Vietnam J. Math.
DOI 10.1007/s10013-016-0237-4

Extragradient-Proximal Methods for Split Equilibrium and Fixed Point Problems in Hilbert Spaces

Bui Van Dinh[1] **· Dang Xuan Son**[2] **· Tran Viet Anh**[3]

Received: 31 May 2015 / Accepted: 19 October 2016

Abstract In this paper, we propose two new extragradient-proximal algorithms for solving split equilibrium and fixed point problems (SEFPP) in real Hilbert spaces, in which the first equilibrium bifunction is pseudomonotone, the second one is monotone, and the fixed point mappings are nonexpansive. By using the extragradient method incorporated with the proximal point algorithm and cutting techniques, we obtain algorithms for solving (SEFPP). Under certain conditions on parameters, the iteration sequences generated by the proposed algorithms are proved to be weakly and strongly convergent to a solution of (SEFPP). Our results improve and extend the previous results given in the literature.

Keywords Split equilibrium problem · Split fixed point problem · Nonexpansive mapping · Weak and strong convergence · Pseudomonotonicity

Mathematics Subject Classification (2010) 47H09 · 47J25 · 65K10 · 65K15 · 90C99

✉ Bui Van Dinh
vandinhb@gmail.com

Dang Xuan Son
dangsonmath@gmail.com

Tran Viet Anh
tranvietanh@outlook.com

[1] Faculty of Information Technology, Le Quy Don Technical University, Hanoi, Vietnam

[2] Hanoi University of Science, Vietnam National University, Hanoi, Vietnam

[3] Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology, Hanoi, Vietnam

Published online: 24 December 2016

1 Introduction

Let us assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces endowed with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. By "$\rightarrow$" we denote the strong convergence, while "$\rightharpoonup$" stands for the weak convergence. Let C and Q be two nonempty closed convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Given bifunctions $f : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ and nonexpansive mappings $S : C \to C$, $T : Q \to Q$, we consider the following split equilibrium and fixed point problem (SEFPP):

$$\text{Find } x^* \in C \text{ such that} \begin{cases} f(x^*, y) \geq 0 \ \ \forall y \in C, \\ Sx^* = x^*, \end{cases} \tag{1}$$

and such that

$$\text{the point } u^* = Ax^* \in Q \text{ solves} \begin{cases} g(u^*, v) \geq 0 \ \ \forall v \in Q, \\ Tu^* = u^*. \end{cases} \tag{2}$$

When looking separately at (SEFPP), (1) (also (2) when $A \equiv I$-the identity mapping) is the problem of finding common elements of the solution set of an equilibrium problem and the set of fixed points of a nonexpansive mapping. The motivation for studying such a problem is its possible application to mathematical models whose constraints can be described by fixed point problems and/or equilibrium problems. This happens, especially, in the practical problems as signal processing, network resource allocation, image recovery [18, 19, 23, 24], and Nash–Cournot oligopolistic equilibrium models in economy [17, 20].

Problem (SEFPP) is quite general, in the sense that it includes, as special cases, many mathematical models which have been studied intensively by several researchers recently: split equilibrium problems (SEP) [12, 16, 27], split variational inequality problems (SVIP) [8], split common fixed point problems (SCFPP) [9, 21, 28], and the split feasibility problems (SFP) which have been used for studying medical image reconstruction, intensity-modulated radiation therapy, sensor networks, and data compression, see [2, 6, 7, 10] and the references quoted therein.

