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Extragradient subgradient methods for solving bilevel equilibrium problems

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Abstract

In this paper, we propose two algorithms for finding the solution of a bilevel equilibrium problem in a real Hilbert space. Under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions, we obtain the strong convergence of the iterative sequence generated by the first algorithm. Furthermore, the strong convergence of the sequence generated by the second algorithm is obtained without a Lipschitz-type condition.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H, and let f and g be bifunctions from $H \times H$ to $\mathbb{R}$ such that $f(x,x) = 0$ and $g(x,x) = 0$ for all $x \in H$. The equilibrium problem associated with g and C is denoted by $EP(C,g)$: Find $x^* \in C$ such that

$$g(x^*,y) \leq 0 \quad \text{for every } y \in C. \tag{1}$$

The solution set of problem (1) is denoted by Ω. The equilibrium problem is very important because many problems arise in applied areas such as the fixed point problem, the (generalized) Nash equilibrium problem in game theory, the saddle point problem, the variational inequality problem, the optimization problem and others.

The basic method for solving the monotone equilibrium problem is the proximal method (see [14, 16, 19]). In 2008, Tran et al. [25] proposed the extragradient algorithm for solving the equilibrium problem by using the strongly convex minimization problem to solve at each iteration. Furthermore, Hieu [9] introduced subgradient extragradient methods for pseudomonotone equilibrium problem and the other methods (see the details in [1, 8, 10, 15, 17, 22, 28]).

In this paper, we consider the equilibrium problem whose constraints are the solution sets of equilibrium problems (bilevel equilibrium problems) : Find $x^* \in \Omega$ such that

$$f(x^*,y) \geq 0 \quad \text{for every } y \in \Omega. \tag{2}$$

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The solution set of problem (2) is denoted by Ω^*. The bilevel equilibrium problems were introduced by Chadli et al. [4] in 2000. This kind of problems is very important and interesting because it is a generalization class of problems such as optimization problems over equilibrium constraints, variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. Furthermore, the particular case of the bilevel equilibrium can be applied to a real word model such as the variational inequality over the fixed point set of a firmly nonexpansive mapping applied to the power control problem of CDMA networks which were introduced by Iiduka [11]. For more on the relation of bilevel equilibrium with particular cases, see [7, 12, 21].

Methods for solving bilevel equilibrium problems have been studied extensively by many authors. In 2010, Moudafi [20] introduced a simple proximal method and proved the weak convergence to a solution of problem (2). In 2014, Quy [23] introduced the algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. For more details and most recent works on the methods for solving bilevel equilibrium problems, we refer the reader to [2, 5, 24]. The authors considered the method for monotone and pseudoparamonotone equilibrium problem. If a bifunction is more generally monotone, we cannot use the above methods for solving bilevel equilibrium problem, for example, the pseudomonotone property.

Inspired by the above work, in this paper, we propose a method for finding the solution for bilevel equilibrium problems where f is strongly monotone and g is pseudomonotone and Lipschitz-type continuous. Firstly, we obtain the convergent sequence by combining an extragradient subgradient method with the Halpern method. Second, we obtain the convergent sequence without Lipschitz-type continuity on bifunction g by combining an Armijo line search with the extragradient subgradient method.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Denote that $x_n \rightharpoonup x$ and $x_n \to x$ are the weak convergence and the strong convergence of a sequence $\{x_n\}$ to x, respectively. For every $x \in H$, there exists a unique element $P_C x$ defined by

$$P_C x = \inf\{\|x - y\| : y \in C\}.$$

It is also known that P_C is a nonexpansive mapping from H onto C, i.e., $\|P_C(x) - P_C(y)\| \leq \|x - y\| \ \forall x, y \in H$. For every $x \in H$ and $y \in C$, we have

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \text{and} \quad \langle x - P_C x, P_C x - y \rangle \geq 0.$$

A bifunction $\psi : H \times H \to \mathbb{R}$ is called:
  (i) β-strongly monotone on C if

$$\psi(x, y) + \psi(y, x) \leq -\beta \|x - y\|^2 \quad \forall x, y \in C;$$

  (ii) monotone on C if

$$\psi(x, y) + \psi(y, x) \leq 0 \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\psi(x,y) \geq 0 \Rightarrow \psi(y,x) \leq 0, \quad \forall x,y \in C.$$

It is easy to check that the monotone bifunction implies the pseudomonotone bifunction. On the other hand, if the bifunction is pseudomonotone, then we cannot guarantee that the bifunction is monotone, for example, let $\phi(x,y) = \frac{2y-x}{1-x}$ for all $x,y \in \mathbb{R}$. It follows that ϕ is pseudomonotone on $\mathbb{R}^+ \setminus \{0\}$ but ϕ is not monotone on $\mathbb{R}^+ \setminus \{0\}$. Let $\psi(x,\cdot)$ be convex for every $x \in H$. For each fixed $x \in H$, the subdifferential of $\psi(x,.)$ at x, denoted by $\partial_2 \psi(x,x)$, is defined by

$$\begin{aligned}
\partial_2 \psi(x,x) &= \left\{ w \in H : \psi(x,y) - \psi(x,x) \geq \langle w, y - x \rangle \ \forall y \in H \right\} \\
&= \left\{ w \in H : \psi(x,y) \geq \langle w, y - x \rangle \ \forall y \in H \right\},
\end{aligned} \tag{3}$$

studied in [13]. In this paper, we consider the bifunctions f and g under the following conditions.

Condition A

(A1) $f(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.

(A2) $f(\cdot,y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.

(A3) f is β-strongly monotone on H.

(A4) For each $x,y \in H$, there exists $L > 0$ such that

$$\|w - v\| \leq L\|x - y\|, \quad \forall w \in \partial_2 f(x,x), v \in \partial_2 f(y,y).$$

(A5) The function $x \mapsto \partial_2 f(x,x)$ is bounded on the bounded subsets of H.

Condition B

(B1) $g(x,\cdot)$ is convex, weakly lower semicontinuous, and subdifferentiable on H for every fixed $x \in H$.

(B2) $g(\cdot,y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.

(B3) g is pseudomonotone on C with respect to Ω, i.e.,

$$g(x,x^*) \leq 0, \quad \forall x \in C, x^* \in \Omega.$$

(B4) g is Lipschitz-type continuous, i.e., there are two positive constants L_1, L_2 such that $g(x,y) + g(y,z) \geq g(x,z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \forall x,y,z \in H$.

(B5) g is jointly weakly continuous on $H \times H$ in the sense that, if $x,y \in H$ and $\{x_n\}, \{y_n\} \in H$ converge weakly to x and y, respectively, then $g(x_n, y_n) \to g(x,y)$ as $n \to +\infty$.

Example 2.1 Let $f,g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = 5y^2 - 7x^2 + 2xy$ and $g(x,y) = 2y^2 - 7x^2 + 5xy$. It follows that f and g satisfy Condition A and Condition B, respectively.

Lemma 2.2 ([3, Propositions 3.1, 3.2]) *If the bifunction g satisfies Assumptions* (B1), (B2), *and* (B3), *then the solution set Ω is closed and convex.*

Remark 2.3 Let the bifunction f satisfy Assumption A and the bifunction g satisfy B. If $\Omega \neq \emptyset$, then the bilevel equilibrium problem (2) has a unique solution, see the details in [23].

