

Critical Angles of the Nyström Method for Double Layer Potential Equation

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Abstract. The Nyström method for double layer potential equations on simple closed piecewise smooth contours is stable if and only if certain operators associated with the opening angles of the corners are invertible. Numerical experiments show that there are opening angles which cause the instability of the method.

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1. Introduction

Boundary integral equations are widely used in approximate solution of partial differential equations — e.g. the Dirichlet problem for the Laplace equation

$$\begin{aligned}\Delta u(x, y) &= 0, \quad (x, y) \in D \\ u(x, y) &= f(x, y), \quad (x, y) \in \Gamma\end{aligned}\tag{1.1}$$

in a simply connected domain $D \subset \mathbb{R}^2$ can be reduced to a double layer potential equation on the boundary Γ of D .

We consider the double layer potential equation

$$(Ax)(t) = x(t) + \frac{1}{\pi} \int_{\Gamma} x(\tau) \frac{d}{dn_{\tau}} \log |t - \tau| d\Gamma_{\tau} + (Tx)(t) = f(t), \quad t \in \Gamma.\tag{1.2}$$

where n_{τ} refers to the outer normal to Γ at the point $\tau \in \Gamma$ and T is a compact operator. The form of T depends on the boundary value problem studied. In particular, if the following integral representation

$$u(z) := \frac{1}{\pi} \int_{\Gamma} x(\tau) \frac{d}{dn_{\tau}} \log |z - \tau| d\Gamma_{\tau}, \quad z \in D,$$

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of the solutions of (1.1) is used, then T is the zero operator [2]. The operator A in (1.2) has been thoroughly studied for various contours and spaces — cf. Refs. [8, 22, 25, 27]. For smooth curves Γ , the double layer potential operator

$$V_{\Gamma}x(t) := \frac{1}{\pi} \int_{\Gamma} x(\tau) \frac{d}{dn_{\tau}} \log |t - \tau| d\Gamma_{\tau} \quad (1.3)$$

acting in an L^p -space is compact. This property of V_{Γ} substantially simplifies the stability analysis of approximation methods. In fact, for smooth curves, the invertibility of A and a good convergence of approximation operators to A ensure the stability of the method. For curves with corners, the operator (1.3) is not compact and stability depends on additional parameters — cf. Refs. [2, 12, 13, 18, 21, 22]. Here we consider the Nyström discretisation based on Gauss-Legendre quadrature formula. This method is easily implemented and demonstrates a good convergence even for contours with a large set of corners — cf. Refs. [4, 5, 20]. Its stability for the Sherman-Lauricella and Muskhelishvili equations has been studied earlier [9–11, 15]. For the double layer potential equation, the stability is proven for sufficiently nice curves — polygons or curves coinciding with polygons in neighbourhoods of corners, or/and for sufficiently nice double layer potential operators — perturbations of the identity by compact and small norm operators [18, 21, 22]. Besides, so far only sufficient stability conditions are available. Our goal here is to establish necessary and sufficient stability conditions of this method for general simple piecewise smooth curves and to provide a procedure for their verification. It turns out that there exist angles which cause instability. Such angles are called critical and if Γ has a critical angle, the Nyström method is not stable regardless of the shape of the curve. In contrast, spline Galerkin methods do not have any critical angles [15, 17] but the complexity of the implementation and computational cost are significantly higher than for Nyström methods. Therefore, in practical computations Nyström type methods are preferable and there are modifications designed to handle the instability induced by non-smooth boundaries [7, 19]. On the other hand, in the interval $[0.1\pi, 1.9\pi]$, the Nyström method for the double layer potential equation has only four critical angles. It is possible that some or all of them are irrational, so we may never encounter the unstable situation during implementation of the method — cf. Remark 3.1 and Fig. 3 below.

Let us make a few technical remarks. We identify every point (x, y) of \mathbb{R}^2 with the point $z = x + iy$ in the complex plane \mathbb{C} . By S_{Γ} we denote the Cauchy singular integral operator on Γ ,

$$(S_{\Gamma}x)(t) := \frac{1}{\pi i} \int_{\Gamma} \frac{x(\tau) d\tau}{\tau - t}$$

and by M the operator of complex conjugation, $M\varphi(t) := \overline{\varphi(t)}$. It is known [23] that the double layer potential operator V_{Γ} can be represented in the form

$$V_{\Gamma} = \frac{1}{2}(S_{\Gamma} + MS_{\Gamma}M),$$

so that the equation (1.2) becomes

$$Ax = \left(I + \frac{1}{2}S_\Gamma + \frac{1}{2}MS_\Gamma M + T \right) x = f. \quad (1.4)$$

We consider this equation in the space $L^2 := L^2(\Gamma)$ of all complex-valued Lebesgue measurable functions f with the norm

$$\|f\|_{L^2} := \left(\int_\Gamma |f(t)|^2 |dt| \right)^{1/2}.$$

The Nyström method we study, is based on the composite Gauss-Legendre quadrature rule

$$\int_0^1 u(s) ds \approx \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p u(s_{lp}) / n, \quad (1.5)$$

where

$$s_{lp} = \frac{l + \varepsilon_p}{n}, \quad l = 0, 1, \dots, n-1, \quad p = 0, 1, \dots, d-1,$$

and w_p and $0 < \varepsilon_0 < \varepsilon_1 < \dots < \varepsilon_{d-1} < 1$ are, respectively, Gauss-Legendre weights and Gauss-Legendre points rescaled to the interval $[0, 1]$. A more detailed description of the approximation method is given in Section 2. The stability of the method is studied in Section 3. We prove that the corresponding Nyström method is stable if and only if some operators B_ω , $\omega \in (0, 2\pi)$ depending on the corners are invertible. These operators belong to a Toeplitz algebra and in general no effective conditions of their invertibility are available. Nevertheless, for $\omega \in [0.1\pi, 1.9\pi]$ a numerical approach introduced in Section 4, allows us to find all irreversible operators B_ω , hence locate the critical angles. Small opening angles and angles close to 2π can also be scrutinised but it requires more efforts and computational cost is high.

2. Nyström Method

Let Γ be a simple piecewise smooth positively oriented contour in the complex plane \mathbb{C} and let $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ be a 1-periodic parametrization of Γ . By \mathfrak{M}_Γ we denote the set of all corner points $\tau_0, \tau_1, \dots, \tau_{q-1}$ of Γ . Assume that

$$\tau_j = \gamma(j/q), \quad j = 0, 1, \dots, q-1, \quad (2.1)$$

the function γ is two times continuously differentiable on each subinterval $(j/q, (j+1)/q)$ and

$$|\gamma'(j/q+0)| = |\gamma'(j/q-0)|, \quad j = 0, 1, \dots, q-1. \quad (2.2)$$

Choose an $n = rq$, $r \in \mathbb{N}$ and a non-negative integer d such that $n > d + 1$. By $S_n^d(\Gamma)$ we denote the space of the smoothest splines of degree d on Γ associated with the parametrization γ . More precisely, let $\varphi^d = \varphi^d(s)$, $s \in \mathbb{R}$ be the uniform cardinal B-spline of degree d .