Let us denote the solution set of the equilibrium problem EP(C, f) by Sol(C, f), i.e.,

$$\mathrm{Sol}(C, f) = \{x^* \in C : \ f(x^*, y) \geq 0 \ \ \forall y \in C\}$$

and the set of fixed points of the mapping S by Fix(S), i.e.,

$$\mathrm{Fix}(S) = \{x^* \in C : \ Sx^* = x^*\}.$$

To find a common element of the set of fixed points of a nonexpansive mapping and the solution set of an equilibrium problem, Tada and Takahashi [33] proposed to combine Mann's iterative scheme [25] for the fixed point map with the proximal point algorithm [29] for equilibrium problem. In detail, the sequences $\{x^k\}$, $\{y^k\}$ are calculated as follows:

$$\begin{cases} x^0 \in C; \{\lambda_k\} \subset (0, \infty); \{\alpha_k\} \subset (0, 1), \\ y^k \in C \text{ such that } f(y^k, y) + \dfrac{1}{\lambda_k}\langle y - y^k, y^k - x^k \rangle \geq 0 \ \ \forall y \in C, \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k) S y^k. \end{cases} \tag{3}$$

It was proved (see [33, Theorem 4.1]) that if f is monotone and the parameters are chosen such that $\{\alpha_k\} \subset [a, b] \subset (0, 1)$ and $\lambda_k \geq \underline{\lambda} > 0 \ \forall k$, then the sequences $\{x^k\}$, $\{y^k\}$ generated by (3) converge weakly to $x^* \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S)$.

For obtaining a solution of (SEP), He [16] introduced an iterative method, which can be considered as a combination of the proximal point algorithm for equilibrium problem and the CQ-algorithm proposed by Byrne [5] for (SFP). The sequences $\{x^k\}$, $\{y^k\}$ are generated by the following scheme

$$\begin{cases} x^0 \in C; \{\lambda_k\} \subset (0, \infty); \mu > 0, \\ y^k \in C \text{ such that } f(y^k, y) + \dfrac{1}{\lambda_k}\langle y - y^k, y^k - x^k \rangle \geq 0 \ \forall y \in C, \\ u^k \in Q \text{ such that } g(u^k, v) + \dfrac{1}{\lambda_k}\langle v - u^k, u^k - Ay^k \rangle \geq 0 \ \forall v \in Q, \\ x^{k+1} = P_C(y^k + \mu A^*(u^k - Ay^k)). \end{cases} \tag{4}$$

Here, A^* is the adjoint operator of A. Under certain conditions on parameters, the author showed that $\{x^k\}$, $\{y^k\}$ generated by (4) converge weakly to a solution of (SEP) provided that f and g are monotone bifunctions on C and Q, respectively.

It should be noted that for each $x^k \in C$, $l_k(x, y) = \frac{1}{\lambda_k}\langle y - x, x - x^k \rangle$ is strongly monotone on C with constant $\tau_k = \frac{1}{\lambda_k}$ (see Definition 1 below). Hence, if f is monotone on C, then the bifunction $f_k(x, y) = f(x, y) + l_k(x, y)$ is strongly monotone with constant τ_k, and therefore, to find y^k in schemes (3) and (4), we can apply some existing methods, see, for instance [3, 30]. However, if f is pseudomonotone on C, the bifunction f_k may not be strongly monotone, even not be pseudomonotone on C; see, [36, Counterexample 2.1], [14, Example 2.8], so we cannot apply the available methods using the monotonicity of bifunction f_k to find y^k directly.

To solve EP(C, f) when f is pseudomonotone, Tran et al. [37] suggested the use of the extragradient algorithm introduced by Korpelevich [22] (see also [13, 15] for more details on extragradient algorithms) for finding saddle points and other related problems. The sequence $\{x^k\}$ is computed as follows:

$$\begin{cases} x^0 \in C; \{\lambda_k\} \subset (0, \infty), \\ y^k = \arg\min \left\{ \lambda_k f(x^k, y) + \dfrac{1}{2}\|y - x^k\|^2 : y \in C \right\}, \\ x^{k+1} = \arg\min \left\{ \lambda_k f(y^k, y) + \dfrac{1}{2}\|y - x^k\|^2 : y \in C \right\}. \end{cases} \tag{5}$$

They pointed out that the sequence $\{x^k\}$ generated by (5) converges weakly to a solution of EP(C, f) under the main assumptions that f is pseudomonotone and Lipschitz-continuous on C.