Lemma 2.4 ([6]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $\phi : C \rightarrow \mathbb{R}$ be a convex, lower semicontinuous, and subdifferentiable function on C. Then x^* is a solution to the convex optimization problem $\min\{\phi(x) : x \in C\}$ if and only if $0 \in \partial\phi(x^*) + N_C(x^*)$, where $\partial\phi(\cdot)$ denotes the subdifferential of ϕ and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.5 ([27]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$, and $\{\xi_n\}$ be a sequence in $\mathbb{R}$ satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\xi_n \quad \forall n \geq 0,$$

where $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \to \infty} \xi_n \leq 0$. Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.6 ([18]) *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that*

$$a_{n_j} < a_{n_{j+1}} \quad \text{for all } j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined, for all $n \geq n_0$, by

$$\tau(n) = \max\{k \leq n | a_k < a_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n \to \infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold:

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}.$$

Lemma 2.7 *Suppose that f is β-strongly monotone on H and satisfies (A4). Let $0 < \alpha < 1$, $0 \leq \eta \leq 1 - \alpha$, and $0 < \mu < \frac{2\beta}{L^2}$. For each $x, y \in H$, $w \in \partial_2 f(x, x)$, and $v \in \partial_2 f(y, y)$, we have*

$$\left\| (1 - \eta)x - \alpha\mu w - \left[(1 - \eta)y - \alpha\mu v\right] \right\| \leq (1 - \eta - \alpha\tau)\|x - y\|,$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.

Proof Let $x, y \in H$, $w \in \partial_2 f(x, x)$, and $v \in \partial_2 f(y, y)$. Thus

$$\left\| (1 - \eta)x - \alpha\mu w - \left[(1 - \eta)y - \alpha\mu v\right] \right\|$$
$$\leq \left\| (1 - \eta)(x - y) - \alpha\mu(w - v) \right\|$$
$$= \left\| (1 - \eta - \alpha)(x - y) + \alpha\left[(x - y) - \mu(w - v)\right] \right\|$$
$$\leq (1 - \eta - \alpha)\|x - y\| + \alpha\left\| (x - y) - \mu(w - v) \right\|. \tag{4}$$

Since $f : H \times H \to \mathbb{R}$ is β-strongly monotone, $w \in \partial_2 f(x,x)$, and $v \in \partial_2 f(y,y)$, we have

$$-\beta \|x - y\|^2 \geq f(x,y) + f(y,x) \geq -\langle x - y, w - v \rangle. \tag{5}$$

From (5) and (A4), we have

$$
\begin{aligned}
\left\| (x - y) - \mu(w - v) \right\|^2 &= \|x - y\|^2 - 2\mu \langle x - y, w - v \rangle + \mu^2 \|w - v\|^2 \\
&\leq \|x - y\|^2 - 2\mu\beta \|x - y\|^2 + \mu^2 L^2 \|x - y\|^2 \\
&= \left(1 - 2\mu\beta + \mu^2 L^2\right) \|x - y\|^2.
\end{aligned}
$$

This implies that

$$\left\| (x - y) - \mu(w - v) \right\| \leq \sqrt{1 - 2\mu\beta + \mu^2 L^2} \, \|x - y\|. \tag{6}$$

By using (4) and (6), we can conclude that

$$
\begin{aligned}
&\left\| (1 - \eta)x - \alpha\mu w - \left[(1 - \eta)y - \alpha\mu v \right] \right\| \\
&\quad \leq (1 - \eta - \alpha)\|x - y\| + \left(\alpha \sqrt{1 - 2\mu\beta + \mu^2 L^2} \right) \|x - y\| \\
&\quad = (1 - \eta - \alpha\tau)\|x - y\|,
\end{aligned}
$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1]$. □

3 The extragradient subgradient Halpern methods

In this section, we propose the algorithm for finding the solution of a bilevel equilibrium problem under the strong monotonicity of f and the pseudomonotonicity and Lipschitz-type continuous conditions on g.

Algorithm 1

Initialization: Choose $x_0 \in H, 0 < \mu < \frac{2\beta}{L^2}$, the sequences $\{\alpha_n\} \subset (0,1), \{\eta_n\}$, and $\{\lambda_n\}$ are such that

$$
\begin{cases}
\lim_{n \to \infty} \alpha_n = 0, & \sum_{n=0}^{\infty} \alpha_n = \infty, \\
0 \leq \eta_n \leq 1 - \alpha_n & \forall n \geq 0, \quad \lim_{n \to \infty} \eta_n = \eta < 1, \\
0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < \min(\frac{1}{2L_1}, \frac{1}{2L_2}).
\end{cases}
$$

Set $n = 0$ and go to Step 1.

Step 1. Compute

$$
y_n = \arg\min_{y \in C} \left\{ \lambda_n g(x_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\},
$$

$$
z_n = \arg\min_{y \in C} \left\{ \lambda_n g(y_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\}.
$$

Step 2. Compute $w_n \in \partial_2 f(z_n, z_n)$ and

$$
x_{n+1} = \eta_n x_n + (1 - \eta_n)z_n - \alpha_n \mu w_n.
$$

Set $n = n + 1$ and go back to Step 1.

Theorem 3.1 *Let bifunctions f and g satisfy Condition A and Condition B, respectively. Assume that $\Omega \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the unique solution of the bilevel equilibrium problem (2).*

Proof Under assumptions of two bifunctions f and g, we get the unique solution of the bilevel equilibrium problem (2), denoted by x^*.

Step 1: Show that

$$\left\| z_n - x^* \right\|^2 \leq \left\| x_n - x^* \right\|^2 - (1 - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1 - 2\lambda_n L_2)\|y_n - z_n\|^2. \tag{7}$$

The definition of y_n and Lemma 2.4 imply that

$$0 \in \partial_2 \left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}(y_n) + N_C(y_n).$$

There are $w \in \partial_2 g(x_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\lambda_n w + y_n - x_n + \bar{w} = 0. \tag{8}$$

Since $\bar{w} \in N_C(y_n)$, we have

$$\langle \bar{w}, y - y_n \rangle \leq 0 \quad \text{for all } y \in C. \tag{9}$$

By using (8) and (9), we obtain $\lambda_n \langle w, y - y_n \rangle \geq \langle x_n - y_n, y - y_n \rangle$ for all $y \in C$. Since $z_n \in C$, we have

$$\lambda_n \langle w, z_n - y_n \rangle \geq \langle x_n - y_n, z_n - y_n \rangle. \tag{10}$$

It follows from $w \in \partial_2 g(x_n, y_n)$ that

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n \rangle \quad \text{for all } y \in H. \tag{11}$$

By using (10) and (11), we get

$$\lambda_n \left\{ g(x_n, z_n) - g(x_n, y_n) \right\} \geq \langle x_n - y_n, z_n - y_n \rangle. \tag{12}$$

Similarly, the definition of z_n implies that

$$0 \in \partial_2 \left\{ \lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}(z_n) + N_C(z_n).$$

There are $u \in \partial_2 g(y_n, z_n)$ and $\bar{u} \in N_C(z_n)$ such that

$$\lambda_n u + z_n - x_n + \bar{u} = 0. \tag{13}$$

Since $\bar{u} \in N_C(z_n)$, we have

$$\langle \bar{u}, y - z_n \rangle \leq 0 \quad \text{for all } y \in C. \tag{14}$$

By using (13) and (14), we obtain $\lambda_n \langle u, y - z_n \rangle \geq \langle x_n - z_n, y - z_n \rangle$ for all $y \in C$. Since $x^* \in C$, we have

$$\lambda_n \langle u, x^* - z_n \rangle \geq \langle x_n - z_n, x^* - z_n \rangle. \tag{15}$$

It follows from $u \in \partial_2 g(y_n, z_n)$ that

$$g(y_n, y) - g(y_n, z_n) \geq \langle u, y - z_n \rangle \quad \text{for all } y \in H. \tag{16}$$

By using (15) and (16), we get

$$\lambda_n \{ g(y_n, x^*) - g(y_n, z_n) \} \geq \langle x_n - z_n, x^* - z_n \rangle. \tag{17}$$