Recall [26] that $\varphi^0 = \chi_{[0,1]}$, where $\chi_{[0,1]}$ is the characteristic function of the interval $[0, 1]$ and for $d \geq 1$, the functions φ^d can be obtained by the recursive formula

$$\varphi^d(s) = \int_{\mathbb{R}} \chi_{[0,1]}(s-x) \varphi^{d-1}(x) dx.$$

For any fixed non-negative integer d and for any $n = md$, we define the functions

$$\psi_{kn}(s) := \psi(ns - k), \quad k \in \mathbb{Z}, \quad s \in \mathbb{R}, \quad (2.3)$$

where $\psi(s) := \varphi^d(s)$, and consider the set

$$\Lambda^d := \{\psi_{kn} : k \geq 0 \text{ and } \text{supp}\{\psi_{kn}\} \subset (0, 1)\}.$$

It follows from Lemma 5.4.1 in Ref. [14] that for $d \geq 1$ the support $\text{supp}\{\psi_{kn}\}$ of the function ψ_{kn} is the interval $(k/n, (k+d+1)/n)$ hence $\psi_{kn} \in \Lambda^d$ if and only if $0 \leq k \leq n-d-1$. If n is sufficiently large, then the restriction $\psi_{kn}|_{[0, (1+d)/n]}$ of ψ_{kn} to $[0, (1+d)/n]$ can be extended to the whole real line \mathbb{R} in a 1-periodic way. Such an extension is denoted by the same symbol ψ_{kn} . Now using a 1-periodic parametrization $\gamma = \gamma(s)$, $s \in \mathbb{R}$ of the contour Γ , we set

$$\tilde{\psi}_{kn}(t) := \psi_{kn}(s), \quad t = \gamma(s) \in \Gamma.$$

If $(d+1)/n < 1$, this definition is correct, and by $S_n^d = S_n^d(\Gamma)$ we denote the corresponding spline space on Γ — i.e. $S_n^d(\Gamma) \subset L^2(\Gamma)$ is the set of all linear combinations of the functions $\tilde{\psi}_{kn}$, $k = 0, 1, \dots, n-1$.

We assume that $n = md$ and $m = qr$, $m, r \in \mathbb{N}$ and consider the set

$$\tau_{lp} = \gamma\left(\frac{l + \varepsilon_p}{n}\right), \quad l = 0, 1, \dots, n-1, \quad p = 0, 1, \dots, d-1,$$

where $0 < \varepsilon_0 < \varepsilon_1 < \dots < \varepsilon_{d-1} < 1$ are the Gauss-Legendre points rescaled to the interval $[0, 1]$. If the integral operator

$$K\varphi(t) := \int_{\Gamma} k(t, \tau) \varphi(\tau) d\tau$$

has a sufficiently smooth kernel k and if φ is a Riemann integrable function, then this operator can be approximated by the quadrature rule (1.5) — viz.

$$\begin{aligned} \int_{\Gamma} k(t, \tau) \varphi(\tau) d\tau &= \int_0^1 k(\gamma(\sigma), \gamma(s)) \varphi(\gamma(s)) \gamma'(s) ds \\ &\approx K^{(\varepsilon, n)} \varphi(t) = \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p k(t, \tau_{lp}) \varphi(\tau_{lp}) \tau'_{lp} / n, \end{aligned} \quad (2.4)$$

with $\tau'_{lp} := \gamma'((l + \varepsilon_p)/n)$.

Let $\mathbf{R}(\Gamma)$ stand for the set of all Riemann integrable functions on Γ and $Q_n^\varepsilon : L_\infty(\Gamma) \mapsto S_n^d(\Gamma)$ be the interpolation projection on the space $S_n^d(\Gamma)$ such that

$$Q_n^\varepsilon x(\tau_{lp}) := x(\tau_{lp}), \quad l = 0, 1, \dots, n-1, \quad p = 0, 1, \dots, d-1,$$

for all $x \in \mathbf{R}(\Gamma)$. If $\varepsilon_p \neq 0.5$, $p = 0, 1, \dots, d-1$, then such projection operator Q_n^ε exists and the sequence $(Q_n^\varepsilon)_{n \in \mathbb{N}} : \mathbf{R}(\Gamma) \mapsto L^2(\Gamma)$ converges strongly to the corresponding embedding operator [24, Chapter 2] — viz.

$$\lim_{n \rightarrow \infty} \|Q_n^\varepsilon f - f\|_{L^2(\Gamma)} = 0, \quad f \in \mathbf{R}(\Gamma). \quad (2.5)$$

We also consider the orthogonal projections $P_n : L^2(\Gamma) \rightarrow S_n^d(\Gamma)$, $n \in \mathbb{N}$ and note that the sequence (P_n) converges strongly to the identity operator — cf. Ref. [24].

It is convenient to drop the compact operator in (1.2) and study the stability of the Nyström method for the equation

$$A_\Gamma x = (I + V_\Gamma)x = f. \quad (2.6)$$

This simplifies notation but does not influence the proof of the main result. Approximate solutions x_n , $n \in \mathbb{N}$ of (2.6) are derived from the operator equations

$$Q_n^\varepsilon A_\Gamma^{(\varepsilon, n)} P_n x_n := Q_n^\varepsilon P_n x_n + Q_n^\varepsilon V_\Gamma^{(\varepsilon, n)} P_n x_n = Q_n^\varepsilon f, \quad x_n \in S_n^d(\Gamma), \quad n \in \mathbb{N},$$

which are equivalent to the following systems of linear algebraic equations:

$$x(\tau_{kr}) + \frac{1}{2\pi i} \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p x(\tau_{lp}) \left(\frac{\tau'_{lp}}{\tau_{lp} - \tau_{kr}} - \frac{\overline{\tau'_{lp}}}{\overline{\tau_{lp}} - \overline{\tau_{kr}}} \right) \frac{1}{n} = f(\tau_{kr})$$

$$k = 0, 1, \dots, n-1, \quad r = 0, 1, \dots, d-1. \quad (2.7)$$

Remark 2.1. Let $k = k(t, \tau)$ denote the kernel of the double layer potential operator V_Γ . Straightforward calculations [16] show that for any $\tau \notin \mathfrak{M}_\Gamma$,

$$\lim_{t \rightarrow \tau} k(t, \tau) = \frac{i \operatorname{Im} \left(\gamma'(s) \overline{\gamma''(s)} \right)}{|\gamma'(s)|^2}, \quad \tau = \gamma(s).$$

Therefore, if $l = k$ and $p = r$, then the expression in the brackets of (2.7) is set to be $\gamma'(s_{lp}) \lim_{t \rightarrow \tau_{lp}} k(t, \tau_{lp})$.

