

# Variation of constant formulas for fractional difference equations

PHAM THE ANH, ARTUR BABIARZ, ADAM CZORNIK, MICHAŁ NIEZABITOWSKI and STEFAN SIEGMUND

In this paper, we establish variation of constant formulas for both Caputo and Riemann-Liouville fractional difference equations. The main technique is the  $\mathcal{L}$ -transform. As an application, we prove a lower bound on the separation between two different solutions of a class of nonlinear scalar fractional difference equations.

**Key words:** fractional difference equation, variation of constant, separation of solutions

## 1. Introduction

Recently, the theory of fractional calculus became very popular and its development is still very fast (see e.g. [22, 25] and the references therein). In the literature, one can find results on theoretical problems as well as practical applications. In the classical framework of differential or difference equations a powerful tool for analyzing properties of dynamical systems is the so-called variation of constant formula which expresses the solution of a nonlinear equation by the solution of a linear approximation and an implicit term involving the nonlinearity

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P.T. Anh is with Department of Mathematics, Le Quy Don Technical University, 236 Hoang Quoc Viet, Ha noi, Vietnam. e-mail: anhpt@lqdtu.edu.vn

A. Babiarz and A. Czornik are with Silesian University of Technology, Faculty of Automatic Control, Electronics and Computer Science, Akademicka 16, 44-100 Gliwice, Poland. e-mails: artur.babiarz@polsl.pl, adam.czornik@polsl.pl

M. Niezabitowski (corresponding author) is with Silesian University of Technology, Faculty of Automatic Control, Electronics and Computer Science, Akademicka 16, 44-100 Gliwice, Poland. e-mail: michal.niezabitowski@polsl.pl and with University of Silesia, Faculty of Mathematics, Physics and Chemistry, Institute of Mathematics, Bankowa 14, 40-007 Katowice, Poland. e-mail: mniezabitowski@us.edu.pl

S. Siegmund is with Technische Universität Dresden, Faculty of Mathematics, Center for Dynamics, Zellescher Weg 12-14, 01069 Dresden, Germany. e-mail: stefan.siegmund@tu-dresden.de

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(see [10]). The Laplace transform method has been utilized to derive a variation of constant formula for linear fractional differential equations in [14].

This paper is devoted to study linear discrete-time fractional systems. In the discrete-time framework four main types of fractional differences are considered: forward/backward Caputo and forward/backward Riemann-Liouville operators (see e.g. [1, 3, 5]). For linear discrete time-invariant fractional systems the stability problem is studied in [4, 15]. In this paper we use the  $\mathcal{Z}$ -transform to establish variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations in Section 2. In Section 3 we use the variation of constant formula to show a separation result for solutions of scalar fractional difference equations.

A reader who is familiar with fractional difference equations may very well skip the next paragraph, in which we recall notation to keep the paper self-contained. Denote by  $\mathbb{R}$  the set of real numbers, by  $\mathbb{Z}$  the set of integers, by  $\mathbb{N} := \mathbb{Z}_{\geq 0}$  the set  $\{0, 1, 2, \dots\}$  of natural numbers including 0, and by  $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$  the set of non-positive integers. For  $a \in \mathbb{R}$  we denote by  $\mathbb{N}_a := a + \mathbb{N}$  the set  $\{a, a + 1, \dots\}$ . By  $\Gamma: \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \rightarrow \mathbb{R}$  we denote the Euler gamma function defined by

$$\Gamma(\alpha) := \lim_{n \rightarrow \infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \quad (\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}). \quad (1)$$

Note that (see [12])

$$\Gamma(\alpha) = \begin{cases} \int_0^\infty x^{\alpha-1} e^{-x} dx & \text{if } \alpha > 0, \\ \frac{\Gamma(\alpha+1)}{\alpha} & \text{if } \alpha < 0 \text{ and } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}. \end{cases} \quad (2)$$

For  $s \in \mathbb{R}$  with  $s+1, s+1-\alpha \notin \mathbb{Z}_{\leq 0}$ , the falling factorial power  $(s)^{(\alpha)}$  is defined by

$$(s)^{(\alpha)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)}. \quad (3)$$

By  $\lceil x \rceil := \min\{k \in \mathbb{Z}: k \geq x\}$  we denote the least integer greater or equal to  $x$  and by  $\lfloor x \rfloor := \max\{k \in \mathbb{Z}: k \leq x\}$  the greatest integer less or equal to  $x$ . Binomial coefficients  $\binom{r}{m}$  can be defined for any  $r, m \in \mathbb{C}$  as described in [12, Section 5.5, formula (5.90)]. For  $r \in \mathbb{R}$  and  $m \in \mathbb{Z}$  the binomial coefficient satisfies [12, Section 5.1, formula (5.1)]

$$\binom{r}{m} = \begin{cases} \frac{r(r-1)\cdots(r-m+1)}{m!} & \text{if } m \in \mathbb{Z}_{\geq 1}, \\ 1 & \text{if } m = 0, \\ 0 & \text{if } m \in \mathbb{Z}_{\leq -1}. \end{cases}$$