Since $x^* \in \Omega$, we have $g(x^*, y_n) \geq 0$. If follows from the pseudomonotonicity of g on C with respect to Ω that $g(y_n, x^*) \leq 0$. This implies that

$$\langle x_n - z_n, z_n - x^* \rangle \geq \lambda_n g(y_n, z_n). \tag{18}$$

Since g is Lipschitz-type continuous, there exist two positive constants L_1, L_2 such that

$$g(y_n, z_n) \geq g(x_n, z_n) - g(x_n, y_n) - L_1 \|x_n - y_n\|^2 - L_2 \|y_n - z_n\|^2. \tag{19}$$

By using (18) and (19), we get

$$\langle x_n - z_n, z_n - x^* \rangle \geq \lambda_n \{ g(x_n, z_n) - g(x_n, y_n) \} - \lambda_n L_1 \|x_n - y_n\|^2 - \lambda_n L_2 \|y_n - z_n\|^2.$$

From (12) and the above inequality, we obtain

$$2 \langle x_n - z_n, z_n - x^* \rangle \geq 2 \langle x_n - y_n, z_n - y_n \rangle - \lambda_n L_1 \|x_n - y_n\|^2 - \lambda_n L_2 \|y_n - z_n\|^2. \tag{20}$$

We know that

$$2 \langle x_n - z_n, z_n - x^* \rangle = \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2,$$
$$2 \langle x_n - y_n, z_n - y_n \rangle = \|x_n - y_n\|^2 + \|z_n - y_n\|^2 - \|x_n - z_n\|^2.$$

From (20), we can conclude that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n L_1) \|x_n - y_n\|^2 - (1 - 2\lambda_n L_2) \|y_n - z_n\|^2.$$

Step 2: The sequences $\{x_n\}, \{w_n\}, \{y_n\}$, and $\{z_n\}$ are bounded.

Since $0 < \lambda_n < a$, where $a = \min(\frac{1}{2L_1}, \frac{1}{2L_2})$, we have

$$(1 - 2\lambda_n L_1) > 0 \quad \text{and} \quad (1 - 2\lambda_n L_2) > 0.$$

It follows from (7) and the above inequalities that

$$\|z_n - x^*\| \leq \|x_n - x^*\| \quad \text{for all } n \in \mathbb{N}. \tag{21}$$

By Lemma 2.7 and (21), we obtain

$$
\begin{aligned}
\left\| x_{n+1} - x^* \right\| &= \left\| \eta_n x_n + (1-\eta_n)z_n - \alpha_n \mu w_n - x^* + \eta_n x^* - \eta_n x^* + \alpha_n \mu v - \alpha_n \mu v \right\| \\
&= \left\| (1-\eta_n)z_n - \alpha_n \mu w_n - (1-\eta_n)x^* + \alpha_n \mu v + \eta\left(x_n - x^*\right) - \alpha_n \mu v \right\| \\
&\leq \left\| (1-\eta_n)z_n - \alpha_n \mu w_n - \left[(1-\eta_n)x^* - \alpha_n \mu v\right] \right\| + \eta_n \left\| x_n - x^* \right\| + \alpha_n \mu \|v\| \\
&\leq (1-\eta_n - \alpha_n \tau)\left\| z_n - x^* \right\| + \eta_n \left\| x_n - x^* \right\| + \alpha_n \mu \|v\| \\
&\leq (1-\eta_n - \alpha_n \tau)\left\| x_n - x^* \right\| + \eta_n \left\| x_n - x^* \right\| + \alpha_n \mu \|v\| \\
&= (1-\alpha_n \tau)\left\| x_n - x^* \right\| + \alpha_n \mu \|v\| \\
&= (1-\alpha_n \tau)\left\| x_n - x^* \right\| + \alpha_n \tau \left(\frac{\mu\|v\|}{\tau} \right),
\end{aligned} \tag{22}
$$

where $w_n \in \partial_2 f(z_n, z_n)$ and $v \in \partial_2 f(x^*, x^*)$. This implies that

$$
\left\| x_{n+1} - x^* \right\| \leq \max\left\{ \left\| x_n - x^* \right\|, \frac{\mu\|v\|}{\tau} \right\}.
$$

By induction, we obtain

$$
\left\| x_n - x^* \right\| \leq \max\left\{ \left\| x_0 - x^* \right\|, \frac{\mu\|v\|}{\tau} \right\}.
$$

Thus the sequence $\{x_n\}$ is bounded. By using (21), we have $\{z_n\}$, and using Condition (A5), we can conclude that $\{w_n\}$ is also bounded.

Step 3: Show that the sequence $\{x_n\}$ converges strongly to x^*.

Since $x \in \Omega^*$, we have $f(x^*, y) \geq 0$ for all $y \in \Omega$. Thus x^* is a minimum of the convex function $f(x^*, \cdot)$ over Ω. By Lemma 2.4, we obtain $0 \in \partial_2 f(x^*, x^*) + N_\Omega(x^*)$. Then there exists $v \in \partial_2 f(x^*, x^*)$ such that

$$
\langle v, z - x^* \rangle \geq 0 \quad \text{for all } z \in \Omega. \tag{23}
$$

Note that

$$
\|x - y\|^2 \leq \|x\|^2 - 2\langle y, x - y \rangle \quad \text{for all } x, y \in H. \tag{24}
$$

From Lemma (2.7) and (24), we obtain

$$
\begin{aligned}
\left\| x_{n+1} - x^* \right\|^2 &= \left\| \eta_n x_n + (1-\eta_n)z_n - \alpha_n \mu w_n - x^* \right\|^2 \\
&= \left\| (1-\eta_n)z_n - \alpha_n \mu w_n - \left[(1-\eta_n)x^* - \alpha_n \mu v\right] \right. \\
&\quad \left. + \eta_n\left(x_n - x^*\right) - \alpha_n \mu v \right\|^2 \\
&\leq \left\| (1-\eta_n)z_n - \alpha_n \mu w_n - \left[(1-\eta_n)x^* - \alpha_n \mu v\right] \right. \\
&\quad \left. + \eta_n\left(x_n - x^*\right) \right\|^2 - 2\alpha_n \mu \langle v, x_{n+1} - x^* \rangle \\
&\leq \left\{ \left\| (1-\eta_n)z_n - \alpha_n \mu w_n - \left[(1-\eta_n)x^* - \alpha_n \mu v\right] \right. \right. \\
&\quad \left. \left. + \eta_n \|\left(x_n - x^*\right)\| \right\}^2 - 2\alpha_n \mu \langle v, x_{n+1} - x^* \rangle
\end{aligned}
$$

$$\leq \left[(1 - \eta_n - \alpha_n \tau) \| z_n - x^* \| + \eta_n \| x_n - x^* \| \right]^2 - 2\alpha_n \mu \langle v, x_{n+1} - x^* \rangle$$

$$\leq (1 - \eta_n - \alpha_n \tau) \| z_n - x^* \|^2 + \eta_n \| x_n - x^* \|^2 - 2\alpha_n \mu \langle v, x_{n+1} - x^* \rangle$$

$$\leq (1 - \eta_n - \alpha_n \tau) \| x_n - x^* \|^2 + \eta_n \| x_n - x^* \|^2 - 2\alpha_n \mu \langle v, x_{n+1} - x^* \rangle$$

$$= (1 - \alpha_n \tau) \| x_n - x^* \|^2 + 2\alpha_n \mu \langle v, x^* - x_{n+1} \rangle.$$

It follows that

$$\| x_{n+1} - x^* \|^2 \leq (1 - \alpha_n \tau) \| x_n - x^* \|^2 + 2\alpha_n \mu \langle v, x^* - x_{n+1} \rangle. \tag{25}$$

Let us consider two cases.

