By the algorithm $\{\rho_k^i\}, \{r_k^j\} \subset [\underline{\rho},\ \bar{\rho}] \subset (0, \min\{\frac{1}{2L_1}, \frac{1}{2L_1}\})$, $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha},\ \bar{\alpha}] \subset (0,\ 1]$, $0 < \mu < \dfrac{1}{\|A\|^2}$, and (3.9), we have

$$\|t^k - p\| \leq \|\bar{z}^k - p\| \leq \|x^k - p\|, \quad \forall k. \tag{3.10}$$

Because $p \in B_0$ and (3.10), we get by induction that $p \in B_k$ for all $k \in \mathbb{N}$, i.e., $\Omega \subset B_k$, so $B_k \neq \emptyset$ for all k.

For each $k \in \mathbb{N}$, define

$$D_k = \{x \in \mathbb{H}_1 : \|x - t^k\| \leq \|x - \bar{z}^k\| \leq \|x - x^k\|\} = H_k^1 \cap H_k^2,$$

then $B_{k+1} = B_k \cap D_k$. Since B_0 and D_k are closed and convex for all k, B_k is closed for all k.

By Step 6 $x^{k+1} \in B_{k+1} \subset B_k$ and $x^k = P_{B_k}(x^g)$, so

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\|, \text{ for all } k.$$

In addition, $x^{k+1} = P_{B_{k+1}}(x^g)$ and $p \in B_{k+1}$, it implies that

$$\|x^{k+1} - x^g\| \leq \|p - x^g\|.$$

Thus,

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\| \leq \|p - x^g\|, \ \forall k.$$

Therefore, $\lim_{k\to+\infty} \|x^k - x^g\|$ exists, consequently $\{x^k\}$ is bounded.

Hence, $\{t^k\}$ and $\{\bar{z}^k\}$ are also bounded.

For all $k > n$, we have that $x^k \in B_k \subset B_n$, $x^n = P_{B_n}(x^g)$. Combining this fact with Lemma 2.1, we get

$$\|x^k - x^n\|^2 \leq \|x^k - x^g\|^2 - \|x^n - x^g\|^2. \tag{3.11}$$

Since $\lim_{k\to+\infty} \|x^k - x^g\|^2$ exists, (3.11) implies that

$$\lim_{k,n\to\infty} \|x^k - x^n\| = 0.$$

So $\{x^k\}$ is a Cauchy sequence, i.e.,

$$\lim_{k\to\infty} x^k = x^*. \tag{3.12}$$

We need showing that $x^* \in \Omega$. From the definitions of B_{k+1} and x^{k+1}, we have

$$\|x^{k+1} - t^k\| \leq \|x^{k+1} - \bar{z}^k\| \leq \|x^{k+1} - x^k\|.$$

Thus,

$$\begin{aligned}\|t^k - x^k\| &\leq \|t^k - x^{k+1}\| + \|x^{k+1} - x^k\| \\ &\leq \|x^k - x^{k+1}\| + \|x^k - x^{k+1}\| \\ &= 2\|x^k - x^{k+1}\|,\end{aligned}$$

Combining with (3.12), we get

$$\lim_{k\to\infty} \|t^k - x^k\| = 0. \tag{3.13}$$

From (3.9), one has

$$\begin{aligned}&\sum_{i=1}^{N} \alpha_k^i\big[(1-2\rho_k^i L_1)\|x^k - y_i^k\|^2 - (1-2\rho_k^i L_2)\|y_i^k - z_i^k\|^2\big] \\ &+\mu \sum_{j=1}^{M} \big[\beta_k^j(1-2r_k^j L_1)\|A\bar{z}^k - v_j^k\|^2 - (1-2r_k^j L_2)\|v_j^k - u_j^k\|^2\big] \leq \|x^k - p\|^2 - \|t^k - p\|^2 \\ &\leq \|t^k - x^k\|\big(\|x^k - p\| + \|t^k - p\|\big).\end{aligned} \tag{3.14}$$

Because $\{\rho_k^i\}, \{r_k^j\} \subset [\underline{\rho}, \ \bar{\rho}] \subset (0, \ \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\})$; $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha}, \ \bar{\alpha}] \subset (0, \ 1]$, $\mu \in (0, \frac{1}{\|A\|})$; (3.12), and (3.13) we deduce from (3.14) that

$$\lim_{k\to+\infty} \|x^k - y_i^k\| = \|y_i^k - z_i^k\| = 0, \ \text{for all } i = 1, 2, ..., N, \tag{3.15}$$

and

$$\lim_{k\to+\infty} \|A\bar{z}^k - v_j^k\| = \|v_j^k - u_j^k\| = 0, \text{ for all } j = 1, 2, ..., M. \tag{3.16}$$