For  $a \in \mathbb{R}$ ,  $\nu \in \mathbb{R}_{\geq 0}$  and a function  $x: \mathbb{N}_a \rightarrow \mathbb{R}^d$ , the  $\nu$ -th delta fractional sum  $\Delta_a^{-\nu}x: \mathbb{N}_{a+\nu} \rightarrow \mathbb{R}^d$  of  $x$  is defined as

$$(\Delta_a^{-\nu}x)(t) := \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-k-1)^{(\nu-1)}x(k) \quad (t \in \mathbb{N}_{a+\nu}).$$

We write  $\Delta^{-\nu}x$  instead of  $\Delta_0^{-\nu}x$ .

The Caputo forward difference  ${}_c\Delta_a^\alpha x: \mathbb{N}_{a+1-\alpha} \rightarrow \mathbb{R}^d$  of  $x$  of order  $\alpha$  is defined as the composition  ${}_c\Delta_a^\alpha := \Delta_a^{-(1-\alpha)} \circ \Delta$  of the  $(1-\alpha)$ -th delta fractional sum with the classical difference operator  $t \mapsto \Delta x(t) := x(t+1) - x(t)$ , i.e.

$$({}_c\Delta_a^\alpha x)(t) := (\Delta_a^{-(1-\alpha)} \Delta x)(t) \quad (t \in \mathbb{N}_{a+1-\alpha}).$$

The Riemann-Liouville forward difference  ${}_{R-L}\Delta_a^\alpha x: \mathbb{N}_{a+1-\alpha} \rightarrow \mathbb{R}^d$  of  $x$  of order  $\alpha$  is defined as  ${}_{R-L}\Delta_a^\alpha := \Delta \circ \Delta_a^{-(1-\alpha)}$ , i.e.

$$({}_{R-L}\Delta_a^\alpha x)(t) := (\Delta \Delta_a^{-(1-\alpha)} x)(t) \quad (t \in \mathbb{N}_{a+1-\alpha}).$$

Similarly, as for the fractional sum, if  $a = 0$  we simply write  ${}_c\Delta^\alpha x$  and  ${}_{R-L}\Delta^\alpha x$ .

Let  $\alpha \in (0, 1)$ . Consider a linear fractional difference equation of the form

$$(\Delta^\alpha x)(n+1-\alpha) = Ax(n) + f(n) \quad (n \in \mathbb{N}), \tag{4}$$

where  $x: \mathbb{N} \rightarrow \mathbb{R}^d$ ,  $\Delta^\alpha$  is either the Caputo  ${}_c\Delta^\alpha$  or Riemann-Liouville  ${}_{R-L}\Delta^\alpha$  forward difference operator of order  $\alpha$ ,  $f: \mathbb{N} \rightarrow \mathbb{R}^d$  and  $A \in \mathbb{R}^{d \times d}$ . For an initial value  $x_0 \in \mathbb{R}^d$ , (4) has a unique solution  $x: \mathbb{N} \rightarrow \mathbb{R}^d$  which satisfies the initial condition  $x(0) = x_0$ . We denote  $x$  by  $\varphi_c(\cdot, x_0)$  or  $\varphi_{R-L}(\cdot, x_0)$ , respectively. If  $f \equiv 0$ , (4) is called homogeneous, and its solutions can be expressed with discrete-time Mittag-Leffler functions. In the literature, different types of discrete-time Mittag-Leffler functions are defined [17, 21, 24]. In [17], for  $\beta \in \mathbb{C}$ , two functions  $E_{(\alpha, \beta)}$  and  $\mathcal{E}_{(\alpha, \beta)}$  are defined by

$$E_{(\alpha, \beta)}(A, n) = \sum_{k=0}^{\infty} A^k \binom{n-k+k\alpha+\beta-1}{n-k} \quad (n \in \mathbb{Z}), \tag{5}$$

and

$$\mathcal{E}_{(\alpha, \beta)}(A, z) = \sum_{k=0}^{\infty} A^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k + \beta)} \quad (z \in \mathbb{C}).$$

These are two different functions, however,

$$E_{(\alpha, 1)}(A, n) = \mathcal{E}_{(\alpha, 1)}(A, n+1-\alpha) \quad (n \in \mathbb{N}),$$

since for  $\beta = 1$ , by setting  $z = n - 1 + \alpha$ ,

$$\begin{aligned} & \frac{(z + (k-1)(\alpha-1))^{(k\alpha)} (z + k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k + \beta)} \\ &= \frac{(z + (k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k + 1)} \\ &= \frac{\Gamma(z + k\alpha - k - \alpha + 2)}{\Gamma(z - k - \alpha + 2)\Gamma(\alpha k + 1)} \\ &= \frac{\Gamma(n + k\alpha - k + 1)}{\Gamma(n - k + 1)\Gamma(\alpha k + 1)}, \end{aligned}$$

and

$$\binom{n - k + k\alpha + \beta - 1}{n - k} = \frac{\Gamma(n - k + k\alpha + 1)}{\Gamma(n - k + 1)\Gamma(k\alpha + \beta)} = \frac{\Gamma(n - k + k\alpha + 1)}{\Gamma(n - k + 1)\Gamma(k\alpha + 1)}.$$