Motivated by these facts, in this paper, we consider (SEFPP) when f is pseudomonotone, g is monotone, and S, T are nonexpansive mappings. More precisely, we propose to use the extragradient algorithm for EPs in $\mathcal{H}_1$ and the proximal point algorithm for EPs in $\mathcal{H}_2$, and Mann's method for fixed point mappings to design the weak convergence algorithm for (SEFPP). We then combine the first one with the hybrid cutting technique [35] to get the strong convergence algorithm. One advantage of our algorithms is that it could be applied for (SEFPP) when the first bifunction is pseudomonotone and at each iteration one only has to solve two strongly convex optimization problems and one regularized equilibrium problem instead of solving two equilibrium problems as in He's methods.

The rest of this paper is organized as follows. In the next section, we recall some preliminary results which will be used later. The first part of the main section is devoted to prove the weak convergence theorem and it corollaries. Then, we combine the proposed method with the hybrid projected method for obtaining the strong convergence theorem. Some special cases of (SEFPP) are also considered.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and let C be a nonempty closed convex subset of $\mathcal{H}$. By P_C, we denote the metric projection onto C. Namely, for each $x \in \mathcal{H}$, $P_C(x)$ is the unique element in C such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C.$$

It is well-known that the metric projection P_C has the following characterizations

Lemma 1 *Suppose that C is a nonempty closed convex subset in $\mathcal{H}$. Then, P_C has the following properties:*

(a) *$P_C(x)$ is a singleton and well defined for every $x \in \mathcal{H}$.*
(b) *For each $x \in \mathcal{H}$, $z = P_C(x)$ if and only if $\langle x - z, y - z \rangle \leq 0 \ \forall y \in C$.*
(c) *$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle \ \forall x, y \in \mathcal{H}$.*
(d) *$\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|x - P_C(x) - y + P_C(y)\|^2 \ \forall x, y \in \mathcal{H}$.*

Lemma 2 *Let $\mathcal{H}$ be a real Hilbert space, then for all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$, we have*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Definition 1 [4, 26, 31] Let $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction, and C be a nonempty, closed and convex subset of $\mathcal{H}$. Let D also be a nonempty subset of C. The bifunction f is said to be:

(a) strongly monotone with constant $\tau > 0$ if

$$f(x, y) + f(y, x) \leq -\tau\|x - y\|^2 \quad \forall x, y \in C;$$

(b) monotone on C if

$$f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in C;$$

(c) pseudomonotone on C if

$$\forall x, y \in C, \quad f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

(d) pseudomonotone on C with respect to D if

$$\forall x^* \in D, \ \forall y \in C, \quad f(x^*, y) \geq 0 \Rightarrow f(y, x^*) \leq 0;$$

(e) Lipschitz-type continuous on C if there exist positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \quad \forall x, y, z \in C.$$

From Definition 1 we have the following properties:

(i) (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).
(ii) If $f(x, y) = \langle F(x), y - x \rangle$ for a mapping $F : \mathcal{H} \to \mathcal{H}$, then the notions of monotonicity for the mapping F corresponds to the notions of monotonicity for the bifunction f, respectively. In addition, if the mapping F is L-Lipschitz on C, i.e., $\|F(x) - F(y)\| \leq L\|x - y\| \ \forall x, y \in C$, then for any $\epsilon > 0$, f is also Lipschitz-type continuous on C (see [26, 37]), for example, with constants $c_1 = \frac{L}{2\epsilon}$, $c_2 = \frac{L\epsilon}{2}$.

Lemma 3 (Opial's condition)[32] *For any sequence* $\{x^k\} \subset \mathcal{H}$ *with* $x^k \rightharpoonup x$*, the inequality*

$$\liminf_{k\to+\infty} \|x^k - x\| < \liminf_{k\to+\infty} \|x^k - y\|$$

holds for each $y \in \mathcal{H}$ *with* $y \neq x$*.*

3 Main Results

Now, let $g : Q \times Q \to \mathbb{R}$ and $f : C \times C \to \mathbb{R}$ be bifunctions. We start this section with the following widely used assumptions in monotone and pseudomonotone equilibrium problems.