From (3.15), (3.16) and $\lim_{k\to+\infty} x^k = x^*$, we have

$$\lim_{k\to+\infty} y_i^k = x^*, \quad \lim_{k\to+\infty} z_i^k = x^*, \quad \lim_{k\to+\infty} \bar{z}^k = x^*, \ \forall i = 1, 2, ..., N \tag{3.17}$$

and

$$\lim_{k\to+\infty} A\bar{z}^k = Ax^*, \quad \lim_{k\to+\infty} v_j^k = Ax^*, \quad \lim_{k\to+\infty} u_j^k = Ax^*, \ \forall j = 1, 2, ..., M. \tag{3.18}$$

Beside that, for each $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$, Lemma 3.1 implies that

$$\rho_k^i[f_i(x^k, y) - f_i(x^k, y_i^k)] \ge \langle y_i^k - x^k, y_i^k - y\rangle, \ \forall y \in C_i, \text{ and}$$

$$r_k^j[g_j(A\bar{z}^k, v) - g_j(A\bar{z}^k, v_j^k)] \ge \langle v_j^k - A\bar{z}^k, v_j^k - v\rangle, \ \forall v \in Q_j.$$

We have

$$\langle y_i^k - x^k, y_i^k - y\rangle \ge -\|y_i^k - x^k\|\|y_i^k - y\| \text{ and}$$

$$\langle v_j^k - A\bar{z}^k, v_j^k - v\rangle \ge -\|v_j^k - A\bar{z}^k\|\|v_j^k - v\|,$$

therefore we get from the last inequalities that

$$f_i(x^k, y) - f_i(x^k, y_i^k) + \frac{1}{\rho_k^i}\|y_i^k - x^k\|\|y_i^k - y\| \ge 0, \text{ and}$$

$$g_j(A\bar{z}^k, v) - g_j(A\bar{z}^k, v_j^k) + \frac{1}{r_k^j}\|v_j^k - A\bar{z}^k\|\|v_j^k - v\| \ge 0.$$

Letting $k \to +\infty$, in combination with (3.17), (3.18) and the continuity of f_i, g_j, yield

$$f_i(x^*, y) - f_i(x^*, x^*) \ge 0, \ \forall y \in C_i, \forall i = 1, 2, ..., N, \text{ and}$$

$$g_j(Ax^*, v) - g_j(Ax^*, Ax^*) \ge 0, \ \forall v \in Q_j, \forall j = 1, 2, ..., M,$$

which means that $x^* \in \cap_{i=1}^N \mathrm{Sol}(C_i, f_i)$, and $Ax^* \in \cap_{j=1}^M \mathrm{Sol}(Q_j, g_j)$, or $x^* \in \Omega$. The proof is completed. □

The following corollary is immediate from Algorithm 2 and Theorem 3.2 when $N = M = 1$.

Corollary 3.3 *Let f, g be bifunctions satisfying assumptions $\mathcal{A}$ on C and Q, respectively. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with its adjoint A^*. Choose $x^0 = x^g \in C$, $B_0 = C$; $\{\rho_k\}$, $\{r_k\} \subset [\underline{\rho}, \bar{\rho}]$, for some $\underline{\rho}$, $\bar{\rho}$ such that*

$0 < \underline{\rho} \leq \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$; $0 < \mu < \dfrac{1}{\|A\|^2}$. *Consider the sequences* $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, *and* $\{v^k\}$, $\{u^k\}$ *defined by*

$$\begin{cases} y^k = \arg\min\left\{\rho_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \arg\min\left\{\rho_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ v^k = \arg\min\left\{r_k g(Az^k, v) + \frac{1}{2}\|v - Az^k\|^2 : v \in Q\right\}, \\ u^k = \arg\min\left\{r_k g(v^k, y) + \frac{1}{2}\|v - Az^k\|^2 : v \in Q\right\}, \\ t^k = P_C(z^k + \mu A^*(u^k - Az^k)) \\ B_{k+1} = \{x \in B_k : \|x - t^k\| \leq \|x - z^k\| \leq \|x - x^k\|\}, \\ x^{k+1} = P_{B_{k+1}}(x^g). \end{cases}$$

Suppose that $\Omega = \{x^* \in Sol(C, f) : Ax^* \in Sol(Q, g)\} \neq \emptyset$, *then the sequences* $\{x^k\}$, $\{y^k\}$ *and* $\{z^k\}$ *converge strongly to an element* $x^* \in \Omega$ *and* $\{v^k\}$, $\{u^k\}$ *converge strongly to* $Ax^* \in Sol(Q, g)$.