Similarly,  $E_{(\alpha,\alpha)}(A, n) = \mathcal{E}_{(\alpha,\alpha)}(A, n + 1 - \alpha)$  for  $n \in \mathbb{N}$ , since for  $\beta = \alpha$ , by setting  $z = n - 1 + \alpha$ ,

$$\begin{aligned} & \frac{(z + (k-1)(\alpha-1))^{(k\alpha)} (z + k(\alpha-1))^{(\alpha-1)}}{\Gamma(\alpha k + \alpha)} \\ &= \frac{\Gamma(z + k\alpha - k - \alpha + 2)}{\Gamma(z - k - \alpha + 2)} \frac{\Gamma(z + k\alpha - k + 1)}{\Gamma(z + k\alpha - k - \alpha + 2)} \frac{1}{\Gamma(\alpha k + \alpha)} \\ &= \frac{\Gamma(n - k + k\alpha + \alpha)}{\Gamma(n - k + 1)\Gamma(\alpha k + \alpha)} \\ &= \binom{n - k + k\alpha + \alpha - 1}{n - k}. \end{aligned}$$

The next remark provides formulas for solutions of homogeneous Caputo and Riemann-Liouville equations in terms of discrete-time Mittag-Leffler functions.

**Remark 1** (a) *The solution of the linear homogeneous Caputo difference equation*

$$({}_c\Delta^\alpha x)(n + 1 - \alpha) = Ax(n), \quad x(0) = x_0 \in \mathbb{R}^d,$$

is given by

$$\varphi_{\mathbb{C}}(n, x_0) = E_{(\alpha)}(A, n)x_0 \quad (n \in \mathbb{N}), \quad (6)$$

with the discrete-time Mittag-Leffler function

$$E_{(\alpha)}(A, n) := E_{(\alpha,1)}(A, n) = \sum_{k=0}^{\infty} A^k \binom{n - k + k\alpha}{n - k} \quad (n \in \mathbb{N}). \quad (7)$$

See e.g. [2].

(b) *The solution of the linear homogeneous Riemann-Liouville difference equation*

$$({}_{\mathbb{R}\text{-L}}\Delta^\alpha x)(n + 1 - \alpha) = Ax(n), \quad x(0) = x_0 \in \mathbb{R}^d,$$

is given by

$$\varphi_{\mathbb{R}\text{-L}}(n, x_0) = E_{(\alpha, \alpha)}(A, n)x_0 \quad (n \in \mathbb{N}), \tag{8}$$

with the discrete-time Mittag-Leffler function

$$E_{(\alpha, \alpha)}(A, n) = \sum_{k=0}^{\infty} A^k \binom{n - k + (k + 1)\alpha - 1}{n - k} \quad (n \in \mathbb{N}). \tag{9}$$

Instead of giving a direct proof, we refer to our main Theorem 1 which implies (6) and (8) for the special case  $f \equiv 0$ .

Note that the sums in the right-hand sides of (5), (7) and (9) for  $n \in \mathbb{Z}$  are taken over only finitely many summands, since  $\binom{r}{m} = 0$  if  $r \in \mathbb{R}$  and  $m \in \mathbb{Z}_{\leq -1}$ , therefore

$$\varphi_{\mathbb{C}}(n, x_0) = \sum_{k=0}^n A^k \binom{n - k + k\alpha}{n - k} x_0 = \sum_{k=0}^n A^k (-1)^{n-k} \binom{-k\alpha - 1}{n - k} x_0$$

and

$$\varphi_{\mathbb{R}\text{-L}}(n, x_0) = \sum_{k=0}^n A^k \binom{n - k + (k + 1)\alpha - 1}{n - k} x_0 = \sum_{k=0}^n A^k (-1)^{n-k} \binom{-k\alpha - \alpha}{n - k} x_0.$$

In the last step we used the following identity for binomial coefficients [12, p. 174]

$$\binom{r}{k} = (-1)^k \binom{k - r - 1}{k} \quad (r \in \mathbb{R}, k \in \mathbb{Z}). \tag{10}$$

### 2. Variation of constant formula

The next theorem presents variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations.