Case 1: There exists n_0 such that $\{\| x_n - x^* \|\}$ is decreasing for $n \geq n_0$. Therefore the limit of sequence $\{\| x_n - x^* \|\}$ exists. By using (21) and (25), we obtain

$$0 \leq \| x_n - x^* \|^2 - \| z_n - x^* \|^2$$

$$\leq -\frac{\alpha_n \tau}{1 - \eta_n} \| z_n - x^* \|^2 - \frac{2\alpha_n \mu}{1 - \eta_n} \langle v, x_{n+1} - x^* \rangle$$

$$+ \frac{1}{1 - \eta_n} \left(\| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2 \right). \tag{26}$$

Since $\lim_{n \to \infty} \eta_n = \eta < 1, \lim_{n \to \infty} \alpha_n = 0$ and the limit of $\{\| x_n - x^* \|\}$ exists, we have

$$\lim_{n \to \infty} \left(\| x_n - x^* \|^2 - \| z_n - x^* \|^2 \right) = 0. \tag{27}$$

From $0 < \lambda_n < a$ and inequality (7), we get

$$(1 - 2a) \| x_n - y_n \|^2 \leq (1 - 2\lambda_n L_1) \| x_n - y_n \|^2 \leq \| x_n - x^* \|^2 - \| z_n - x^* \|^2.$$

By using (27), we obtain $\lim_{n \to \infty} \| x_n - y_n \| = 0$. Next, we show that

$$\limsup_{n \to \infty} \langle v, x^* - x_{n+1} \rangle \leq 0. \tag{28}$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle v, x^* - x_{n+1} \rangle = \limsup_{k \to \infty} \langle v, x^* - x_{n_k} \rangle.$$

Since $\{x_{n_k}\}$ is bounded, we may assume that $\{x_{n_k}\}$ converges weakly to some $\bar{x} \in H$. Therefore

$$\limsup_{n \to \infty} \langle v, x^* - x_{n+1} \rangle = \limsup_{k \to \infty} \langle v, x^* - x_{n_k} \rangle = \langle v, x^* - \bar{x} \rangle. \tag{29}$$

Since $\lim_{n \to \infty} \| x_n - y_n \| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $y_{n_k} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed and thus $\bar{x} \in C$. Next, we show that $\bar{x} \in \Omega$. From the definition of $\{y_n\}$ and Lemma 2.4, we obtain

$$0 \in \partial_2 \left\{ \lambda_n g(x_n, y) + \frac{1}{2} \| x_n - y_n \|^2 \right\} (y_n) + N_C(y_n).$$

There exist $\bar{w} \in N_C(y_n)$ and $w \in \partial_2 g(x_n, y_n)$ such that

$$\lambda_n w + y_n - x_n + \bar{w} = 0. \tag{30}$$

Since $\bar{w} \in N_C(y_n)$, we have $\langle \bar{w}, y - y_n \rangle \leq 0$ for all $y \in C$. From (30), we obtain

$$\lambda_n \langle w, y - y_n \rangle \geq \langle x_n - y_n, y - y_n \rangle \quad \text{for all } y \in C. \tag{31}$$

Since $w \in \partial_2 g(x_n, y_n)$, we have

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n \rangle \quad \text{for all } y \in H. \tag{32}$$

Combining (31) and (32), we get

$$\lambda_n \big\{ g(x_n, y) - g(x_n, y_n) \big\} \geq \langle x_n - y_n, y - y_n \rangle \quad \text{for all } y \in C. \tag{33}$$

Taking $n = n_k$ and $k \to \infty$ in (33), the assumption of λ_n and (B5), we obtain $g(\bar{x}, y) \geq 0$ for all $y \in C$. This implies that $\bar{x} \in \Omega$. By inequality (23), we obtain $\langle v, \bar{x} - x^* \rangle \geq 0$. It follows from (29) that

$$\limsup_{n \to \infty} \langle v, x^* - x_{n+1} \rangle \leq 0. \tag{34}$$

We can write inequality (25) in the following form:

$$\big\| x_{n+1} - x^* \big\|^2 \leq (1 - \alpha_n \tau) \big\| x_n - x^* \big\|^2 - \alpha_n \tau \xi_n,$$

where $\xi_n = \frac{2\mu}{\tau} \langle v, x^* - x_{n+1} \rangle$. It follows from (34) that $\limsup_{n \to \infty} \xi_n \leq 0$. By Lemma 2.5, we can conclude that $\lim_{n \to \infty} \| x_n - x^* \|^2 = 0$. Hence $x_n \to x^*$ as $n \to \infty$.

   Case 2: There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\| x_{n_j} - x^* \| \leq \| x_{n_j+1} - x^* \|$ for all $j \in \mathbb{N}$. By Lemma 2.6, there exists a nondecreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ such that $\lim_{n \to \infty} \tau(n) = \infty$, and for each sufficiently large $n \in \mathbb{N}$, we have

$$\big\| x_{\tau(n)} - x^* \big\| \leq \big\| x_{\tau(n)+1} - x^* \big\| \quad \text{and} \quad \big\| x_n - x^* \big\| \leq \big\| x_{\tau(n)+1} - x^* \big\|. \tag{35}$$

Combining (22) and (35), we have

$$\begin{aligned}
\big\| x_{\tau(n)} - x^* \big\| &\leq \big\| x_{\tau(n)+1} - x^* \big\| \\
&\leq (1 - \eta_{\tau(n)} - \alpha_{\tau(n)} \tau) \| z_{\tau(n)-x^*} \| + \eta_{\tau(n)} \big\| x_{\tau(n)} - x^* \big\| + \alpha_{\tau(n)} \mu \| v \|.
\end{aligned} \tag{36}$$

From (21) and (36), we get

$$0 \leq \big\| x_{\tau(n)} - x^* \big\| - \big\| z_{\tau(n)} - x^* \big\| \leq -\frac{\alpha_{\tau(n)} \tau}{1 - \eta_{\tau(n)}} \big\| z_{\tau(n)} - x^* \big\| + \frac{\alpha_{\tau(n)} \mu}{1 - \eta_{\tau(n)}} \| v \|. \tag{37}$$

Since $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \eta_n = \eta < 1$, $\{z_n\}$ is bounded, and (37), we have $\lim_{n \to \infty} (\| x_{\tau(n)} - x^* \| - \| z_{\tau(n)} - x^* \|) = 0$. It follows from the boundedness of $\{x_n\}$ and $\{z_n\}$ that

$$\lim_{n \to \infty} \big(\big\| x_{\tau(n)} - x^* \big\|^2 - \big\| z_{\tau(n)} - x^* \big\|^2 \big) = 0. \tag{38}$$

By using the assumption of $\{\lambda_n\}$, we get the following two inequalities:

$$1 - 2\lambda_{\tau(n)}L_1 > 1 - 2aL_1 > 0 \quad \text{and} \quad 1 - 2\lambda_{\tau(n)}L_2 > 1 - 2aL_2 > 0.$$

From (7), we obtain

$$
\begin{aligned}
\left\| z_{\tau(n)} - x^* \right\|^2 &\le \left\| x_{\tau(n)} - x^* \right\|^2 - (1 - 2\lambda_{\tau(n)}L_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\
&\quad - (1 - 2\lambda_{\tau(n)}L_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\
&\le \left\| x_{\tau(n)} - x^* \right\|^2 - (1 - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\
&\quad - (1 - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
0 &< (1 - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 + (1 - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\
&\le \left\| x_{\tau(n)} - x^* \right\|^2 - \left\| z_{\tau(n)} - x^* \right\|^2.
\end{aligned}
$$

It follows from (38) and the above inequality that

$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{39}$$

Note that $\|x_{\tau(n)} - z_{\tau(n)}\| \le \|x_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - z_{\tau(n)}\|$. From (39), we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{40}$$