Applying Algorithm 2 and Theorem 2 to MSSVIP, we get the following strong convergence algorithm for solving multiple set split pseudo-monotone variational inequality problems without using any product space.

Corollary 3.4 *Suppose that* F_i, G_j *are mappings satisfying assumptions* $\mathcal{A}$ *and* $A : \mathbb{H}_1 \to \mathbb{H}_2$ *be a bounded linear operator with its adjoint* A^*. *Take* $x^0 = x^g \in C = \cap_{i=1}^N C_i$; $\{\rho_k^i\}$, $\{r_k^j\} \subset [\underline{\rho}, \bar{\rho}]$, *for some* $\underline{\rho}$, $\bar{\rho}$ *such that* $0 < \underline{\rho} \leq \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$; $\{\alpha_k^i\}, \{\beta_k^j\} \subset [\underline{\alpha}, \bar{\alpha}] \subset (0, 1]$, *for all* $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, *and* $\sum_{i=1}^N \alpha_k^i = \sum_{j=1}^M \beta_k^j = 1$; $0 < \mu < \dfrac{1}{\|A\|^2}$; $B_0 = C$. *Consider the sequences* $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ *and* $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ *defined by*

$$\begin{cases} \begin{cases} y_i^k = P_{C_i}(x^k - \rho_k^i F_i(x^k)) \\ z_i^k = P_{C_i}(x^k - \rho_k^i F_i(y_i^k)), \end{cases} i = 1, 2, ..., N. \\ \bar{z}^k = \sum_{i=1}^N \alpha_k^i z_i^k, \\ \hat{v}^k = A\bar{z}^k, \\ \begin{cases} v_j^k = P_{Q_j}(\hat{v}^k - r_k^j G_j(\hat{v}^k)) \\ u_j^k = P_{Q_j}(\hat{v}^k - r_k^j G_j(v_j^k)), \end{cases} j = 1, 2, ..., M. \\ \bar{u}^k = \sum_{j=1}^M \beta_k^j u_j^k \\ t^k = P_C(\bar{z}^k + \mu A^*(\bar{u}^k - \hat{v}^k)) \\ x^{k+1} = P_{B_{k+1}}(x^g), \text{ where } B_{k+1} = \{x \in B_k : \|x - t^k\| \leq \|x - \bar{z}^k\| \leq \|x - x^k\|\} \end{cases}$$

If $\Omega = \{x^* \in \bigcap_{i=1}^N Sol(C_i, F_i) : Ax^* \in \bigcap_{j=1}^M Sol(Q_j, G_j)\} \neq \emptyset$, *then the sequences* $\{x^k\}$, $\{y_i^k\}$, $\{z_i^k\}$, $i = 1, 2, ..., N$ *converge strongly to an element* $x^* \in \Omega$, *and* $\{v_j^k\}$, $\{u_j^k\}$, $j = 1, 2, ..., M$ *converge strongly to* $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, G_j)$.

4 A numerical example

In this section, we consider the (MSSEP) when $\mathbb{H}_1 = \mathbb{R}^n$, and $\mathbb{H}_2 = \mathbb{R}^m$, the linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ is given by $m \times n$ matrix $A = (a_{ls})_{m\times n} \in \mathbb{R}^{m\times n}$. The bifunctions f_i, $i = 1, 2, ..., N$, g_j, $j = 1, 2, ..., M$, are given as follows

$$f_i(x, y) = (P^i x + Q^i y + b^i)^T (y - x), \forall x, y \in \mathbb{R}^n, i = 1, 2, ..., N,$$
$$g_j(u, v) = (U^j u + V^j v + w^j)^T (u - v), \forall u, v \in \mathbb{R}^m, j = 1, 2, ..., M,$$