Let us find approximate solutions of the equation (2.6) with continuous and discontinuous right-hand sides for two different contours $\mathcal{L}_1 = \mathcal{L}_1(\omega)$, $\mathcal{L}_2 = \mathcal{L}_2(\omega)$, $\omega \in (0, 2\pi)$, the graphs of which are given in Fig. 1. The curves \mathcal{L}_1 and \mathcal{L}_2 have, respectively, one and two corners of the same magnitude $\omega \in (0, 2\pi)$ and are defined by

$$\mathcal{L}_j := \{t \in \mathbb{C} : t = \gamma_j(s), \quad s \in [0, 1]\}, \quad j = 1, 2,$$

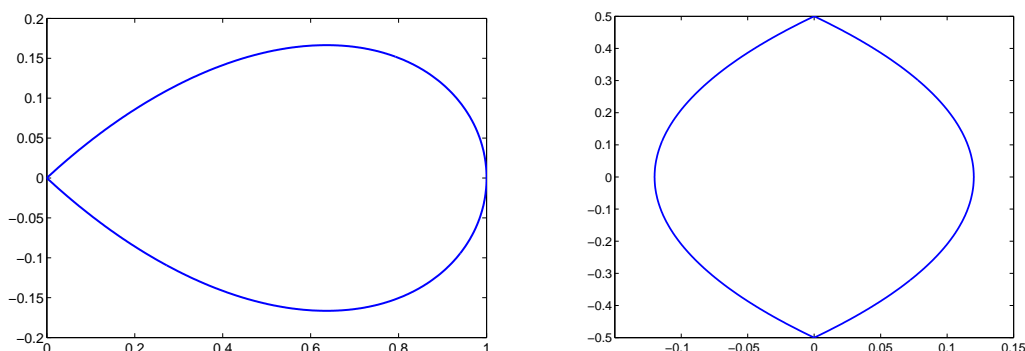


Figure 1: Left: one-corner curve \mathcal{L}_1 . Right: two-corner curve \mathcal{L}_2 .

Table 1: Convergence of the Nyström method.

n	$E_n^{(f_1, \mathcal{L}_1)}$	$E_n^{(f_1, \mathcal{L}_2)}$	$E_n^{(f_2, \mathcal{L}_1)}$	$E_n^{(f_2, \mathcal{L}_2)}$
32	2.5×10^{-3}	2.6×10^{-3}	1.5×10^{-2}	2.0×10^{-2}
96	8.3×10^{-4}	1.1×10^{-3}	7.5×10^{-3}	1.3×10^{-2}
256	3.1×10^{-4}	2.1×10^{-4}	4.0×10^{-3}	7.3×10^{-3}

where

$$\gamma_1(s) = \sin(\pi s) \exp(i\omega(s - 0.5)), \quad s \in [0, 1],$$

$$\gamma_2(s) = \begin{cases} -\frac{1}{2} \cot(\omega/2) + \frac{1}{2 \sin(\omega/2)} \exp(i\omega(2s - 0.5)) & \text{if } 0 \leq s \leq 1/2; \\ \frac{1}{2} \cot(\omega/2) - \frac{1}{2 \sin(\omega/2)} \exp(i\omega(2s - 1.5)) & \text{if } 1/2 < s \leq 1. \end{cases}$$

Further, let f_1 and f_2 be the following functions

$$f_1(z) = -z|z|,$$

and

$$f_2(z) = \begin{cases} -1 + iz & \text{if } \text{Im } z < 0 \\ 1 + iz & \text{if } \text{Im } z \geq 0 \end{cases}$$

The function f_1 is continuous everywhere, while f_2 has discontinuities one of which is located at the angular point of \mathcal{L}_1 .

The double layer potential equation with the right-hand sides f_1 and f_2 is solved by the Nyström method (2.7) with $d = 16$, and approximate solutions are shown in Fig. 2. Table 1 demonstrates the convergence of the method. We note that

$$E_n^{(f_k, \mathcal{L}_j)} := \frac{\|x_{2n} - x_n\|_2}{\|x_{2n}\|_2},$$

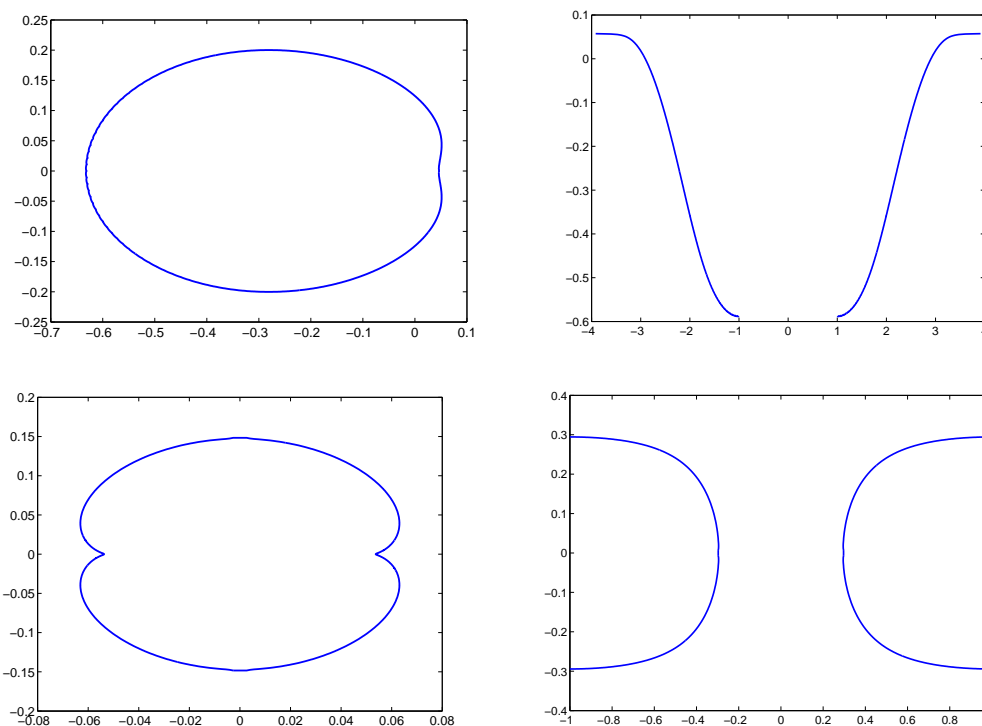


Figure 2: Approximate solutions of equation (2.6) obtained by Nyström method (2.7) (Real part versus imaginary part), $n = 512$, $d = 16$. First row: Solutions of (2.6) on \mathcal{L}_1 . Second row: Solutions of (2.6) on \mathcal{L}_2 . Left: Solutions for continuous r.-h.s. $f_1(z)$. Right: Solutions for discontinuous r.-h.s. $f_2(z)$.

where x_n is the approximate solution of the equation (2.6) for the contour $\mathcal{L}_j(0.3\pi)$, $j = 1, 2$ with the right-hand side f_k , $k = 1, 2$.