Assumptions $\mathcal{A}$

(A_1) $g(u, u) = 0$ for all $u \in Q$;
(A_2) g is monotone on Q;
(A_3) for each $u, v, w \in Q$

$$\limsup_{\lambda\downarrow 0} g(\lambda w + (1-\lambda)u, v) \leq g(u, v);$$

(A_4) $g(u, \cdot)$ is convex and lower semicontinuous on Q for each $u \in Q$.

Assumptions $\mathcal{B}$

(B_1) $f(x, x) = 0$ for all $x \in C$.
(B_2) f is pseudomonotone on C with respect to $\mathrm{Sol}(C, f)$.
(B_3) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x^k\}$, $\{y^k\}$ $\subset C$ converge weakly to x and y, respectively, then $f(x^k, y^k) \to f(x, y)$ as $k \to \infty$.
(B_4) $f(x, \cdot)$ is convex, subdifferentiable on C for all $x \in C$.
(B_5) f is Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$.

The following lemma is well-known in theory of monotone equilibrium problems.

Lemma 4 [4] *Let* g *satisfy Assumption* $\mathcal{A}$*. Then, for each* $\alpha > 0$ *and* $u \in \mathcal{H}$*, there exists* $w \in Q$ *such that*

$$g(w, v) + \frac{1}{\alpha}\langle v - w, w - u\rangle \geq 0 \quad \forall v \in Q.$$

The following lemmas give us a connection between monotone equilibrium problems and fixed point problems.

Lemma 5 [11] *Under assumptions of Lemma 4, the mapping* T_α^g *defined on* $\mathcal{H}$ *by*

$$T_\alpha^g(u) = \left\{ w \in Q : g(w, v) + \frac{1}{\alpha}\langle v - w, w - u\rangle \geq 0 \ \forall v \in Q \right\},$$

has the following properties:

(i) T_α^g *is single-valued.*

(ii) T^g_α *is firmly nonexpansive, i.e., for any* $u, v \in \mathcal{H}$,

$$\|T^g_\alpha(u) - T^g_\alpha(v)\|^2 \le \langle T^g_\alpha(u) - T^g_\alpha(v), u - v\rangle.$$

(iii) $\mathrm{Fix}(T^g_\alpha) = \mathrm{Sol}(Q, g)$.
(iv) $\mathrm{Sol}(Q, g)$ *is closed and convex.*

Lemma 6 [16] *Under assumptions of Lemma 5, for* $\alpha, \beta > 0$ *and* $u, v \in \mathcal{H}$, *one has*

$$\|T^g_\alpha(u) - T^g_\beta(v)\| \le \|v - u\| + \frac{|\beta - \alpha|}{\beta}\|T^g_\beta(v) - v\|.$$

Now, we are in a position to present our main results.

Theorem 1 (Weak convergence theorem) *Let* C, Q *be two nonempty closed convex subsets in* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively. Let* $S : C \to C$; $T : Q \to Q$ *be nonexpansive mappings, and bifunctions* g, f *satisfy Assumptions* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively. Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator with its adjoint* A^*. *Take* $x^1 \in C$; $\{\lambda_k\} \subset [a, b]$ *for some* $a, b \in (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$; $0 < \alpha < 1$; $0 < \mu < \frac{1}{\|A\|^2}$; $\{\alpha_k\} \subset (0, +\infty)$ *with* $\liminf_{k\to\infty} \alpha_k > 0$, *and consider the sequences* $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, $\{t^k\}$, *and* $\{u^k\}$ *defined by*

$$\begin{cases} y^k = \arg\min\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \arg\min\left\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ t^k = (1 - \alpha)z^k + \alpha S z^k, \\ u^k = T^g_{\alpha_k} A t^k, \\ x^{k+1} = P_C(t^k + \mu A^*(T u^k - A t^k)). \end{cases} \tag{6}$$

If $\Omega = \{x^* \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S) : Ax^* \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)\} \neq \emptyset$, *then the sequences* $\{x^k\}$ *and* $\{z^k\}$ *converge weakly to an element* $p \in \Omega$ *and* $\{u^k\}$ *converges weakly to* $Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)$.