where $P^i = (p^i_{ls})_{n\times n}$, $Q^i = (q^i_{ls})_{n\times n}$, and $U^j = (u^j_{ls})_{m\times m}$, $V^j = (v^j_{ls})_{m\times m}$ are symmetric positive semidefinite matrices such that $P^i - Q^i$ and $U^j - V^j$ are also positive semidefinite matrices, $b^i \in \mathbb{R}^n$, $w^j \in \mathbb{R}^m$, for all $i = 1, 2, ..., N$, and $j = 1, 2, ..., M$. The bifunctions f_i and g_j have the form of the one arising from a Nash-Cournot oligopolistic electricity market equilibrium model [14, 36]. In this case, these f_i are convex in the second variable, Lipschitz-type continuous with constants $L^1_i = L^2_i = \frac{1}{2}\|P^i - Q^i\|$, and the positive semidefinition of $P^i - Q^i$ implies that f^i are monotone, for all $i = 1, 2, ..., N$ (see, [42]). Similarly, we have that g_j are convex in the second variable, Lipschitz-type continuous with constants $\bar{L}^1_j = \bar{L}^2_j = \frac{1}{2}\|U^j - V^j\|$, and those g^j are monotone, for all $j = 1, 2, ..., M$. By choosing $L_1 = \max\{L^1_i, \bar{L}^1_j : i = 1, 2, ..., N, j = 1, 2, ..., M\}$ and $L_2 = \max\{L^2_i, \bar{L}^2_j : i = 1, 2, ..., N, j = 1, 2, ..., M\}$, the bifunctions f_i and g_j are Lipschitz-type continuous with constants L_1 and L_2, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$ (see, Remark 2.1).

In order to ensure that the intersection of the solution set Ω of (MSSEP) is nonempty, we further assume that $b^i = 0$, $w^j = 0$ and the constraint sets C_i of EP(C_i, f_i) and the constraint sets Q_j of EP(Q_j, g_j) contain the original of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, for all $i = 1, 2, ..., N$, $j = 1, 2, ..., M$.

We tested the proposed algorithms for this example in which $C_i = C, \forall i = 1, ..., N$, C is the box $C = \prod_{i=1}^n[-5, 5]$, with $n = 5, 10, 20, 50$; $N = 10, 20, 50, 100$, and $Q_j = Q, \forall j = 1, ..., M$, Q is the box $Q = \prod_{j=1}^m[-20, 20]$, with $m = 5, 10, 20$; $M = 5, 20, 40$. The matrix $A = (a_{ls})_{m\times n}$ is randomly generated in the interval $[-2, 2]$. Similarly, P^i, and $P^i - Q^i$ are matrices of the form $D^T D$ with $D = (d_{ls})_{n\times n}$ being randomly generated in the interval $[-5, 5]$; U^j, and $U^j - V^j$ are matrices of the form $W^T W$ with $W = (w_{ls})_{m\times m}$ being randomly generated in the interval $[-5, 5]$. Starting point is chosen as a guess solution $x^g = (1, 1, ..., 1)^T \in \mathbb{R}^n$ and the parameters: $\mu = \frac{1}{2\|A\|^2}$; $\rho^i_k = r^j_k = \rho = \frac{1}{4L}$, with $L = \max\{L_1, L_2\}$, $\alpha^i_k = \frac{1}{N}$, $\beta^j_k = \frac{1}{M}$, $\forall k$, $\forall i = 1, 2, ..., N$, and $\forall j = 1, 2, ..., M$.

At Iteration k, in Step 1 of Algorithms 1 and 2, to get y^k_i we need to solve the following optimization programs

$$\arg\min\{f_i(x^k, y) + \frac{1}{2\rho}\|y - x^k\|^2 : y \in C\}$$

or the following convex quadratic problems

$$\arg\min\{\frac{1}{2}y^T \bar{P}^i y + \bar{p}^i y : y \in C\}, \tag{4.19}$$

where $\bar{P}^i = 2Q^i + \frac{1}{\rho}I_n$ and $\bar{p}^i = P^i x^k - Q^i x^k - \frac{1}{\rho}x^k$.

Problem (4.19) can be solved effectively, for instance, by using the Matlab Optimization Toolbox.

Similarly, z_i^k solves the following quadratic program

$$\arg\min\{\frac{1}{2}y^T \bar{Q}^i y + \bar{q}^i y : y \in C\},$$

where $\bar{Q}^i = \bar{P}^i$ and $\bar{q}^i = P^i y_i^k - Q^i y_i^k - \frac{1}{\rho}x^k$.

By the same way, v_j^k, u_j^k in Step 3 of Algorithms 1 and 2 can be computed by using the Matlab Optimization Toolbox.

It has been showed that Algorithm 2 is strongly convergent. Theoretically, it is useful in infinite dimensional Hilbert spaces, although it is not easy to construct the sets B_k in general. However, in this example, we can compute those sets B_k as follows.