The convergence can be improved by using modifications of the Nyström method [3–5, 20] but the focus of this paper is on the stability and critical angles.

3. Stability

It is well-known that the invertibility of A is a necessary condition for applicability of many numerical methods [19, 24]. It depends on the contour Γ , compact operator T and the space where A acts. In this section we describe a more general kind of invertibility — viz. the Fredholmness of the operator A . These results are not new but during the proof we also introduce various operators needed in the analysis of Nyström method.

Let γ be a parametrization of a piecewise smooth curve Γ satisfying the relations (2.1)–(2.2). By ω_j we denote the angle between two semi-tangents to Γ at the corner τ_j and by β_j the angle between the right semi-tangent and the real axis \mathbb{R} . We also consider contour

$$\Gamma_j := \mathbb{R}^- e^{i(\beta_j + \omega_j)} \cup \mathbb{R}^+ e^{i\beta_j},$$

where \mathbb{R}^- and \mathbb{R}^+ are positive semi-axes directed to and away from the origin, respectively. By $L^2(\Gamma_j)$ we denote L^2 -space on Γ_j , and with each corner point $\tau_j \in \Gamma$ we associate the operator $A_{\Gamma_j} := I + V_{\Gamma_j} : L^2(\Gamma_j) \rightarrow L^2(\Gamma_j)$, where V_{Γ_j} is the double layer potential operator on Γ_j .

The application of Theorem 1.9.5 in [14] to the operator (1.4) leads to the following result.

Proposition 3.1. *Let Γ be a simple closed piecewise smooth curve in the complex plane \mathbb{C} . The operator A is Fredholm if and only if the operators $A_{\Gamma_j} : L^2(\Gamma_j) \rightarrow L^2(\Gamma_j)$ are invertible for all $j = 0, 1, \dots, q - 1$.*

Let \mathbb{M} and \mathbb{M}^{-1} denote, respectively, direct and inverse Mellin transforms — i.e.

$$(\mathbb{M}f)(z) = \int_0^{+\infty} x^{1/p+\alpha-zi-1} f(x)dx, \quad (\mathbb{M}^{-1}f)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x^{zi-1/p-\alpha} f(z)dz.$$

The Mellin convolution operator $\mathcal{M}(b)$ with the symbol b is defined by

$$\mathcal{M}(b)x(\sigma) = ((\mathbb{M}^{-1}b\mathbb{M})x)(\sigma),$$

and for some classes of symbols b , this operator can be represented in the integral form

$$\mathcal{M}(b)x(\sigma) = \int_0^{+\infty} \mathbf{k}\left(\frac{\sigma}{s}\right)x(s)\frac{ds}{s}, \tag{3.1}$$

with the kernel $\mathbf{k} = \mathbb{M}^{-1}b$.

Let $\mathcal{N}_\omega : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ be the operator defined by

$$(\mathcal{N}_\omega(\phi))(\sigma) = \frac{1}{\pi i} \int_0^{+\infty} \frac{\phi(s)ds}{s - \sigma e^{i\omega}}.$$

It is easily seen that A_{Γ_j} is isometrically isomorphic to the matrix operator

$$A_{\omega_j} = \begin{pmatrix} I & (1/2)(\mathcal{N}_\omega - \mathcal{N}_{2\pi-\omega}) \\ (1/2)(\mathcal{N}_\omega - \mathcal{N}_{2\pi-\omega}) & I \end{pmatrix}, \tag{3.2}$$

acting on the space $L^2(\mathbb{R}^+)^2 := L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, and the corresponding isomorphism is $A \mapsto \eta A \eta^{-1}$ with the mapping $\eta : L^2(\Gamma_j) \mapsto L^2(\mathbb{R}^+)^2$ defined by

$$\eta(f)(s) = (f(se^{i(\beta_j+\omega_j)}), f(se^{i\beta_j}))^T, \quad s \in \mathbb{R}^+.$$

It is well-known [14, 19, 25] that \mathcal{N}_ω is the Mellin convolution operator $\mathcal{M}(\mathbf{n}_\omega)$ with the symbol

$$\mathbf{n}_{\omega_j}(y) = \frac{e^{(\pi-\omega_j)y}}{\sinh \pi y}, \quad y = z + \frac{i}{2}, \quad z \in \mathbb{R}, \tag{3.3}$$

which leads to the formula

$$\text{smb}\left((1/2)(\mathcal{N}_{\omega_j} - \mathcal{N}_{2\pi - \omega_j})\right) = \frac{\sinh(\pi - \omega_j)y}{\sinh \pi y}, \quad y = z + \frac{i}{2}, \quad z \in \mathbb{R},$$

so that

$$\text{smb}A_{\omega_j}(y) = \begin{pmatrix} 1 & \frac{\sinh(\pi - \omega_j)y}{\sinh \pi y} \\ \frac{\sinh(\pi - \omega_j)y}{\sinh \pi y} & 1 \end{pmatrix}. \quad (3.4)$$

The Mellin operator $(1/2)(\mathcal{N}_{\omega_j} - \mathcal{N}_{2\pi - \omega_j})$ can be also represented in the integral form (3.1) with the kernel

$$\mathbf{k}(z) = \mathbf{k}_{\omega_j}(z) = \frac{1}{\pi i} \frac{iz \sin \omega_j}{(1 - ze^{i\omega_j})(1 - ze^{-i\omega_j})} \quad (3.5)$$

Corollary 3.1. *If Γ is a simple closed piecewise smooth contour satisfying the conditions of Section 2, then the operator $A : L^2(\Gamma) \rightarrow L^2(\Gamma)$ of (1.4) is Fredholm.*

Proof. The matrix Mellin operator A_{ω_j} is invertible in $L^2(\Gamma)$ if and only if its symbol (3.4) is invertible. The determinant of $\text{smb}A_{\omega_j}$ is $1 - \sinh^2(\pi - \omega_j)y / \sinh^2 \pi y$ and is equal to zero if and only if $\sinh(\pi - \omega_j)y = \sinh \pi y$ or $\sinh(\pi - \omega_j)y = -\sinh \pi y$. Let us consider the first of these equations. Separation of the real and imaginary parts produces the equations

$$\begin{aligned} \cosh\left((\pi - \omega_j)z\right) \sin \frac{\pi - \omega_j}{2} &= \cosh(\pi z) \\ \sinh\left((\pi - \omega_j)z\right) \cos \frac{\pi - \omega_j}{2} &= 0, \end{aligned}$$

where $z \in \mathbb{R}$. Since $\cos((\pi - \omega_j)/2) \neq 0$ for any $\omega_j \in (0, 2\pi)$, the second equation is satisfied only if $z = 0$ or $\pi - \omega_j = 0$. For $z = 0$, the first equation takes the form $\sin((\pi - \omega_j)/2) = 1$ and has no solution in the interval $(0, 2\pi)$. On the other hand, if $\pi - \omega_j = 0$, the first equation transforms into $\cosh(\pi z) = 0$ and is also not solvable. Thus the symbol of A_{ω_j} does not vanish on the line $\mathbb{R} + i/2$, hence A_{ω_j} are invertible for all $j = 0, 1, \dots, q-1$ and so are the operators A_{Γ_j} . The application of Proposition 3.1 finishes the proof. \square