**Theorem 1** (a) *The solution of the linear Caputo difference equation*

$$({}_{\mathbb{C}}\Delta^\alpha x)(n + 1 - \alpha) = Ax(n) + f(n) \quad (n \in \mathbb{N}),$$

with initial condition  $x(0) = x_0 \in \mathbb{R}^d$ , is given by

$$\varphi_{\mathbb{C}}(n, x_0) = E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha, \alpha)}(A, n-k-1)f(k) \quad (n \in \mathbb{N}). \quad (11)$$

(b) The solution of the linear Riemann-Liouville difference equation

$$({}_{\text{R-L}}\Delta^\alpha x)(n+1-\alpha) = Ax(n) + f(n) \quad (n \in \mathbb{N}),$$

with initial condition  $x(0) = x_0 \in \mathbb{R}^d$ , is given by

$$\varphi_{\text{R-L}}(n, x_0) = E_{(\alpha, \alpha)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha, \alpha)}(A, n-k-1)f(k) \quad (n \in \mathbb{N}). \quad (12)$$

In order to prepare the proof of Theorem 1, we summarize some results about the  $\mathcal{Z}$ -transform of a sequence  $x: \mathbb{N} \rightarrow \mathbb{R}$ , which is defined by

$$\mathcal{Z}[x](z) = \sum_{i=0}^{\infty} x(i)z^{-i} \quad (z \in \mathbb{C}, |z| > R),$$

for  $R = \limsup_{i \rightarrow \infty} |x(i)|^{1/i}$ , see e.g. [10, Chapter 6] and [13]. The  $\mathcal{Z}$ -transform of  $\mathbb{R}^d$  or  $\mathbb{R}^{d \times d}$  valued sequences is defined component-wise.

The next lemma is devoted to the  $\mathcal{Z}$ -transform of discrete-time Mittag-Leffler functions and fractional differences.

**Lemma 6** Let  $A \in \mathbb{R}^{d \times d}$ ,  $x: \mathbb{N} \rightarrow \mathbb{R}$ . Then

$$(i) \quad \mathcal{Z}[E_{(\alpha, \beta)}(A, \cdot)](z) = \left(\frac{z}{z-1}\right)^\beta \left(I - \frac{1}{z} \left(\frac{z}{z-1}\right)^\alpha A\right)^{-1},$$

$$(ii) \quad \mathcal{Z}[E_{(\alpha, \beta)}(A, \cdot - 1)](z) = \frac{1}{z} \left(\frac{z}{z-1}\right)^\beta \left(I - \frac{1}{z} \left(\frac{z}{z-1}\right)^\alpha A\right)^{-1},$$

$$(iii) \quad \mathcal{Z}[({}_c\Delta^\alpha x)(\cdot + 1 - \alpha)] = z \left(\frac{z}{z-1}\right)^{-\alpha} \left[\mathcal{Z}[x](z) - \frac{z}{z-1}x(0)\right],$$

$$(iv) \quad \mathcal{Z}[({}_{\text{R-L}}\Delta^\alpha x)(\cdot + 1 - \alpha)] = z \left(\frac{z}{z-1}\right)^{-\alpha} \mathcal{Z}[x](z) - zx(0).$$

**Proof.** (i) The proof is similar to [20, Proposition 2]. By the definition of the  $\mathcal{L}$ -transform, we have

$$\begin{aligned} \mathcal{L}[E_{(\alpha,\beta)}(A, \cdot)](z) &= \sum_{n=0}^{\infty} E_{(\alpha,\beta)}(A, n) \frac{1}{z^n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A^k (-1)^{n-k} \binom{-k\alpha - \beta}{n-k} \frac{1}{z^n} \\ &= \sum_{k=0}^{\infty} A^k \sum_{n=0}^{\infty} (-1)^{n-k} \binom{-k\alpha - \beta}{n-k} \frac{1}{z^n}. \end{aligned}$$

With  $s = n - k$ , we get

$$\begin{aligned} \mathcal{L}[E_{(\alpha,\beta)}(A, \cdot)](z) &= \sum_{k=0}^{\infty} A^k \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - \beta}{s} \frac{1}{z^{s+k}} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{z}A\right)^k \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - \beta}{s} \frac{1}{z^s} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{z}A\right)^k \left(1 - \frac{1}{z}\right)^{-k\alpha - \beta} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{z}A\right)^k \left(\frac{z}{z-1}\right)^{k\alpha + \beta}. \end{aligned}$$

Hence, we obtain

$$\mathcal{L}[E_{(\alpha,\beta)}(A, \cdot)](z) = \left(\frac{z}{z-1}\right)^{\beta} \left(I - \frac{1}{z} \left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}.$$

(ii) By the definition of the  $\mathcal{L}$ -transform, we have

$$\begin{aligned} \mathcal{L}[E_{(\alpha,\beta)}(A, \cdot - 1)](z) &= \sum_{n=0}^{\infty} E_{(\alpha,\beta)}(A, n-1) \frac{1}{z^n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A^k (-1)^{n-1-k} \binom{-k\alpha - \beta}{n-1-k} \frac{1}{z^n} \\ &= \sum_{k=0}^{\infty} A^k \sum_{n=0}^{\infty} (-1)^{n-1-k} \binom{-k\alpha - \beta}{n-1-k} \frac{1}{z^n}. \end{aligned}$$

With  $s = n - 1 - k$ , we get

$$\begin{aligned} \mathcal{L}[E_{(\alpha,\beta)}(A, \cdot - 1)](z) &= \sum_{k=0}^{\infty} A^k \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - \beta}{s} \frac{1}{z^{s+k+1}} \\ &= \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - \beta}{s} \frac{1}{z^s} \\ &= \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k \left(1 - \frac{1}{z}\right)^{-k\alpha - \beta} \\ &= \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k \left(\frac{z}{z-1}\right)^{k\alpha + \beta}. \end{aligned}$$

Hence, we obtain

$$\mathcal{L}[E_{(\alpha,\beta)}(A, \cdot - 1)](z) = \frac{1}{z} \left(\frac{z}{z-1}\right)^{\beta} \left(I - \frac{1}{z} \left(\frac{z}{z-1}\right)^{\alpha} A\right)^{-1}.$$

(iii) This is [18, Corollary 9].