Remark 1 Since f satisfies assumption (B_4), $\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2$ is strongly convex with modulus 1 on C, y^k is well defined. Assumptions $\mathcal{A}$ and Lemma 4 imply that u^k is also well defined. Therefore, the sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, $\{t^k\}$, and $\{u^k\}$ determined by (6) are well defined.

The following lemma obtained in [1] is needed for proving Theorem 1.

Lemma 7 [1] *Suppose that* $x^* \in \mathrm{Sol}(C, f)$, $f(x, \cdot)$ *is convex and subdifferentiable on* C *for all* $x \in C$, *and* f *is pseudomonotone on* C. *Then, we have*

(i) $\lambda_k[f(x^k, y) - f(x^k, y^k)] \ge \langle y^k - x^k, y^k - y\rangle \ \forall y \in C$.
(ii) $\|z^k - x^*\|^2 \le \|x^k - x^*\|^2 - (1 - 2\lambda_k c_1)\|x^k - y^k\|^2 - (1 - 2\lambda_k c_2)\|y^k - z^k\|^2 \ \forall k$.

Now, let us prove Theorem 1.

Proof The proof of Theorem 1 is divided into several steps.

Step 1. $\lim_{k\to\infty} \|x^k - x^*\|$ *exists for all* $x^* \in \Omega$.

Take $x^* \in \Omega$, i.e., $x^* \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S)$ and $Ax^* \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)$. By the definition of t^k, we have

$$\begin{aligned} \|t^k - x^*\| &= \|(1-\alpha)z^k + \alpha S z^k - x^*\| \\ &= \|(1-\alpha)(z^k - x^*) + \alpha(Sz^k - Sx^*)\| \\ &\le (1-\alpha)\|z^k - x^*\| + \alpha\|Sz^k - Sx^*\| \\ &\le (1-\alpha)\|z^k - x^*\| + \alpha\|z^k - x^*\| \\ &= \|z^k - x^*\|. \end{aligned}$$

From $\{\lambda_k\} \subset [a, b] \subset \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$ and Lemma 7, we get

$$\begin{aligned} \|z^k - x^*\|^2 &\le \|x^k - x^*\|^2 - (1 - 2\lambda_k c_1)\|x^k - y^k\|^2 - (1 - 2\lambda_k c_2)\|y^k - z^k\|^2 \\ &\le \|x^k - x^*\|^2. \end{aligned}$$

Thus,

$$\|t^k - x^*\| \le \|z^k - x^*\| \le \|x^k - x^*\|. \tag{7}$$

It follows from Lemma 5 that

$$\begin{aligned} \|T^g_{\alpha_k} At^k - Ax^*\|^2 &= \|T^g_{\alpha_k} At^k - T^g_{\alpha_k} Ax^*\|^2 \\ &\le \langle T^g_{\alpha_k} At^k - T^g_{\alpha_k} Ax^*, At^k - Ax^*\rangle \\ &= \langle T^g_{\alpha_k} At^k - Ax^*, At^k - Ax^*\rangle \\ &= \frac{1}{2}[\|T^g_{\alpha_k} At^k - Ax^*\|^2 + \|At^k - Ax^*\|^2 \\ &\quad -\|T^g_{\alpha_k} At^k - At^k\|^2]. \end{aligned}$$

Hence,

$$\|T^g_{\alpha_k} At^k - Ax^*\|^2 \le \|At^k - Ax^*\|^2 - \|T^g_{\alpha_k} At^k - At^k\|^2.$$