We proceed with the stability of the method (2.7). Let (A_n) be a bounded sequence of linear bounded operators $A_n : S_n^d(\Gamma) \mapsto S_n^d(\Gamma)$. The set \mathcal{S} of such sequences equipped with componentwise operations of addition, multiplication, involution and multiplication by scalars, and with the norm

$$\|(A_n)\| := \sup_{n \in \mathbb{N}} \|A_n\|,$$

becomes a C^* -algebra.

Definition 3.1. The sequence $(A_n) \in \mathcal{S}$ is called stable if there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators $A_n P_n : S_n^d(\Gamma) \mapsto S_n^d(\Gamma)$ are invertible and the norms $\|(A_n P_n)^{-1} P_n\|_{n \geq n_0}$ are uniformly bounded.

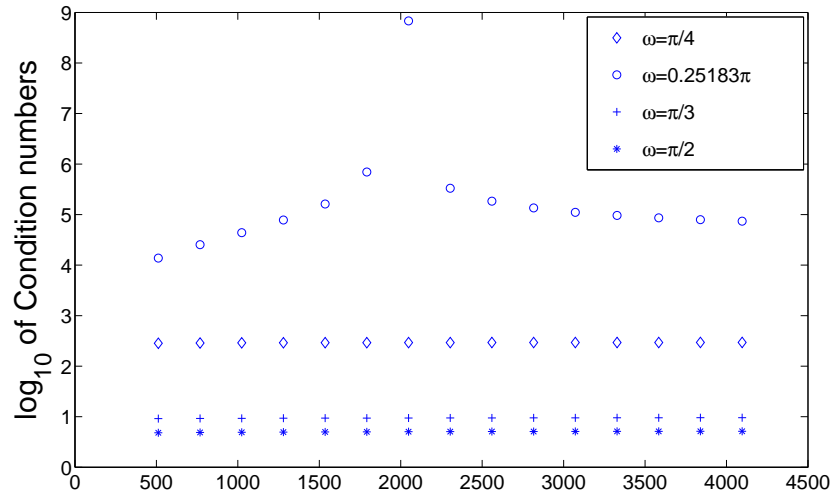


Figure 3: The Nyström method:condition numbers versus the number of discretisation points for contour $\mathcal{L}_1(\omega)$ with $\omega = 0.25183\pi, \pi/3, \pi/4, \pi/2$.

Remark 3.1. The stability of approximation methods is directly connected to the condition numbers of the corresponding matrix sequences. Fig. 3 shows the condition numbers for the matrices representing the Nyström method for the double layer potential operator on the contour $\mathcal{L}_1(\omega)$ with $\omega = 0.25183\pi, \pi/3, \pi/4, \pi/2$. An abnormality of the graph in the case $\omega = 0.25183\pi$ is caused by the proximity of this point to a critical angle. Nevertheless, since 0.25183π is only an approximation of that angle, the sequence of condition numbers stabilises as n grows.

It is well known that the stability of the approximation method (A_n) is equivalent to the invertibility of the coset $(A_n) + \mathcal{G}$ in the quotient algebra \mathcal{T}/\mathcal{G} — cf. Refs. [14, 19, 24, 25], where \mathcal{G} is the set of all bounded sequences uniformly convergent to zero,

$$\mathcal{G} = \{(G_n) \in \mathcal{T} : \lim_{n \rightarrow \infty} \|G_n\| = 0\}.$$

In many cases, quotient algebra \mathcal{T}/\mathcal{G} is too large to treat the invertibility problem efficiently. Therefore, one often considers a smaller algebra $\mathcal{A} \subset \mathcal{T}$ of sequences containing the approximation method studied, simultaneously expanding \mathcal{G} to a larger ideal \mathcal{I} in such a way that the initial problem is equivalent to the invertibility of the corresponding coset in the quotient algebra \mathcal{A}/\mathcal{I} . More precisely, let $\mathcal{B}(L^2(\Gamma))$ be the set of all linear bounded operators acting in the space $L^2(\Gamma)$. By $\mathcal{A} \subset \mathcal{T}$ we denote the close subalgebra of \mathcal{T} containing all sequences (A_n) such that there is an operator $A \in \mathcal{B}(L^2(\Gamma))$ for which the strong limits

$$s\text{-}\lim_{n \rightarrow \infty} A_n P_n = A, \quad s\text{-}\lim_{n \rightarrow \infty} (A_n)^* P_n = A^*$$

exist. Moreover, if $\mathcal{K}(L^2(\Gamma))$ is the set of all compact operators on $L^2(\Gamma)$, then the family

of sequences

$$\mathcal{J} = \{(J_n) : J_n = P_n K P_n + G_n, K \in \mathcal{K}(L^2(\Gamma)), G_n \in \mathcal{G}\}.$$

is a closed two-sided ideal of \mathcal{A} .

Theorem 3.1 (cf. Refs. [14,24,25]). *A sequence $(A_n) \in \mathcal{A}$ is stable if and only if the operator A is invertible in $\mathcal{B}(L^2(\Gamma))$ and the coset $(A_n) + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{A} / \mathcal{J}$.*