Let I_n be the unit matrix of order n and $A_0 = \begin{pmatrix} I_n \\ -I_n \end{pmatrix}$ be the $2n \times n$ matrix; $b_0 = (5, 5, ..., 5)^T \in \mathbb{R}^{2n}$. Then, it is clear that

$$B_0 = C = \prod_{i=1}^{n}[-5, 5] = \{x \in \mathbb{R}^n : A_0 x \leq b_0\}.$$

Now, having B_0, we describe how to construct the set B_1 in the *Iteration 0*. Indeed, set

$$\begin{aligned} H_0^1 &= \{x \in \mathbb{R}^n : \|x - t^0\| \leq \|x - \bar{z}^0\|\} \\ &= \{x \in \mathbb{R}^n : \langle \bar{z}^0 - t^0, x\rangle \leq \frac{1}{2}\left(\|\bar{z}^0\|^2 - \|t^0\|^2\right)\}, \end{aligned}$$

$$\begin{aligned} H_0^2 &= \{x \in \mathbb{R}^n : \|x - \bar{z}^0\| \leq \|x - x^0\|\} \\ &= \{x \in \mathbb{R}^n : \langle x^0 - \bar{z}^0, x\rangle \leq \frac{1}{2}\left(\|x^0\|^2 - \|\bar{z}^0\|^2\right)\}. \end{aligned}$$

Let $\bar{B}_0$ be the matrix of the size $2 \times n$ and $\bar{b}_0$ be the vector defined as

$$\bar{B}_0 = \begin{pmatrix} (\bar{z}^0 - t^0)^T \\ (x^0 - \bar{z}^0)^T \end{pmatrix}, \ \bar{b}_0 = \begin{pmatrix} \frac{1}{2}\left(\|\bar{z}^0\| - \|t^0\|^2\right) \\ \frac{1}{2}\left(\|x^0\|^2 - \|\bar{z}^0\|^2\right) \end{pmatrix}.$$

Then, we have $H_0^1 \cap H_0^2 = \{x \in \mathbb{R}^n : \bar{B}_0 x \leq \bar{b}_0\}$. By setting

$$A_1 = \begin{pmatrix} A_0 \\ \bar{B}_0 \end{pmatrix}, \ b_1 = \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix}.$$

By definition of B_1, we have

$$B_1 = B_0 \cap H_0^1 \cap H_0^2 = \{x \in \mathbb{R}^n : A_1 x \leq b_1\}.$$

Similarly, at *Iteration k*, we have already had the set

$$B_k = \{x \in \mathbb{R}^n : A_k x \leq b_k\},$$

we can compute B_{k+1} as follows:

Since $B_{k+1} = B_k \cap H_k^1 \cap H_k^2$, where

$$\begin{aligned} H_k^1 &= \{x \in \mathbb{R}^n : \|x - t^k\| \leq \|x - \bar{z}^k\|\} \\ &= \{x \in \mathbb{R}^n : \langle \bar{z}^k - t^k, x\rangle \leq \frac{1}{2}\big(\|\bar{z}^k\|^2 - \|t^k\|^2\big)\}, \\ H_k^2 &= \{x \in \mathbb{R}^n : \|x - \bar{z}^k\| \leq \|x - x^k\|\} \\ &= \{x \in \mathbb{R}^n : \langle x^k - \bar{z}^k, x\rangle \leq \frac{1}{2}\big(\|x^k\|^2 - \|\bar{z}^k\|^2\big)\}. \end{aligned}$$

Let $\bar{B}_k$ be the matrix of the size $2 \times n$ and $\bar{b}_k$ be the vector such that

$$\bar{B}_k = \begin{pmatrix} (\bar{z}^k - t^k)^T \\ (x^k - \bar{z}^k)^T \end{pmatrix}, \ \bar{b}_k = \begin{pmatrix} \frac{1}{2}\big(\|\bar{z}^k\| - \|t^k\|^2\big) \\ \frac{1}{2}\big(\|x^k\|^2 - \|\bar{z}^k\|^2\big) \end{pmatrix}.$$

Then, we have $H_k^1 \cap H_k^2 = \{x \in \mathbb{R}^n : \bar{B}_k x \leq \bar{b}_k\}$.