By using the definition of x_{n+1} and Lemma 2.7, we obtain

$$
\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\| &= \left\| \eta_{\tau(n)}x_{\tau(n)} + (1 - \eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}\mu t_{\tau(n)} - x_{\tau(n)} \right\| \\
&= \left\| (1 - \eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}\mu t_{\tau(n)} \right. \\
&\qquad - \left[(1 - \eta_{\tau(n)})x_{\tau(n)} - \alpha_{\tau(n)}w_{\tau(n)} \right] - \alpha_{\tau(n)}w_{\tau(n)} \Big\| \\
&\le \left\| (1 - \eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}t_{\tau(n)} \right. \\
&\qquad - \left[(1 - \eta_{\tau(n)})x_{\tau(n)} - \alpha_{\tau(n)}w_{\tau(n)} \right] \Big\| + \alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\le (1 - \eta_{\tau(n)} - \alpha_{\tau(n)}\tau)\|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\le \|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)}\|w_{\tau(n)}\|,
\end{aligned}
$$

where $t_{\tau(n)} \in \partial_2 f(z_{\tau(n)}, z_{\tau(n)})$ and $w_{\tau(n)} \in \partial_2 f(x_{\tau(n)}, x_{\tau(n)})$. Since $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{w_{\tau(n)}\}$ and (40), we have $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$. As proved in the first case, we can conclude that

$$\limsup_{n\to\infty}\langle v, x^* - x_{\tau(n)+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{\tau(n)}\rangle \le 0. \tag{41}$$

Combining (25) and (35), we obtain

$$\left\| x_{\tau(n)+1} - x^* \right\|^2 \le (1 - \alpha_{\tau(n)}\tau) \left\| x_{\tau(n)} - x^* \right\|^2 + 2\alpha_{\tau(n)}\mu\langle v, x^* - x_{\tau(n)+1}\rangle$$
$$\le (1 - \alpha_{\tau(n)}\tau) \left\| x_{\tau(n)+1} - x^* \right\|^2 + 2\alpha_{\tau(n)}\mu\langle v, x^* - x_{\tau(n)+1}\rangle.$$

By using (35) again, we have

$$\left\| x_n - x^* \right\|^2 \le \left\| x_{\tau(n)+1} - x^* \right\|^2 \le \frac{2\mu}{\tau}\langle v, x^* - x_{\tau(n)+1}\rangle. \tag{42}$$

From (41), we can conclude that $\limsup_{n\to\infty} \|x_n - x^*\|^2 \le 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

4 The extragradient subgradient methods with line searches

In this section, we introduce the algorithm for finding the solution of a bilevel equilibrium problem without the Lipschitz condition for the bifunction g.

Algorithm 2

Initialization: Choose $x_0 \in C$, $0 < \mu < \frac{2\beta}{L^2}$, $\rho \in (0, 2)$, $\gamma \in (0, 1)$, the sequences $\{\lambda_n\}$, $\{\xi_n\}$, and $\{\alpha_n\} \subset (0, 1)$ such that

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, & \sum_{n=0}^{\infty} \alpha_n = \infty, & \sum_{n=0}^{\infty} \alpha_n^2 < \infty, \\ \lambda_n \in [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty), & \xi_n \in [\underline{\xi}, \overline{\xi}] \subset (0, 2). \end{cases}$$

Set $n = 0$, and go to Step 2.

  Step 1. Compute

$$y_n = \arg\min_{y\in C}\left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}.$$

If $y_n = x_n$, then set $u_n = x_n$ and go to Step 4. Otherwise, go to Step 2.

  Step 2. (Armijo line search rule) Find m as the smallest positive integer number satisfying

$$\begin{cases} z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n, \\ g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \ge \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2. \end{cases}$$

Set $z_n = z_{n,m}$ and $\gamma_n = \gamma^m$.

  Step 3. Choose $t_n \in \partial_2 g(z_n, x_n)$ and compute $u_n = P_C(x_n - \xi_n \sigma_n t_n)$ where $\sigma_n = \frac{g(z_n, x_n)}{\|t_n\|^2}$

  Step 4. Compute $w_n \in \partial_2 f(u_n, u_n)$ and

$$x_{n+1} = P_C(u_n - \alpha_n \mu w_n).$$

Set $n = n + 1$, and go back to Step 1.

Lemma 4.1 ([26]) *Suppose that $y_n \ne x_n$ for some $n \in \mathbb{N}$. Then the line search corresponding to x_n and y_n (Step 2) is well defined, $g(z_n, x_n) > 0$, and $0 \notin \partial_2 g(z_n, x_n)$.*

Lemma 4.2 ([26]) *Let $g : \Theta \times \Theta \to \infty$ be a bifunction satisfying conditions* (B1) *on C and* (B5) *on Θ, where Θ is an open convex set containing C. Let $\bar{x}, \bar{y} \in \Theta$ and $\{x_n\}, \{y_n\}$ be two sequences in Θ converging weakly to $\bar{x}, \bar{y} \in \Theta$, respectively. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that*

$$\partial_2 g(x_n, y_n) \subset \partial_2 g(\bar{x}, \bar{y}) + \frac{\varepsilon}{\eta} B$$

for all $n \geq n_\varepsilon$, where B denotes the closed unit ball in H.

Lemma 4.3 ([8]) *Let the bifunction g satisfy assumption* (B1) *on $C \times C$,* (B5) *on $\Theta \times \Theta$. Suppose that $\{x_n\}$ is a bounded sequence in $C, \rho > 0$ and $\{y_n\}$ is a sequence such that*

$$y_n = \arg\min_{y \in C} \left\{ g(x_n, y) + \frac{\rho}{2} \|y - x_n\|^2 \right\}.$$

Then $\{y_n\}$ is also bounded.

Theorem 4.4 *Let the bifunction f satisfy Condition A and g satisfy Conditions* (B1)–(B3) *and* (B5). *Assume that $\Omega \neq \emptyset$. Then the sequences $\{x_n\}$ generated by Algorithm 2 converge strongly to the unique solution of the bilevel equilibrium problem* (2).

Proof Let x^* be the unique solution of the bilevel equilibrium problem (2). Then we have $x^* \in \Omega$ and there exists $v \in \partial_2 f(x^*, x^*)$ such that

$$\langle v, z - x^* \rangle \geq 0 \quad \text{for all } z \in \Omega. \tag{43}$$

Step 1: Show that

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \xi_n(2 - \xi_2)(\sigma_n \|t_n\|)^2. \tag{44}$$

By the definition of u_n, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|P_C(x_n - \xi_n \sigma_n t_n) - P_C(x^*)\|^2 \\
&\leq \|x_n - \xi_n \sigma_n t_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - 2\xi_n \sigma_n \langle t_n, x_n - x^* \rangle + (\xi_n \sigma_n \|t_n\|)^2.
\end{aligned} \tag{45}$$

Since $t_n \in \partial_2 g(z_n, x_n)$ and $g(z_n, \cdot)$ is convex on C, we have $g(z_n, x^*) - g(z_n, x_n) \geq \langle t_n, x^* - x_n \rangle$. It follows that

$$\langle t_n, x_n - x^* \rangle \geq g(z_n, x_n) - g(z_n, x^*). \tag{46}$$

Since g is pseudomonotone on C with respect to Ω, we have $g(z_n, x^*) \leq 0$. It follows from (46) and the definition of σ_n that

$$\langle t_n, x_n - x^* \rangle \geq g(z_n, x_n) = \sigma_n \|t_n\|^2. \tag{47}$$

Combining (45) with (47), we obtain

$$
\begin{aligned}
\left\|u_n - x^*\right\|^2 &\leq \left\|x_n - x^*\right\|^2 - 2\xi_n\sigma_n\left(\sigma_n\|t_n\|^2\right) + \left(\xi_n\sigma_n\|t_n\|\right)^2 \\
&= \left\|x_n - x^*\right\|^2 - 2\xi_n\left(\sigma_n\|t_n\|\right)^2 + \xi_n{}^2\left(\sigma_n\|t_n\|\right)^2 \\
&= \left\|x_n - x^*\right\|^2 - \xi_n(2 - \xi_n)\left(\sigma_n\|t_n\|\right)^2.
\end{aligned}
$$

Step 2: The sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{w_n\}$ are bounded.