Combining this fact with the nonexpansiveness of the mapping T, one has

$$\begin{aligned} \|Tu^k - Ax^*\|^2 &= \|TT^g_{\alpha_k} At^k - TAx^*\|^2 \\ &\le \|T^g_{\alpha_k} At^k - Ax^*\|^2 \\ &\le \|At^k - Ax^*\|^2 - \|T^g_{\alpha_k} At^k - At^k\|^2. \end{aligned} \tag{8}$$

Using (8), we obtain

$$\begin{aligned} \langle A(t^k - x^*), Tu^k - At^k\rangle &= \langle A(t^k - x^*) + Tu^k - At^k - (Tu^k - At^k), Tu^k - At^k\rangle \\ &= \langle Tu^k - Ax^*, Tu^k - At^k\rangle - \|Tu^k - At^k\|^2 \\ &= \frac{1}{2}[\|Tu^k - Ax^*\|^2 + \|Tu^k - At^k\|^2 - \|At^k - Ax^*\|^2] \\ &\quad -\|Tu^k - At^k\|^2 \\ &= \frac{1}{2}[(\|Tu^k - Ax^*\|^2 - \|At^k - Ax^*\|^2) - \|Tu^k - At^k\|^2] \\ &\le -\frac{1}{2}\|T^g_{\alpha_k} At^k - At^k\|^2 - \frac{1}{2}\|Tu^k - At^k\|^2. \end{aligned} \tag{9}$$

It implies from (9) that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &= \|P_C(t^k+\mu A^*(Tu^k-At^k))-P_C(x^*)\|^2\\ &\le \|(t^k-x^*)+\mu A^*(Tu^k-At^k)\|^2\\ &= \|t^k-x^*\|^2+\|\mu A^*(Tu^k-At^k)\|^2+2\mu\langle t^k-x^*,\, A^*(Tu^k-At^k)\rangle\\ &\le \|t^k-x^*\|^2+\mu^2\|A^*\|^2\|Tu^k-At^k\|^2+2\mu\langle A(t^k-x^*), Tu^k-At^k\rangle\\ &\le \|t^k-x^*\|^2+\mu^2\|A^*\|^2\|Tu^k-At^k\|^2-\mu\|T^g_{\alpha_k}At^k-At^k\|^2\\ &\quad -\mu\|Tu^k-At^k\|^2\\ &= \|t^k-x^*\|^2-\mu(1-\mu\|A^*\|^2)\|Tu^k-At^k\|^2-\mu\|T^g_{\alpha_k}At^k-At^k\|^2.\end{aligned}$$

According to the definition of u^k, the last inequality becomes

$$\|x^{k+1}-x^*\|^2 \le \|t^k-x^*\|^2-\mu(1-\mu\|A\|^2)\|Tu^k-At^k\|^2-\mu\|u^k-At^k\|^2. \tag{10}$$

From (7), (10) and $0<\mu<\frac{1}{\|A\|^2}$, we get

$$\|x^{k+1}-x^*\| \le \|t^k-x^*\| \le \|z^k-x^*\| \le \|x^k-x^*\| \tag{11}$$

and

$$\mu(1-\mu\|A\|^2)\|Tu^k-At^k\|^2+\mu\|u^k-At^k\|^2 \le \|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2. \tag{12}$$

From (11), we can conclude that $\lim_{k\to\infty}\|x^k-x^*\|$ exists, and from (12), we obtain that

$$\begin{aligned}&\lim_{k\to\infty}\|x^k-x^*\| = \lim_{k\to\infty}\|t^k-x^*\| = \lim_{k\to\infty}\|z^k-x^*\| \quad \text{and}\\ &\lim_{k\to\infty}\|Tu^k-At^k\| = \lim_{k\to\infty}\|u^k-At^k\| = 0.\end{aligned} \tag{13}$$

Step 2. $\lim_{k\to\infty}\|z^k-x^k\| = \lim_{k\to\infty}\|Sz^k-z^k\| = \lim_{k\to\infty}\|t^k-x^k\| = 0$.