We use this result to study the applicability of the Nyström method to the double layer potential equations. Note that the sequence of approximation operators $(A_n)_{n \in \mathbb{N}}$ corresponding to the Nyström method strongly converges to the operator A_Γ — cf. Ref. [24, Chapter 2] and Ref. [19, Chapter 5]. Similar statement is valid for the sequence of adjoint operators. To show the invertibility of the coset $(A_n) + \mathcal{J}$ in the quotient algebra $\mathcal{T} / \mathcal{J}$ we use a local principle. With each point $\tau \in \Gamma$ we associate a simpler sequence of approximation operators (A_n^τ) , an algebra \mathcal{A}_τ and ideal \mathcal{J}_τ such that the invertibility of the coset $(A_n) + \mathcal{J}$ in $\mathcal{T} / \mathcal{J}$ is equivalent to the invertibility of the coset $(A_n) + \mathcal{J}_\tau$ in $\mathcal{A}_\tau / \mathcal{J}_\tau$ for all $\tau \in \Gamma$. For more detail we refer the reader to Refs. [14,19,24,25]. If $\tau \notin \mathfrak{M}_\Gamma$ and $U_\tau \subset \Gamma$ is a neighbourhood of τ such that $\mathfrak{M}_\Gamma \cap U_\tau = \emptyset$ and if f_τ is a function continuous on Γ such that

$$f_\tau(t) = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{if } t \in \Gamma \setminus U_\tau, \end{cases}$$

then the operator $f_\tau V_\Gamma f_\tau \in \mathcal{K}(L^2(\Gamma))$ — cf. Corollary 4.6.3 in [25]. Therefore, the sequence (A_n^τ) is locally equivalent to the sequence generated by the projections (P_n) , so that the corresponding coset containing the sequence (A_n^τ) is invertible. Thus one only has to identify and study the cosets associated with the corners of Γ . To this end, for each corner points $\tau_j \in \Gamma$ we consider the corresponding approximation method for the operator A_{ω_j} of (3.2), approximating the integrals $\int_{\Gamma_j} x(\tau) d\tau$ similarly to (1.5) — i.e.

$$\begin{aligned} \int_{\Gamma_j} x(\tau) d\tau \approx & \sum_{l=-\infty}^{-1} \sum_{p=0}^{d-1} w_p x\left(\frac{l + \varepsilon_p}{n} e^{i(\beta_j + \omega_j)}\right) \frac{e^{i(\beta_j + \omega_j)}}{n} \\ & + \sum_{l=0}^{+\infty} \sum_{p=0}^{d-1} w_p x\left(\frac{l + \varepsilon_p}{n} e^{i\beta_j}\right) \frac{e^{i\beta_j}}{n}, \end{aligned} \tag{3.6}$$

where w_p and ε_p are as in (1.5). We also need spline spaces on the contours Γ_j and \mathbb{R}^+ . Let $S_n^{\beta_j, \omega_j}$ be the smallest subspace of $L^2(\Gamma_j)$ which contains all functions

$$\tilde{\varphi}_{kn}(t) = \begin{cases} \begin{cases} \psi_{kn}(s) & \text{if } t = se^{i\beta_j} \\ 0 & \text{if } t = se^{i(\beta_j + \omega_j)} \end{cases}, & k \geq 0, \\ \begin{cases} \psi_{k-d,n}(s) & \text{if } t = se^{i(\beta_j + \omega_j)} \\ 0 & \text{if } t = se^{i\beta_j} \end{cases}, & k < 0, \end{cases}$$

with ψ_{kn} defined by (2.3).

The spline space $S_n(\mathbb{R}^+) = S_n^d(\mathbb{R}^+)$ is constructed analogously but we let $\beta_j = 0$ and use the splines $\tilde{\varphi}_{kn}$ for $k \geq 0$ only. Moreover, let \tilde{P}_n and \hat{P}_n be the orthogonal projections from $L^2(\Gamma_j)$ onto $S_n^{\beta_j, \omega_j}$ and from $L^2(\mathbb{R}^+)$ onto $S_n(\mathbb{R}^+)$, respectively. By $\mathbf{R}_2(\Gamma_j)$ we denote the set of functions on Γ_j which are Riemann integrable on each finite part of Γ_j and such that

$$\|f\|_{\mathbf{R}} = \|f\|_{L^2(\Gamma_j)} + \left(\sum_{k=0}^{+\infty} \sup_{t \in e^{i(\beta_j + \omega_j)}[k, k+1]} |f(t)|^2 \right)^{1/2} + \left(\sum_{k=0}^{+\infty} \sup_{t \in e^{i\beta_j}[k, k+1]} |f(t)|^2 \right)^{1/2} < +\infty.$$

Considering now the integral equation

$$A_{\Gamma_j} x = f, \quad f \in \mathbf{R}_2(\Gamma_j),$$

we replace x by an element $x_n \in S_n^{\beta_j, \omega_j}$, apply quadratures (3.6) to the corresponding integrals and use the interpolation projections $\tilde{Q}_n^\varepsilon : \mathbf{R}(\Gamma_j) \mapsto S_n^{\beta_j, \omega_j}$ defined by

$$\begin{aligned} \tilde{Q}_n^\varepsilon x(\tau_{lp}) &= x(\tau_{lp}), \quad l \in \mathbb{Z}, p = 0, 1, \dots, d-1; \\ \tau_{lp} &= \begin{cases} \frac{l + \varepsilon_p}{n} e^{i(\beta_j + \omega_j)} & \text{if } l < 0, \\ \frac{l + \varepsilon_p}{n} e^{i\beta_j} & \text{if } l \geq 0. \end{cases} \end{aligned}$$

As the result, we obtain the operator equations

$$\tilde{Q}_n^\varepsilon A_{\Gamma_j}^{(\varepsilon, n)} \tilde{P}_n x_n = \tilde{Q}_n^\varepsilon f, \quad x_n \in S_n^{\beta_j, \omega_j}, \quad n \in \mathbb{N}, \tag{3.7}$$

equivalent to the following infinite systems of linear algebraic equations,

$$\begin{aligned} x_n(\tau_{kr}) &+ \frac{1}{2\pi i} \sum_{l=-\infty}^{-1} \sum_{p=0}^{d-1} w_p x_n(\tau_{lp}) \left(\frac{e^{i(\beta_j + \omega_j)}}{\tau_{lp} - \tau_{kr}} - \frac{e^{-i(\beta_j + \omega_j)}}{\overline{\tau_{lp}} - \overline{\tau_{kr}}} \right) \frac{1}{n} \\ &+ \frac{1}{2\pi i} \sum_{l=0}^{\infty} \sum_{p=0}^{d-1} w_p x_n(\tau_{lp}) \left(\frac{e^{i\beta_j}}{\tau_{lp} - \tau_{kr}} - \frac{e^{-i\beta_j}}{\overline{\tau_{lp}} - \overline{\tau_{kr}}} \right) \frac{1}{n} \\ &= f(\tau_{kr}), \quad k \in \mathbb{Z}, r = 0, 1, \dots, d-1, \end{aligned}$$

where the expressions in the round brackets are set to be 0 if $k = l$ and $r = p$ simultaneously. Note that operators (3.7) can be written in a different form by using the integral representation (3.1) of the Mellin convolution operators $\mathcal{M}(\mathbf{n}_{\omega_j})$. More precisely, let $\hat{Q}_n^\varepsilon, n \in \mathbb{N}$ be the interpolation operators defined on the positive semi-axis.