(iv) This is [19, Proposition 8]. □

**Proof.** [Proof of Theorem 1](a) Applying the  $\mathcal{L}$ -transform to equation (4) with the Caputo forward difference operator, we get

$$\begin{aligned} z \left(\frac{z}{z-1}\right)^{-\alpha} \left[ \mathcal{L}[\mathfrak{C}(\cdot, x_0)](z) - \frac{z}{z-1} x_0 \right] \\ = A \mathcal{L}[\mathfrak{C}(\cdot, x_0)](z) + \mathcal{L}[f](z). \end{aligned}$$

Using Lemma 6(i), we obtain

$$\begin{aligned} \mathcal{L}[\mathfrak{C}(\cdot, x_0)](z) &= \mathcal{L}[E_{(\alpha)}(A, \cdot)(z)x_0] \\ &\quad + \left( z \left(\frac{z}{z-1}\right)^{-\alpha} I - A \right)^{-1} \mathcal{L}[f](z). \end{aligned}$$

For notational clarity, we write  $\mathcal{L}^{-1}[z \mapsto w(z)] := \mathcal{L}^{-1}[w]$  for applying the inverse of the  $\mathcal{L}$ -transform to a function  $w(\cdot)$ , and get

$$\begin{aligned} \mathfrak{C}(n, x_0) &= E_{(\alpha)}(A, n)x_0 \\ &\quad + \mathcal{L}^{-1} \left[ z \mapsto \left( z \left(\frac{z}{z-1}\right)^{-\alpha} I - A \right)^{-1} \mathcal{L}[f](z) \right] (n) \quad (n \in \mathbb{N}). \end{aligned}$$



Using

$$\begin{aligned} & \mathcal{L}^{-1} \left[ z \mapsto \left( z \left( \frac{z}{z-1} \right)^{-\alpha} I - A \right)^{-1} \right] (n) \\ &= \mathcal{L}^{-1} \left[ z \mapsto \frac{1}{z} \left( \frac{z}{z-1} \right)^\alpha \left( I - \frac{1}{z} \left( \frac{z}{z-1} \right)^\alpha A \right)^{-1} \right] (n) \quad (n \in \mathbb{N}), \end{aligned}$$

and the abbreviation  $g(\cdot) := E_{(\alpha,\alpha)}(A, \cdot - 1)$ , we have from Lemma 6(ii),

$$\mathcal{L}[g](z) = \mathcal{L}[E_{(\alpha,\alpha)}(A, \cdot - 1)](z) = \frac{1}{z} \left( \frac{z}{z-1} \right)^\alpha \left( I - \frac{1}{z} \left( \frac{z}{z-1} \right)^\alpha A \right)^{-1}.$$

Hence, we get

$$\begin{aligned} \varphi_{\mathbb{C}}(n, x_0) &= E_{(\alpha)}(A, n)x_0 + \mathcal{L}^{-1} [z \mapsto \mathcal{L}[g](z)\mathcal{L}[f](z)](n) \\ &= E_{(\alpha)}(A, n)x_0 + (g * f)(n) \\ &= E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^n g(n-k)f(k) \\ &= E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^n E_{(\alpha,\alpha)}(A, n-k-1)f(k) \quad (n \in \mathbb{N}). \end{aligned}$$

By definition of the discrete-time Mittag-Leffler function and since  $\binom{r}{m} = 0$  if  $r \in \mathbb{R}$  and  $m \in \mathbb{Z}_{\leq -1}$ , we have  $E_{(\alpha,\alpha)}(A, -1) = 0$ , and therefore

$$\varphi_{\mathbb{C}}(n, x_0) = E_{(\alpha)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(A, n-k-1)f(k) \quad (n \in \mathbb{N}).$$

(b) Applying the  $\mathcal{L}$ -transform to equation (4) with the Riemann-Liouville forward difference operator, we get

$$\begin{aligned} & z \left( \frac{z}{z-1} \right)^{-\alpha} \mathcal{L}[\varphi_{\text{R-L}}(\cdot, x_0)](z) - zx_0 \\ &= A \mathcal{L}[\varphi_{\text{R-L}}(\cdot, x_0)](z) + \mathcal{L}[f](z). \end{aligned}$$