From (13) and the inequality

$$\|Tu^k-u^k\| \le \|Tu^k-At^k\|+\|u^k-At^k\|,$$

we get

$$\lim_{k\to\infty}\|Tu^k-u^k\| = 0. \tag{14}$$

Besides that,

$$\|z^k-x^*\|^2 \le \|x^k-x^*\|^2-(1-2\lambda_k c_1)\|x^k-y^k\|^2-(1-2\lambda_k c_2)\|y^k-z^k\|^2,$$

so

$$(1-2\lambda_k c_1)\|x^k-y^k\|^2+(1-2\lambda_k c_2)\|y^k-z^k\|^2 \le \|x^k-x^*\|^2-\|z^k-x^*\|^2. \tag{15}$$

Since $\{\lambda_k\}\subset[a,b]\subset(0,\min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$ and (15), one has

$$\begin{aligned}(1-2bc_1)\|x^k-y^k\|^2 &\le \|x^k-x^*\|^2-\|z^k-x^*\|^2,\\ (1-2bc_2)\|y^k-z^k\|^2 &\le \|x^k-x^*\|^2-\|z^k-x^*\|^2.\end{aligned} \tag{16}$$

From (13), we get $\lim_{k\to\infty}(\|x^k-x^*\|^2-\|z^k-x^*\|^2)=0$. Combining this fact with (16), we get

$$\lim_{k\to\infty}\|x^k-y^k\|=0, \qquad \lim_{k\to\infty}\|y^k-z^k\|=0. \tag{17}$$

It is clear that

$$\|z^k-x^k\| \le \|x^k-y^k\|+\|y^k-z^k\|,$$

so, we get from (17) that

$$\lim_{k\to\infty} \|z^k - x^k\| = 0. \tag{18}$$

Using $t^k = (1-\alpha)z^k + \alpha S z^k$, Lemma 2 and the nonexpansiveness of S, we have

$$\begin{aligned}
\|t^k - x^*\|^2 &= \|(1-\alpha)z^k + \alpha S z^k - x^*\|^2 \\
&= \|(1-\alpha)(z^k - x^*) + \alpha(S z^k - x^*)\|^2 \\
&= (1-\alpha)\|z^k - x^*\|^2 + \alpha\|S z^k - x^*\|^2 - \alpha(1-\alpha)\|S z^k - z^k\|^2 \\
&= (1-\alpha)\|z^k - x^*\|^2 + \alpha\|S z^k - S x^*\|^2 - \alpha(1-\alpha)\|S z^k - z^k\|^2 \\
&\le (1-\alpha)\|z^k - x^*\|^2 + \alpha\|z^k - x^*\|^2 - \alpha(1-\alpha)\|S z^k - z^k\|^2 \\
&= \|z^k - x^*\|^2 - \alpha(1-\alpha)\|S z^k - z^k\|^2.
\end{aligned} \tag{19}$$

Therefore,

$$\alpha(1-\alpha)\|S z^k - z^k\|^2 \le \|z^k - x^*\|^2 - \|t^k - x^*\|^2.$$

Combining the last inequality with (13), we conclude that

$$\lim_{k\to\infty} \|S z^k - z^k\| = 0. \tag{20}$$

We have

$$\|t^k - x^k\| \le \|t^k - z^k\| + \|z^k - x^k\| = \alpha\|S z^k - z^k\| + \|z^k - x^k\|.$$

Therefore, we deduce from (18) and (20) that

$$\lim_{k\to\infty} \|t^k - x^k\| = 0. \tag{21}$$

Step 3. *The sequences* $\{x^k\}$, $\{z^k\} \rightharpoonup p \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S)$ *and* $\{u^k\} \rightharpoonup Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)$.