Lemma 3.1. *If \mathbf{k}_ω is the function (3.5), then the sequence $(\tilde{Q}_n^\varepsilon A_{\Gamma_j}^{(\varepsilon,n)} \tilde{P}_n)_{n \in \mathbb{N}}$ is stable if and only if so is the sequence $(\widehat{A}_{\omega_j}^{\varepsilon,n} \text{diag}(\widehat{P}_n, \widehat{P}_n))$,*

$$\widehat{A}_{\omega_j}^{\varepsilon,n} \text{diag}(\widehat{P}_n, \widehat{P}_n) = \begin{pmatrix} \widehat{P}_n & \widehat{Q}_n^\varepsilon \mathcal{M}^{(\varepsilon,n)}(\mathbf{k}_{\omega_j}) \widehat{P}_n \\ \widehat{Q}_n^{1-\varepsilon} \mathcal{M}^{(1-\varepsilon,n)}(\mathbf{k}_{\omega_j}) \widehat{P}_n & \widehat{P}_n \end{pmatrix}.$$

Proof. Let $\eta : L^2(\Gamma_j) \mapsto L^2(\mathbb{R}^+)^2$ be the isomorphism defined in Section 3. It is easily seen that

$$\eta \tilde{Q}_n^\varepsilon \eta^{-1} = \text{diag}(\widehat{Q}_n^\varepsilon, \widehat{Q}_n^{1-\varepsilon}), \quad \eta \tilde{P}_n \eta^{-1} = \text{diag}(\widehat{P}_n, \widehat{P}_n)$$

and

$$\eta A_{\Gamma_j}^{(\varepsilon,n)} \eta^{-1} = \eta \tilde{P}_n \eta^{-1} + \eta V_{\Gamma_j}^{(\varepsilon,n)} \eta^{-1} = \begin{pmatrix} \widehat{P}_n & \mathcal{M}^{(\varepsilon,n)}(\mathbf{k}_{\omega_j}) \\ \mathcal{M}^{(1-\varepsilon,n)}(\mathbf{k}_{\omega_j}) & \widehat{P}_n \end{pmatrix}.$$

The obvious identity $\eta(\tilde{Q}_n^\varepsilon A_{\Gamma_j}^{(\varepsilon,n)} \tilde{P}_n) \eta^{-1} = (\eta \tilde{Q}_n^\varepsilon \eta^{-1})(\eta A_{\Gamma_j}^{(\varepsilon,n)} \eta^{-1})(\eta \tilde{P}_n \eta^{-1})$ completes the proof. \square

Let l_2 be the space of sequences $(\xi_j)_{j=0}^\infty$ of complex numbers $\xi_j, j = 0, 1, \dots$ such that $\sum_{j=0}^\infty |\xi_j|^2 < +\infty$. We now define the operators $E_n : l_2 \mapsto S_n(\mathbb{R}^+)$ and $E_{-n} : S_n(\mathbb{R}^+) \mapsto l_2$ by

$$E_n((\xi_j)_{j=0}^\infty) = \sum_{j=0}^{+\infty} \xi_j \tilde{\varphi}_{jn}(t), \quad E_{-n} \left(\sum_{j=0}^{+\infty} \xi_j \tilde{\varphi}_{jn}(t) \right) = (\xi_j)_{j=0}^\infty.$$

The operators E_n and E_{-n} are bounded [6] and there is a constant C such that

$$\|E_n\| \|E_{-n}\| \leq C \text{ for all } n \in \mathbb{N}.$$

This allows to rewrite the stability conditions for the sequence $(\tilde{Q}_n^\varepsilon A_{\Gamma_j}^{(\varepsilon,n)} \tilde{P}_n)_{n \in \mathbb{N}}$ in a more convenient form. By \widehat{E}_n and \widehat{E}_{-n} we denote the diagonal operators

$$\widehat{E}_n := \text{diag}(E_n, E_n), \quad \widehat{E}_{-n} := \text{diag}(E_{-n}, E_{-n}).$$

Corollary 3.2. *The sequence $(\tilde{Q}_n^\varepsilon A_{\Gamma_j}^{(\varepsilon,n)} \tilde{P}_n)_{n \in \mathbb{N}}$ is stable if and only if the operator $B_{\omega_j, \varepsilon} = \widehat{E}_{-1} \widehat{A}_{\omega_j}^{\varepsilon,1} \text{diag}(\widehat{P}_1, \widehat{P}_1) \widehat{E}_1$ is invertible.*

Proof. Straightforward calculations show that the entries of the approximation operator $\widehat{E}_{-n} \widehat{A}_{\omega_j}^{\varepsilon,n} \text{diag}(\widehat{P}_n, \widehat{P}_n) \widehat{E}_n$ do not depend on n . Indeed, consider for example, the sequence $(E_{-n} \widehat{Q}_n^\varepsilon \mathcal{M}^{(\varepsilon,n)}(\mathbf{k}_{\omega_j}) \widehat{P}_n E_n)$. If $x_n \in \text{im } \widehat{P}_n$, then

$$\begin{aligned} (\mathcal{M}^{(\varepsilon,n)}(\mathbf{k}_{\omega_j}) x_n)(\sigma) &= \sum_{l=0}^{+\infty} \sum_{p=0}^{d-1} w_p \mathbf{k}_{\omega_j} \left(\frac{\sigma}{l+\varepsilon_p} \right) \frac{1}{l+\varepsilon_p} \frac{1}{n} x_n \left(\frac{l+\varepsilon_p}{n} \right) \\ &= \sum_{p=0}^{d-1} w_p \sum_{l=0}^{\infty} \mathbf{k}_{\omega_j} \left(\frac{\sigma}{l+\varepsilon_p} \right) \frac{1}{l+\varepsilon_p} x_n \left(\frac{l+\varepsilon_p}{n} \right), \end{aligned}$$

and applying the interpolation operators \tilde{Q}_n^ε , we obtain

$$\begin{aligned} (\tilde{Q}_n^\varepsilon \mathcal{M}^{(\varepsilon,n)}(\mathbf{k}_{\omega_j})x_n) \left(\frac{k + \varepsilon_r}{n} \right) &= \sum_{p=0}^{d-1} w_p \sum_{l=0}^{\infty} \mathbf{k}_{\omega_j} \left(\frac{k + \varepsilon_r}{n} \right) \frac{1}{l + \varepsilon_p} x_n \left(\frac{l + \varepsilon_p}{n} \right) \\ &= \sum_{p=0}^{d-1} w_p \sum_{l=0}^{\infty} \mathbf{k}_{\omega_j} \left(\frac{k + \varepsilon_r}{l + \varepsilon_p} \right) \frac{1}{l + \varepsilon_p} x_n \left(\frac{l + \varepsilon_p}{n} \right). \end{aligned}$$

Thus the sequence in question is constant and is stable if and only if one its member, say $E_{-1} \hat{A}_{\omega_j}^{\varepsilon,1} \text{diag}(\hat{P}_1, \hat{P}_1) E_1$, is invertible. \square