Using Lemma 6(i), we obtain

$$\begin{aligned} \mathcal{L}[\varphi_{\text{R-L}}(\cdot, x_0)](z) &= \mathcal{L}[E_{(\alpha, \alpha)}(A, \cdot)(z)x_0] \\ &\quad + \left( z \left( \frac{z}{z-1} \right)^{-\alpha} I - A \right)^{-1} \mathcal{L}[f](z). \end{aligned}$$

Applying the inverse of the  $\mathcal{L}$ -transform yields

$$\begin{aligned} \varphi_{\text{R-L}}(n, x_0) &= E_{(\alpha, \alpha)}(A, n)x_0 \\ &\quad + \mathcal{L}^{-1} \left[ z \mapsto \left( z \left( \frac{z}{z-1} \right)^{-\alpha} I - A \right)^{-1} \mathcal{L}[f](z) \right] (n) \quad (n \in \mathbb{N}). \end{aligned}$$

Using

$$\begin{aligned} &\mathcal{L}^{-1} \left[ z \mapsto \left( z \left( \frac{z}{z-1} \right)^{-\alpha} I - A \right)^{-1} \right] \\ &= \mathcal{L}^{-1} \left[ z \mapsto \frac{1}{z} \left( \frac{z}{z-1} \right)^{\alpha} \left( I - \frac{1}{z} \left( \frac{z}{z-1} \right)^{\alpha} A \right)^{-1} \right] \end{aligned}$$

and the abbreviation  $g(\cdot) := E_{(\alpha, \alpha)}(A, \cdot - 1)$ , we have from Lemma 6(ii),

$$\mathcal{L}[g](z) = \mathcal{L}[E_{(\alpha, \alpha)}(A, \cdot - 1)](z) = \frac{1}{z} \left( \frac{z}{z-1} \right)^{\alpha} \left( I - \frac{1}{z} \left( \frac{z}{z-1} \right)^{\alpha} A \right)^{-1}.$$

Hence, we get

$$\begin{aligned} \varphi_{\text{R-L}}(n, x_0) &= E_{(\alpha, \alpha)}(A, n)x_0 + \mathcal{L}^{-1} [z \mapsto \mathcal{L}[g](z) \mathcal{L}[f](z)](n) \\ &= E_{(\alpha, \alpha)}(A, n)x_0 + (g * f)(n) \\ &= E_{(\alpha, \alpha)}(A, n)x_0 + \sum_{k=0}^n g(n-k)f(k) \\ &= E_{(\alpha, \alpha)}(A, n)x_0 + \sum_{k=0}^n E_{(\alpha, \alpha)}(A, n-k-1)f(k) \quad (n \in \mathbb{N}). \end{aligned}$$

By definition of the discrete-time Mittag-Leffler function,  $E_{(\alpha, \alpha)}(A, -1) = 0$ , and therefore

$$\varphi_{\text{R-L}}(n, x_0) = E_{(\alpha, \alpha)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{\alpha, \alpha}(A, n-k-1)f(k) \quad (n \in \mathbb{N}). \quad \square$$

Theorem 1 can be applied to a nonlinear equation yielding an implicit solution representation by the variation of constant formula. Let  $x: \mathbb{N} \rightarrow \mathbb{R}^d$  be a solution of the nonlinear fractional difference equation

$$(\Delta^\alpha x)(n + 1 - \alpha) = Ax(n) + g(x(n)) \quad (n \in \mathbb{N}),$$

where  $\Delta^\alpha$  is either the Caputo  ${}_c\Delta^\alpha$  or Riemann-Liouville  ${}_{R-L}\Delta^\alpha$  forward difference operator of order  $\alpha$ ,  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $A \in \mathbb{R}^{d \times d}$ . Then  $x$  is also a solution of the (nonautonomous) linear fractional difference equation (4) with

$$f: \mathbb{N} \rightarrow \mathbb{R}^d, \quad n \mapsto g(x(n)).$$

By Theorem 1,  $x$  satisfies the implicit equation

$$x(n) = E_{(\alpha, \beta)}(A, n)x_0 + \sum_{k=0}^{n-1} E_{(\alpha, \alpha)}(A, n - k - 1)g(x(k)) \quad (n \in \mathbb{N}) \quad (13)$$

with  $\beta = 1$  or  $\beta = \alpha$ , respectively.

### 3. Scalar solution separation

Consider scalar nonlinear fractional difference equations of the form

$$(\Delta^\alpha x)(n + 1 - \alpha) = \lambda x(n) + f(x(n)) \quad (n \in \mathbb{N}), \quad (14)$$

where  $x: \mathbb{N} \rightarrow \mathbb{R}$ ,  $\Delta^\alpha$  is either the Caputo  ${}_c\Delta^\alpha$  or Riemann-Liouville  ${}_{R-L}\Delta^\alpha$  forward difference operator of a real order  $\alpha \in (0, 1)$ ,  $\lambda > 0$ , and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous, i.e. there is a constant  $L > 0$  such that

$$|f(x) - f(y)| \leq L|x - y| \quad (x, y \in \mathbb{R}). \quad (15)$$

Solutions of initial value problems (14),  $x(0) \in \mathbb{R}$ , exist on  $\mathbb{N}$  (see e.g. [26, Section 3]).