Since $\xi_n \in [\underline{\xi}, \overline{\xi}] \subset (0, 2)$ and (44), we have

$$
\left\|u_n - x^*\right\| \leq \left\|x_n - x^*\right\|. \tag{48}
$$

By the definition of x_{n+1}, we get

$$
\begin{aligned}
\left\|x_{n+1} - x^*\right\| &= \left\|P_C(u_n - \alpha_n\mu w_n) - P_C\left(x^*\right)\right\| \\
&\leq \left\|u_n - \alpha_n\mu w_n - x^*\right\| \\
&\leq \left\|\left(u_n - x^*\right) - \alpha_n\mu(w_n - v) - \alpha_n\mu v\right\| \\
&= \left\|\left(u_n - x^*\right) - \alpha_n\mu(w_n - v)\right\| + \alpha_n\mu\|v\|. \tag{49}
\end{aligned}
$$

From Lemma 2.7, (48), and (49), we can conclude that

$$
\begin{aligned}
\left\|x_{n+1} - x^*\right\| &= (1 - \alpha_n\tau)\left\|u_n - x^*\right\| + \alpha_n\mu\|v\| \\
&\leq (1 - \alpha_n\tau)\left\|x_n - x^*\right\| + \alpha_n\tau\left(\frac{\mu\|v\|}{\tau}\right). \tag{50}
\end{aligned}
$$

This implies that

$$
\left\|x_{n+1} - x^*\right\| \leq \max\left\{\left\|x_n - x^*\right\|, \frac{\mu\|v\|}{\tau}\right\}.
$$

By induction, we obtain

$$
\left\|x_n - x^*\right\| \leq \max\left\{\left\|x_0 - x^*\right\|, \frac{\mu\|v\|}{\tau}\right\}.
$$

Thus the sequence $\{x_n\}$ is bounded. Hence we can conclude from (48) and Lemma 4.3 that $\{y_n\}$ and $\{u_n\}$ are bounded, respectively. From condition (A4), we have $\{w_n\}$ is also bounded.

Step 3: We show that if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $\bar{x}$ and $\lim_{k\to\infty}(\sigma_{n_k}\|t_{n_k}\|) = 0$, then we have $\bar{x} \in \Omega$.

Firstly, we will show that $\{t_{n_k}\}$ is bounded. Since $\{z_n\}$ is bounded, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging weakly to $\bar{z}$ By using Lemma 4.2, for any $\varepsilon > 0$, there exist $\eta > 0$ and k_0 such that

$$
\partial_2 g(z_{n_k}, x_{n_k}) \subset \partial_2 g(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta}B
$$

for all $k \geq k_0$. Since $\{t_{n_k}\} \in \partial_2 g(z_{n_k}, x_{n_k})$, we have $\{t_{n_k}\}$ is bounded. Next, we show that $\|x_{n_k} - y_{n_k}\| \to 0$. Without loss of generality, we can assume that $x_{n_k} \neq y_{n_k}$ for all $k \in \mathbb{N}$. By Lemma 4.1, we obtain $g(z_{n_k}, x_{n_k}) > 0$ and $t_{n_k} \neq 0$. Since $\lim_{k \to \infty}(\sigma_{n_k} \|t_{n_k}\|) = 0$ and $\{t_{n_k}\}$ is bounded, we have

$$\lim_{k \to \infty} g(z_{n_k}, x_{n_k}) = \lim_{k \to \infty} \left(\sigma_{n_k} \|t_{n_k}\|\right) \|t_{n_k}\| = 0. \tag{51}$$

It follows from the convexity of $g(z_{n_k}, \cdot)$ that

$$\gamma_{n_k} g(z_{n_k}, y_{n_k}) + (1 - \gamma_{n_k}) g(z_{n_k}, x_{n_k}) \geq g(z_{n_k}, z_{n_k}) = 0.$$

This implies that

$$\gamma_{n_k} \left[g(z_{n_k}, x_{n_k}) - g(z_{n_k}, y_{n_k}) \right] \leq g(z_{n_k}, x_{n_k}). \tag{52}$$

By the Armijo line search, we get $\frac{\rho}{2\lambda_n} \|x_n - y_n\|^2 \leq g(z_{n_k}, x_{n_k}) - g(z_{n_k}, y_{n_k})$, and (52) implies that

$$\frac{\rho \gamma_{n_k}}{2\lambda_{n_k}} \|x_n - y_n\|^2 \leq \gamma_{n_k} \left[g(z_{n_k}, x_{n_k}) - g(z_{n_k}, y_{n_k}) \right] \leq g(z_{n_k}, x_{n_k}). \tag{53}$$

Combining (51) with (53), we obtain

$$\lim_{k \to \infty} \gamma_{n_k} \|x_n - y_n\|^2 = 0. \tag{54}$$

Then we consider two cases.

Case 1. $\limsup_{k \to \infty} \gamma_{n_k} > 0$. There exist $\bar{\gamma} > 0$ and a subsequence of $\{\gamma_{n_k}\}$ denoted again by $\{\gamma_{n_k}\}$ such that $\gamma_{n_k} > \bar{\gamma}$ for all k. So we get from (54) that

$$\lim_{k \to \infty} \|x_{n_k} - y_{n_k}\| = 0. \tag{55}$$

Since $x_{n_k} \rightharpoonup \bar{x}$ and (55), we have $y_{n_k} \rightharpoonup \bar{x}$. On the other hand, by the definition of y_{n_k}, we have

$$\lambda_{n_k} \left\{ g(x_{n_k}, y) - g(x_{n_k}, y_{n_k}) \right\} \geq \langle x_{n_k} - y_{n_k}, y - y_{n_k} \rangle \quad \text{for all } y \in C. \tag{56}$$

Therefore

$$\lambda_{n_k} \left\{ g(x_{n_k}, y) - g(x_{n_k}, y_{n_k}) \right\} \geq -\|x_{n_k} - y_{n_k}\| \|y - y_{n_k}\| \quad \text{for all } y \in C.$$

Letting $k \to \infty$ in the above inequality, using (55) and the jointly weak continuity of g, we have $g(\bar{x}, y) - g(\bar{x}, \bar{x}) \geq 0$ for all $y \in C$. So, $g(\bar{x}, y) \geq 0$ for all $y \in C$. Hence $\bar{x} \in \Omega$.

Case 2. $\limsup_{k \to \infty} \gamma_{n_k} = 0$. From the boundedness of $\{y_n\}$, there exists $\{y_{n_k}\} \subseteq \{y_n\}$ such that $y_{n_k} \rightharpoonup \bar{y}$. Let $\{m_k\}$ be the sequence of the smallest non-negative integers such that $z_{n_k} = (1 - \gamma^{m_k}) x_{n_k} + \gamma^{m_k} y_{n_k}$ and

$$g(z_{n_k}, x_{n_k}) - g(z_{n_k}, y_{n_k}) \geq \frac{\rho}{2\lambda_{n_k}} \|x_{n_k} - y_{n_k}\|^2.$$

Since $\gamma^{m_k} \to 0$, we have $m_k > 0$. It follows from the Armijo line search that, for $m_k - 1$, we have $\bar{z}_{n_k} = (1 - \gamma^{m_k-1})x_{n_k} + \gamma^{m_k-1}y_{n_k}$ and

$$g(\bar{z}_{n_k}, x_{n_k}) - g(\bar{z}_{n_k}, y_{n_k}) < \frac{\rho}{2\lambda_{n_k}} \|x_{n_k} - y_{n_k}\|^2. \tag{57}$$

On the other hand, by the definition of y_n, we have

$$\lambda_n \{g(x_n, y) - g(x_n, y_n)\} \geq \langle x_n - y_n, y - y_n \rangle \quad \text{for all } y \in C. \tag{58}$$

Letting $n = n_k$ and $y = x_{n_k}$ in (58), we get

$$-\lambda_{n_k}g(x_{n_k}, y_{n_k}) \geq \|x_{n_k} - y_{n_k}\|^2. \tag{59}$$

Combining (57) with (59), we obtain

$$g(\bar{z}_{n_k}, x_{n_k}) - g(\bar{z}_{n_k}, y_{n_k}) < -\frac{\rho}{2}g(x_{n_k}, y_{n_k}). \tag{60}$$

Since $x_{n_k} \rightharpoonup \bar{x}, y_{n_k} \rightharpoonup \bar{y}$, and $\gamma_{n_k} \to 0$, we have $\bar{z}_{n_k} \rightharpoonup \bar{x}$. From (60) and g is jointly weakly continuous on $H \times H$, we get $-g(\bar{x}, \bar{y}) < -\frac{\rho}{2}g(\bar{x}, \bar{y})$. Since $\rho \in (0, 2)$, we have $g(\bar{x}, \bar{y}) \geq 0$. Taking $k \to \infty$ in (59), we obtain $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$. By Case 1, it is immediate that $\bar{x} \in \Omega$.