Theorem 3.2. *If the operator $A : L^2(\Gamma) \mapsto L^2(\Gamma)$ is invertible, then the Nyström method for A is stable if and only if all the operators $B_{\omega_j, \varepsilon}, j = 0, 1, \dots, q - 1$ are invertible.*

Proof. Let \mathcal{C} denote the smallest closed C^* -algebra containing the sequences $(P_n S_\Gamma P_n), (P_n M P_n)$ and $(P_n f P_n)$, where $f \in C(\Gamma)$, and let \mathcal{I} be the ideal defined in Theorem 3.1. Then $(A^{(\varepsilon,n)} P_n) \in \mathcal{C}$ and \mathcal{C}/\mathcal{I} is a C^* -subalgebra of \mathcal{A}/\mathcal{I} . Therefore, the coset $(A^{(\varepsilon,n)} P_n) + \mathcal{I}$ is invertible in \mathcal{A}/\mathcal{I} if and only if it is invertible in \mathcal{C}/\mathcal{I} . However, the algebra \mathcal{C}/\mathcal{I} has a nice centre and the invertibility of the coset $(A^{(\varepsilon,n)} P_n) + \mathcal{I}$ in \mathcal{C}/\mathcal{I} can be established by the Allan's local principle — cf. Ref. [1] and [14, Theorem 1.9.5]. Following the proof of Theorem 3.4 in Ref. [11] we show that for any $\tau = \tau_j \in \mathfrak{M}_\Gamma$ this coset is invertible if and only if the corresponding operator $B_{\omega_j, \varepsilon}$ is invertible. On the other hand, it was already mentioned that for $\tau \notin \mathfrak{M}_\Gamma$, the corresponding coset is always invertible, and application of Theorem 3.1 completes the proof. \square

4. The Invertibility of Local Operators

Theorem 3.2 connects the stability of the Nyström method with the invertibility of the operators $B_{\omega_j, \varepsilon}, j = 0, 1, \dots, q - 1$. A more detailed study of these operators shows that they belong to a Toeplitz algebra with no efficient invertibility conditions available. Similar problems arise in approximation methods for the Sherman–Lauricella and Muskhelishvili equations. A numerical approach to the invertibility of related operators has been proposed in [9, 10], with the invertibility connected to the stability of the approximation method for operators on model curves with all corners of the same magnitude ω . Looking at the condition numbers of the corresponding method, one can decide which opening angles ω are "critical" — i.e. which angles cause instability.

Theorem 4.1. *Let $\mathcal{L} = \mathcal{L}(\omega)$ be any curve $\mathcal{L}_1(\omega)$ or $\mathcal{L}_2(\omega), \omega \in (0, 2\pi)$ defined in Section 3 and $A_{\mathcal{L}(\omega)}$ be the operator (2.6) considered on the contour $\mathcal{L}(\omega)$. The operator $B_{\omega, \varepsilon}$ is invertible if and only if the Nyström method $(Q_n^\varepsilon A_{\mathcal{L}(\omega)}^{(\varepsilon,n)} P_n)$ is stable.*

This statement is a reformulation of Theorem 3.2 with respect to the chosen operator and special contours. Of course, we take into account the invertibility of $A_{\mathcal{L}(\omega)} :$

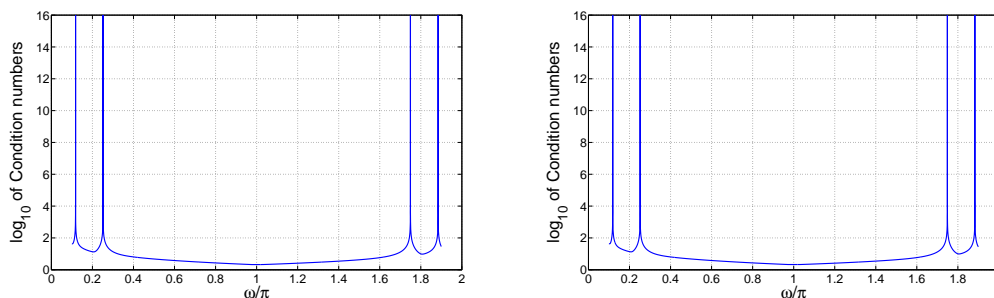


Figure 4: Condition numbers vs. opening angles in case $n = 128$, $d = 16$. Left: one corner contour. Right: two corner contour.

$L^2(\mathcal{L}(\omega)) \rightarrow L^2(\mathcal{L}(\omega))$ — cf. Ref. [27]. Let us emphasise that \mathcal{L}_1 and \mathcal{L}_2 have distinct shapes and a different number of corners. However, if the magnitude of the corners is the same, the corresponding numerical experiments shall produce the same results. In what follows, we vary the parameter ω in the interval $(0.1\pi, 1.9\pi)$ and obtain two families of contours with one and two corners. In order to find critical angles, we divide the interval $[0.1\pi, 1.9\pi]$ by the points $\omega_k = \pi(0.1 + 0.001k)$, and for each point ω_k we compute the condition number of the Nyström method for $n = 128$ and the Gauss–Legendre quadrature with $d = 16$ — viz. there are 128 panels on the curves and each panel has 16 mesh points. Computing the corresponding condition numbers at the points ω_k , we first detect "suspicious" points, in the neighbourhoods of which condition numbers grow rapidly. Thereafter, the initial mesh is refined in neighborhoods of such points and condition numbers are recalculated. The procedure is repeated until the related condition numbers reach 10^{16} . The outcome of these computations is presented in Fig. 4. Thus using both contours we found that the corresponding graphs have four peaks in the interval $(0.1, 1.9)$ — viz.

The case of one corner geometry — curve \mathcal{L}_1 .

$$0.11781222\pi, \quad 0.25164815\pi, \quad 1.74949877\pi, \quad 1.88430019\pi.$$

The case of two corner geometry — curve \mathcal{L}_2 .

$$0.11780844\pi, \quad 0.25164706\pi, \quad 1.74840993\pi, \quad 1.88390254\pi.$$

Let us emphasise that for both curves \mathcal{L}_1 and \mathcal{L}_2 , the results coincide up to three significant numbers. The peaks of the graphs are connected with four possible critical angles in the interval $(0.1\pi, 1.9\pi)$.

The numerical experiments have been conducted in MATLAB environment (version 7.9.0) and executed on an Acer Veriton M680 workstation equipped with a Intel Core i7 vPro 870 processor and 8GB of RAM.

5. Conclusion

We established necessary and sufficient conditions for stability of the Nyström method for double layer potential equations on simple piecewise smooth contours. In particular,

we found four open angles in the interval $[0.1\pi, 1.9\pi]$ which cause the instability. Thus if Γ has at least one corner of such magnitude then, regardless of the shape of the curve, the Nyström method is not stable even if the operator A of (1.2) is invertible. In numerical experiments we used curves with different number of corners of the same magnitude and all of them produced the same critical angles, consistent with theoretical analysis.

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