By setting

$$A_{k+1} = \begin{pmatrix} A^k \\ \bar{B}_k \end{pmatrix}, \ b_{k+1} = \begin{pmatrix} b_k \\ \bar{b}_k \end{pmatrix}.$$

Thus,

$$B_{k+1} = B_k \cap H_k^1 \cap H_k^2 = \{x \in \mathbb{R}^n : A_{k+1}x \leq b_{k+1}\}.$$

Therefore, at *Iteration* k, to compute x^{k+1}, we need to find projection of x^g on to B_{k+1}, i.e., we have to solve the following quadratic optimization problem:

$$\arg\min\{\frac{1}{2}y^T y - x^{g^T} y : y \in B_{k+1}\}. \tag{4.20}$$

Problem (4.20) can be solved effectively by using the Matlab Optimization Toolbox.

We implement Algorithm 1 and Algorithm 2 for this problem in Matlab R2013 running on a Desktop with Intel(R) Core(TM) 2Duo CPU E8400 3 GHz, and 3 GB Ram. To terminate the Algorithms, we use the stopping criteria which is defined as follows: or the number of iteration is greater than 1000: $ITER > 1000$, or the

Table 1 Results computed with Algorithm 1

N.P	S(n/N; m/M)	TOL.	CPU(s)	ITER.	S(n/N; m/M)	CPU(s)	ITER.
5	(5/10; 5/5)	10^{-3}	4.3406	16	(5/10; 10/20)	5.6500	13
		10^{-5}	6.5219	27		10.9469	29
5	(5/10; 20/40)	10^{-3}	9.2875	13	(10/20; 5/5)	6.3031	18
		10^{-5}	14.9688	23		9.0719	30
5	(10/20; 10/20)	10^{-3}	9.5844	16	(10/20; 20/40)	13.9313	17
		10^{-5}	13.3312	27		21.6313	27
5	(20/50; 5/5)	10^{-3}	13.6594	20	(20/50; 10/20)	18.0656	19
		10^{-5}	18.7906	32		21.3750	31
5	(20/50; 20/40)	10^{-3}	24.6625	17	(50/100; 5/5)	103.3469	21
		10^{-5}	29.8688	29		132.3188	32
5	(50/100; 10/20)	10^{-3}	111.5719	20	(50/100; 20/40)	116.7188	20
		10^{-5}	145.9375	32		167.9781	32

Table 2 Results computed with Algorithm 2

N.P	S(n/N; m/M)	TOL.	CPU(s)	ITER.	S(n/N; m/M)	CPU(s)	ITER.
5	(5/10; 5/5)	10^{-3}	11.9063	78	(5/10; 10/20)	27.3469	95
		10^{-5}	17.9188	120		53.4688	173
5	(5/10; 20/40)	10^{-3}	51.7719	106	(10/20; 5/5)	69.6688	272
		10^{-5}	84.4469	182		141.8500	523
5	(10/20; 10/20)	10^{-3}	84.9250	208	(10/20; 20/40)	137.3719	242
		10^{-5}	160.8844	417		266.0250	460
2	(20/50; 5/5)	10^{-3}	497.2433	1000	(20/50; 10/20)	602.8650	1000
		10^{-5}	502.4063	1000		626.7500	1000
1	(20/50; 20/40)	10^{-3}	790.3064	1000	(50/100; 5/5)	2487.4930	1000
		10^{-5}	810.1094	1000		2583.6	1000
1	(50/100; 10/20)	10^{-3}	3598.4	1000	(50/100; 20/40)	4011.2	1000
		10^{-5}	3668.2	1000		4270.5	1000

tolerance: $TOL = \|x^{k+1} - x^k\| < \epsilon$ with a tolerance $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$. The computation results on Algorithm 1 and Algorithm 2 are reported in Tables 1 and 2 respectively, where

N.P: the number of the tested problems;
*S(**n/N; m/M**)*: the size of the tested problems;
CPU(s): the average CPU-computation times (in second);
ITER: the average number of iterations.

From the computed results reported in Tables 1 and 2, we can see that the computational time and the number of iterations computed by Algorithm 1 is less than those by Algorithm 2, especially when the size of the tested problem is large.

5 Conclusions

We have introduced a multiple set split equilibrium problem (MSSEP) in real Hilbert spaces and proposed two new parallel extragradient algorithms for solving it without using any product space. The weak and strong convergence of iteration sequences generated by the algorithms to a solution of MSSEP are obtained under main assumption that the bifunctions are Lipschitz-type continuous and pseudo-monotone with respect to their solution sets. A numerical example in which the equilibrium bifunctions have a form of the one arising from Nash Cournot equilibrium model is also provided to illustrate the convergence of proposed algorithms.

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