The next theorem presents a lower bound on the separation between two solutions.

**Theorem 2** Consider equation (14) and assume that  $f$  satisfies (15) with  $L \in [0, \lambda)$ .

(a) Caputo difference equations: solutions of

$$({}_c\Delta^\alpha x)(n + 1 - \alpha) = \lambda x(n) + f(x(n)) \quad (16)$$

satisfy the estimate

$$|\varphi_{\mathbb{C}}(n, x) - \varphi_{\mathbb{C}}(n, y)| \geq E_{(\alpha)}(\lambda - L, n)|x - y| \quad (x, y \in \mathbb{R}, n \in \mathbb{N}).$$

(b) Riemann-Liouville difference equation: solutions of

$$({}_{\mathbb{R}\text{-L}}\Delta^{\alpha}x)(n + 1 - \alpha) = \lambda x(n) + f(x(n)) \quad (17)$$

satisfy the estimate

$$|\varphi_{\mathbb{R}\text{-L}}(n, x) - \varphi_{\mathbb{R}\text{-L}}(n, y)| \geq E_{(\alpha, \alpha)}(\lambda - L, n)|x - y| \quad (x, y \in \mathbb{R}, n \in \mathbb{N}).$$

In the proof of the above theorem we will use the following lemma on monotonicity with respect to the initial conditions of scalar equations.

**Lemma 7** Consider equation (14) and assume that  $f$  satisfies (15) with  $L \in [0, \lambda)$ .

(a) If  $x \leq y$ , then  $\varphi_{\mathbb{C}}(n, x) \leq \varphi_{\mathbb{C}}(n, y)$  for  $n \in \mathbb{N}$ .

(b) If  $x \leq y$ , then  $\varphi_{\mathbb{R}\text{-L}}(n, x) \leq \varphi_{\mathbb{R}\text{-L}}(n, y)$  for  $n \in \mathbb{N}$ .

**Proof.** Define  $h(x) := Lx + f(x)$ . Then equation (14) can be rewritten as

$$(\Delta^{\alpha}x)(n + 1 - \alpha) = (\lambda - L)x(n) + h(x(n)) \quad (n \in \mathbb{N}). \quad (18)$$

Moreover, for  $x \leq y$

$$\begin{aligned} h(y) - h(x) &= Ly + f(y) - (Lx + f(x)) \\ &= f(y) - f(x) + L(y - x) \\ &\geq -L(y - x) + L(y - x) \\ &= 0, \end{aligned}$$

i.e.,  $h$  is monotonically increasing.

(a) By Theorem 1(a) and (13), for  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} \varphi_{\mathbb{C}}(n, y) - \varphi_{\mathbb{C}}(n, x) &= E_{(\alpha)}(\lambda - L, n)(y - x) \\ &\quad + \sum_{k=0}^{n-1} E_{(\alpha, \alpha)}(\lambda - L, n - k - 1)(h(\varphi_{\mathbb{C}}(k, y)) - h(\varphi_{\mathbb{C}}(k, x))) \quad (n \in \mathbb{N}). \quad (19) \end{aligned}$$

By (10) we have for  $\alpha > 0, \beta \geq 0$

$$\begin{aligned} & \binom{n-k+k\alpha+\beta-1}{n-k} \\ &= (-1)^{n-k} \binom{-(k\alpha+\beta)}{n-k} \\ &= (-1)^{n-k} \frac{(-(k\alpha+\beta))(-k\alpha-\beta+1)\cdots(-k\alpha-\beta+n-k-1)}{1\cdot 2\cdots(n-k)} \\ &= \frac{(k\alpha+\beta)(k\alpha+\beta+1)\cdots(k\alpha+\beta+n-k-1)}{1\cdot 2\cdots(n-k)} > 0. \end{aligned}$$

Substituting into the above inequality  $\beta = 0$  and  $\beta = 1$  and taking into account that  $\lambda - L > 0$ , we have  $E_{(\alpha)}(\lambda - L, n) > 0$  and  $E_{(\alpha,\alpha)}(\lambda - L, n) > 0$  for all  $n \in \mathbb{N}$ , respectively. Hence,  $x \leq y$  implies  $\varphi_{\mathbb{C}}(n, x) \leq \varphi_{\mathbb{C}}(n, y)$  for  $n \in \mathbb{N}$ .

(b) By Theorem 1(b) and (13), for  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} & \varphi_{\mathbb{R-L}}(n, y) - \varphi_{\mathbb{R-L}}(n, x) \\ &= E_{(\alpha,\alpha)}(\lambda - L, n)(y - x) \\ &+ \sum_{k=0}^{n-1} E_{(\alpha,\alpha)}(\lambda - L, n - k - 1)(h(\varphi_{\mathbb{R-L}}(k, y)) - h(\varphi_{\mathbb{R-L}}(k, x))) \quad (n \in \mathbb{N}). \quad (20) \end{aligned}$$

Since  $\lambda - L > 0$ , we have  $E_{(\alpha,\alpha)}(\lambda - L, n) > 0$  for all  $n \in \mathbb{N}$ . Hence,  $x \leq y$  implies  $\varphi_{\mathbb{R-L}}(n, x) \leq \varphi_{\mathbb{R-L}}(n, y)$  for  $n \in \mathbb{N}$ . □

We are now in a position to prove Theorem 2.