Step 4: Show that the sequence $\{x_n\}$ converges strongly to x^*. By using the definition of x_{n+1} and (44), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_C(u_n - \alpha_n\mu w_n - P_C(x^*))\|^2 \\
&\leq \|u_n - \alpha_n\mu w_n - x^*\|^2 \\
&= \|u_n - x^*\|^2 - 2\alpha_n\mu\langle w_n, u_n - x^*\rangle + (\alpha_n\mu\|w_n\|)^2 \\
&\leq \|x_n - x^*\|^2 - \xi_n(2 - \xi_n)(\sigma_n\|t_n\|)^2 \\
&\quad - 2\alpha_n\mu\langle w_n, u_n - x^*\rangle + (\alpha_n\mu\|w_n\|)^2 \\
&\leq \|x_n - x^*\|^2 - (\sigma_n\|t_n\|)^2 - 2\alpha_n\mu\langle w_n, u_n - x^*\rangle \\
&\quad + (\alpha_n\mu\|w_n\|)^2.
\end{aligned} \tag{61}
$$

Setting $a_n = \|x_n - x^*\|^2$. It follows from the boundedness of $\{w_n\}$ and $\{u_n\}$ that

$$|\langle w_n, u_n - x^*\rangle| \leq M_1 \quad \text{and} \quad \|w_n\|^2 \leq M_2. \tag{62}$$

Combining (61), (62) with the definition of a_n, we get

$$a_{n+1} - a_n + (\sigma_n\|t_n\|)^2 \leq 2\alpha_n\mu M_1 + \alpha_n^2\mu^2 M_2. \tag{63}$$

Let us consider two cases.

Case 1: There exists n_0 such that $\{a_n\}$ is decreasing for $n \geq n_0$. Therefore the limit of $\{a_n\}$ exists, denoted by a. It follows that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$. From (63) and the definition of

α_n, we have

$$\lim_{n \to \infty} \left(\sigma_n \| t_n \| \right) = 0. \tag{64}$$

By the definitions of x_{n+1} and u_n, we obtain

$$
\begin{aligned}
\| u_n - x_n \|^2 &= \left\| P_C(x_n - \sigma_n t_n) - P_C(x_n) \right\| \\
&\leq \| x_n - \sigma_n t_n - x_n \| \\
&= \sigma_n \| t_n \|.
\end{aligned} \tag{65}
$$

It follows that $\lim_{n \to \infty} \| u_n - x_n \|^2 = 0$, which implies that

$$\lim_{n \to \infty} \left\| u_n - x^* \right\|^2 = a. \tag{66}$$

Since $\{u_n\} \subseteq C$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to some $\bar{u} \in C$ and satisfies the equality

$$\liminf_{n \to \infty} \langle u_n - x^*, v \rangle = \lim_{k \to \infty} \langle u_{n_k} - x^*, v \rangle.$$

Since $u_{n_k} \rightharpoonup \bar{u} \in C$ and (43), we have

$$\liminf_{n \to \infty} \langle u_n - x^*, v \rangle = \lim_{k \to \infty} \langle u_{n_k} - x^*, v \rangle = \langle \bar{u} - x^*, v \rangle \geq 0. \tag{67}$$

Since $w_n \in \partial_2 f(u_n, u_n), v \in \partial_2 f(x^*, x^*)$ and f is β-strongly monotone on C, we have

$$
\begin{aligned}
\langle w_n, x^* - u_n \rangle &\leq f(u_n, x^*) \leq -\beta \| u_n - x^* \|^2 - f(x^*, u_n) \\
&= -\beta \| u_n - x^* \|^2 - \langle u_n - x^*, v \rangle.
\end{aligned}
$$

This implies that $\langle w_n, u_n - x^* \rangle \geq \beta \| u_n - x^* \|^2 + \langle u_n - x^*, v \rangle$. Combining (66), (67) with the above inequality, we get

$$\liminf_{n \to \infty} \langle w_n, u_n - x^* \rangle \geq \beta a. \tag{68}$$

Assume that $a > 0$. Choose $\varepsilon = \frac{1}{2} \beta a$. There exists n_0 such that

$$\langle u_n - x^*, w_n \rangle \geq \beta a - \varepsilon = \frac{1}{2} \beta a \quad \text{for all } n \geq n_0.$$

It follows from (61) and the above inequality that $a_{n+1} - a_n \leq -\alpha_n \mu \beta a^2 + \alpha_n^2 \mu M_2$. Summing this inequality from n_0 to n, we obtain

$$a_{n+1} - a_{n_0} \leq -\mu \beta a^2 \sum_{k=n_0}^{n} \alpha_k + \mu M_2 \sum_{k=n_0}^{n} \alpha_n^2. \tag{69}$$

Since $\sum_{k=n_0}^{\infty} \alpha_k = \infty$ and $\sum_{k=n_0}^{\infty} \alpha_n^2 < \infty$, we can conclude from (69) that $\liminf_{n \to \infty} a_n = -\infty$, which is a contradiction. So, $a = 0$, and we can conclude that $\lim_{n \to \infty} \| x_n - x^* \|^2 = 0$.

Case 2: There exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \leq a_{n_j+1}$ for all $j \in \mathbb{N}$. Let $\{\tau(n)\}$ be a nondecreasing sequence defined in Lemma 2.6. Thus

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}. \tag{70}$$

From (63) and the definition of α_n, we have

$$\lim_{n \to \infty} \sigma_{\tau(n)} \|t_{\tau(n)}\| = 0. \tag{71}$$

By the definition of $u_{\tau(n)}$, we obtain

$$
\begin{aligned}
\|u_{\tau(n)} - x_{\tau(n)}\|^2 &= \left\| P_C(x_{\tau(n)} - \sigma_{\tau(n)} t_{\tau(n)}) - P_C(x_{\tau(n)}) \right\| \\
&\leq \|x_{\tau(n)} - \sigma_{\tau(n)} t_{\tau(n)} - x_{\tau(n)}\| \\
&= \sigma_{\tau(n)} \|t_{\tau(n)}\|.
\end{aligned} \tag{72}
$$

This implies that $\lim_{n \to \infty} \|u_{\tau(n)} - x_{\tau(n)}\|^2 = 0$. Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{x_{\tau(n)_k}\} \subseteq \{x_{\tau(n)}\}$ such that $x_{\tau(n)_k} \rightharpoonup \bar{x} \in C$ and thus $u_{\tau(n)_k} \rightharpoonup \bar{x} \in C$. From (71) and Step 3 of this proof, we get $\bar{x} \in \Omega$. Next, we will show that $u_{\tau(n)_k} \to x^*$. It follows from (61) that

$$
\begin{aligned}
2\alpha_{\tau(n)_k} \mu \langle w_{\tau(n)_k}, u_{\tau(n)_k} - x^* \rangle &\leq a_{\tau(n)_k} - a_{\tau(n)_k+1} - \left(\sigma_{\tau(n)_k} \|t_{\tau(n)_k}\| \right)^2 \\
&\quad + \left(\alpha_{\tau(n)_k} \mu \|w_{\tau(n)_k}\| \right)^2 \leq (\alpha_{\tau(n)_k} \mu)^2 M_2.
\end{aligned}
$$