**Proof.** [Proof of Theorem 2] Assume that  $x < y$  and  $L \in [0, \lambda)$ .

By Lemma 7, equations (19) and (20), and the fact that  $h$  is monotonically increasing, we get

$$\varphi_{\mathbb{C}}(n, y) - \varphi_{\mathbb{C}}(n, x) \geq E_{(\alpha)}(\lambda - L, n)(y - x) \quad (n \in \mathbb{N}),$$

and

$$\varphi_{\mathbb{R-L}}(n, y) - \varphi_{\mathbb{R-L}}(n, x) \geq E_{(\alpha\alpha)}(\lambda - L, n)(y - x) \quad (n \in \mathbb{N}),$$

respectively. □

As an application of Theorem 2 to equations (14) with trivial solution, we get that the Lyapunov exponent of non-zero solutions is nonnegative.

**Corollary 4** Consider equation (14) with  $\lambda > 0$  and assume that  $f$  satisfies (15) with  $L \in [0, \lambda)$ . Then for  $x_0 \in \mathbb{R} \setminus \{0\}$  the nontrivial solutions of the Caputo and Riemann-Liouville difference equations (16) and (17) satisfy

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln |\varphi_{\mathbb{C}}(n, x_0)| \geq \begin{cases} \lambda - L & \text{if } \lambda - L > 1, \\ 0 & \text{if } 0 < \lambda - L \leq 1, \end{cases} \quad (21)$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln |\varphi_{\mathbb{R-L}}(n, x_0)| \geq \begin{cases} \lambda - L & \text{if } \lambda - L > 1, \\ 0 & \text{if } 0 < \lambda - L \leq 1, \end{cases} \quad (22)$$

respectively.

**Proof.** Recall from [5, p. 656] and [12, pp. 165], that for all  $\alpha > 0, \beta > 0$ ,

$$\begin{aligned} & \binom{n-k+k\alpha+\beta-1}{n-k} \\ &= (-1)^{n-k} \binom{-(k\alpha+\beta)}{n-k} \\ &= (-1)^{n-k} \frac{(-(k\alpha+\beta))(-k\alpha+\beta+1)\cdots(-k\alpha+\beta+n-k-1)}{1 \cdot 2 \cdots (n-k)} \\ &= \frac{(k\alpha+\beta)(k\alpha+\beta+1)\cdots(k\alpha+\beta+n-k-1)}{1 \cdot 2 \cdots (n-k)}. \end{aligned}$$

Hence for  $\beta = 1$ , we have

$$\binom{n-k+k\alpha}{n-k} \geq 1.$$

Choosing  $x = x_0, y = 0$ , from Theorem 2,

$$\begin{aligned} |\varphi_{\mathbb{C}}(n, x_0)| &\geq |E_{\alpha}(\lambda - L, n)| |x_0| \\ &\geq \sum_{k=0}^n (\lambda - L)^k |x_0|. \end{aligned}$$

It remains to verify, that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \sum_{k=n_0}^n q^k = \begin{cases} q & \text{if } q > 1, \\ 0 & \text{if } 0 < q \leq 1. \end{cases} \quad (23)$$

From the last two inequalities we obtain (21).

For the Riemann-Liouville case, with  $n_0 := \left\lceil \frac{1 - \alpha}{\alpha} \right\rceil$ , we have  $k\alpha + \alpha \geq 1$  for all  $k \geq n_0$ . As a consequence, for  $n > n_0$ ,

$$\binom{n - k + k\alpha + \alpha - 1}{n - k} < 1 \quad (k \in \{0, 1, \dots, n_0 - 1\}),$$

and

$$\binom{n - k + k\alpha + \alpha - 1}{n - k} \geq 1 \quad (k \in \{n_0, n_0 + 1, \dots, n\}).$$

Therefore

$$\begin{aligned} |\varphi_{R-L}(n, x_0)| &\geq |E_{\alpha, \alpha}(\lambda - L, n)| |x_0| \\ &\geq \sum_{k=n_0}^n (\lambda - L)^k |x_0|. \end{aligned}$$

Combining the last inequality with (23), we obtain (22). □

#### 4. Conclusions

We used the  $\mathcal{L}$ -transform to establish variation of constant formulas for Caputo and Riemann-Liouville fractional difference equations. Using this formula we provided a lower bound for the norm of differences between two different solutions of a scalar Caputo or Riemann-Liouville time-varying linear equation. In particular, this result implies that the classical Lyapunov exponent is not an appropriate tool for stability analysis of fractional equations.

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