This implies that

$$\langle w_{\tau(n)_k}, u_{\tau(n)_k} - x^* \rangle \leq \frac{\alpha_{\tau(n)_k} \mu M_2}{2}. \tag{73}$$

Since f is β-strongly monotone on C and $w_{\tau(n)} \in \partial_2 f(u_{\tau(n)_k}, u_{\tau(n)_k})$, we have

$$\langle x^* - u_{\tau(n)_k}, w_{\tau(n)_k} \rangle \leq f(u_{\tau(n)_k}, x^*) \leq -\beta \|u_{\tau(n)_k} - x^*\|^2 - f(x^*, u_{\tau(n)_k}).$$

It follows from (73) and the above inequality that

$$
\begin{aligned}
\|u_{\tau(n)_k} - x^*\|^2 &\leq \frac{1}{\beta} \left[\langle w_{\tau(n)_k}, u_{\tau(n)_k} - x^* \rangle - f(x^*, u_{\tau(n)_k}) \right] \\
&\leq \frac{1}{\beta} \left[\frac{\alpha_{\tau(n)_k} \mu M_2}{2} - f(x^*, u_{\tau(n)_k}) \right].
\end{aligned} \tag{74}
$$

Taking $k \to \infty$, by using $u_{\tau(n)_k} \rightharpoonup \bar{x}$ and $\alpha_{\tau(n)_k} \to 0$, we get

$$\limsup_{k \to \infty} \|u_{\tau(n)_k} - x^*\|^2 \leq -f(x^*, \bar{x}) \leq 0.$$

Therefore $\lim_{k \to \infty} \|u_{\tau(n)_k} - x^*\|^2 = 0$. Then, it is easy to see that $\lim_{n \to \infty} \|u_{\tau(n)} - x^*\|^2 = 0$. By the definition of x_{n+1}, we have

$$
\begin{aligned}
\|x_{\tau(n)+1} - x^*\| &\leq \|u_{\tau(n)} - \alpha_{\tau(n)} \mu w_{\tau(n)} - x^*\| \\
&\leq \|u_{\tau(n)} - x^*\| + \alpha_{\tau(n)} \mu \|w_{\tau(n)}\|.
\end{aligned} \tag{75}
$$

Since $\{w_{\tau(n)}\}$ is bounded, and from the definition of $\alpha_{\tau(n)}$, we have $\lim_{n\to\infty}\|x_{\tau(n)+1}-x^*\|^2 = 0$. This means that $\lim_{n\to\infty}a_{\tau(n)+1} = 0$. It follows from (50) that $\lim_{n\to\infty}a_n = 0$. Hence $\lim_{n\to\infty}\|x_n-x^*\|^2 = 0$. This completes the proof. □

5 Numerical examples

Let $H = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n : -5 \le x_i \le 5, \forall i \in \{1,2,\dots,n\}\}$. Let the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g(x,y) = \langle Px + Qy, y - x\rangle \quad \text{for all } x, y \in \mathbb{R}^n,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P - Q$ is positive semidefinite. Then g is pseudomonotone on $\mathbb{R}^n$. Indeed, let $g(x,y) \ge 0$ for every $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}
g(y,x) \le g(x,y) + g(y,x) &= \langle Px + Qy, y - x\rangle + \langle Py + Qx, x - y\rangle \\
&= -\langle (P-Q)(x-y), x - y\rangle \le 0.
\end{aligned}$$

Next, we obtain that g is Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{2}\|P - Q\|$. Indeed, for each $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned}
g(x,y) + g(y,z) - g(x,z) &= \langle Px + Qy, y - x\rangle + \langle Py + Qz, z - y\rangle - \langle Px + Qz, z - x\rangle \\
&= \langle (P-Q)(x-y), y - z\rangle \\
&\ge -2\frac{\|P-Q\|}{2}\|x - y\|\|y - z\| \\
&\ge -\frac{\|P-Q\|}{2}\|x - y\|^2 - \frac{\|P-Q\|}{2}\|y - z\|^2,
\end{aligned}$$

where $\|P - Q\|$ is the spectral norm of the matrix $\|P - Q\|$, that is, the square root of the largest eigenvalue of the positive semidefinite matrix $(P - Q)^T(P - Q)$. It is easy to check that $\Omega \ne \emptyset$. Furthermore, we define the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$f(x,y) = \langle Ax + By, y - x\rangle \quad \text{for all } x, y \in \mathbb{R}^n,$$

with A and B being positive definite matrices defined by

$$B = N^T N + nI_n \quad \text{and} \quad A = B + M^T M + nI_n,$$

where M, N are randomly $n \times n$ matrices and I_n is the identity matrix. Then we have f is n-strongly monotone on $\mathbb{R}^n$. Indeed, let $x, y \in \mathbb{R}^n$, we get

$$\begin{aligned}
f(x,y) + f(y,x) &= \langle Ax + By, y - x\rangle + \langle Ay + Bx, x - y\rangle \\
&= -\langle (A-B)(x-y), x - y\rangle \\
&= -\langle M^T M + nI_n(x-y), x - y\rangle \\
&= -\langle M^T M(x-y), x - y\rangle - \langle nI_n(x-y), x - y\rangle
\end{aligned}$$

Table 1 The results computed on Algorithm 1

n	N.P.	Average iteration	Average times
5	10	72	0.7219
10	10	255	2.3859
50	10	1224	12.6235
100	10	1541	37.8766
500	10	1910	465.510

$$= -\left\| M(x - y) \right\|^2 - n\|x - y\|^2$$
$$\leq -n\|x - y\|^2.$$

Moreover, $\partial_2 f(x, x) = \{(A + B)x\}$ and $\|(A + B)x - (A + B)y\| \leq \|A + B\|\|x - y\|$ for all $x, y \in \mathbb{R}^n$. Thus the mapping $x \to \partial_2 g(x, x)$ is bounded and $\|A + B\|$-Lipschitz continuous on every bounded subset of H. In this example, we consider the quadratic optimization

$$\min_{x \in C} \left\{ \frac{1}{2} x^T H x + f^T x \right\}, \tag{76}$$

where H is a matrix, f and x are vectors. From the subproblem of solving y_k and z_k in Algorithm 1, we can consider problem (76).

We have tested for this example where $n = 5, 10, 50, 100$, and 500. Starting point x_0 is a randomly initial point. Take the parameters

$$\alpha_k = \frac{1}{k + 4}, \qquad \eta_k = \frac{k + 1}{3(k + 4)}, \qquad \lambda_k = \frac{1}{2\|P - Q\|}, \qquad \mu = \frac{2}{\|A + B\|^2}.$$

We have implemented Algorithm 1 for this problem in Matlab R2015 running on a Desktop with Intel(R) Core(TM) i5-7200u CPU 2.50 GHz, and 4 GB RAM, and we used the stopping criteria $\|x_{k+1} - x_k\| < \varepsilon$ with $\varepsilon = 0.001$ is a tolerance to cease the algorithm. Denote that

- N.P: the number of the tested problems.
- Average iteration: the average number of iterations.
- Average times: the average CPU-computation times (in s).

The computation results are reported in the following tables.

From the numerical result Table 1, we see that the sequence generated by our algorithms is convergent and effective for solving the solution of bilevel equilibrium problems.

6 Conclusions

We have proposed two iterative algorithms for finding the solution of a bilevel equilibrium problem in a real Hilbert space. The sequence generated by our algorithms converges strongly to the solution. Furthermore, we reported the numerical result to support our algorithm.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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