By Step 1, $\lim_{k\to\infty} \|x^k - x^*\|$ exists and the sequence $\{x^k\}$ is bounded. Consequently, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that x^{k_j} converges weakly to some $p \in C$ as $j \to \infty$. Then, it follows from (21) that $t^{k_j} \rightharpoonup p$, $At^{k_j} \rightharpoonup Ap$.

Since $\lim_{k\to\infty} \|u^k - At^k\| = 0$, we deduce that $u^{k_j} \rightharpoonup Ap$. Remember that $\{u^k\} \subset Q$, so $Ap \in Q$.

In addition, $\lim_{k\to\infty} \|z^k - x^k\| = 0$ and $x^{k_j} \rightharpoonup p$. Hence, $z^{k_j} \rightharpoonup p$.

If $Sp \ne p$, then, by Opial's condition and (20), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|z^{k_j} - p\| &< \liminf_{j\to\infty} \|z^{k_j} - Sp\| \\
&= \liminf_{j\to\infty} \|z^{k_j} - S z^{k_j} + S z^{k_j} - Sp\| \\
&\le \liminf_{j\to\infty} (\|z^{k_j} - S z^{k_j}\| + \|S z^{k_j} - Sp\|) \\
&= \liminf_{j\to\infty} \|S z^{k_j} - Sp\| \\
&\le \liminf_{j\to\infty} \|z^{k_j} - p\|.
\end{aligned}$$

This is a contradiction. Thus, $Sp = p$, i.e., $p \in \mathrm{Fix}(S)$.

From Lemma 7, we have

$$\lambda_{k_j}[f(x^{k_j}, y) - f(x^{k_j}, y^{k_j})] \ge \langle y^{k_j} - x^{k_j}, y^{k_j} - y\rangle \quad \forall y \in C.$$

Letting $j \to \infty$, we get $f(p, y) \geq 0$ for all $y \in C$ and thus $p \in \mathrm{Sol}(C, f)$. Therefore,

$$p \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S). \tag{22}$$

Next, we show that

$$Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T).$$

First, if $TAp \neq Ap$, then by the Opial's condition and (14), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|u^{k_j} - Ap\| &< \liminf_{j\to\infty} \|u^{k_j} - TAp\| \\
&= \liminf_{j\to\infty} \|u^{k_j} - Tu^{k_j} + Tu^{k_j} - TAp\| \\
&\leq \liminf_{j\to\infty} (\|u^{k_j} - Tu^{k_j}\| + \|Tu^{k_j} - TAp\|) \\
&= \liminf_{j\to\infty} \|Tu^{k_j} - TAp\| \\
&\leq \liminf_{j\to\infty} \|u^{k_j} - Ap\|.
\end{aligned}$$

This is a contradiction. Thus, $Ap \in \mathrm{Fix}(T)$.

On the other hand, $\mathrm{Sol}(Q, g) = \mathrm{Fix}(T^g_\alpha)$. So, if $T^g_\alpha Ap \neq Ap$, then combining (13), the Opial's condition, and Lemma 6, we have

$$\begin{aligned}
\liminf_{j\to\infty} \|At^{k_j} - Ap\| &< \liminf_{j\to\infty} \|At^{k_j} - T^g_\alpha Ap\| \\
&= \liminf_{j\to\infty} \|At^{k_j} - u^{k_j} + u^{k_j} - T^g_\alpha Ap\| \\
&\leq \liminf_{j\to\infty} (\|At^{k_j} - u^{k_j}\| + \|T^g_\alpha Ap - u^{k_j}\|) \\
&= \liminf_{j\to\infty} \|T^g_\alpha Ap - u^{k_j}\| \\
&= \liminf_{j\to\infty} \|T^g_\alpha Ap - T^g_{\alpha_{k_j}} At^{k_j}\| \\
&\leq \liminf_{j\to\infty} \left\{ \|At^{k_j} - Ap\| + \frac{|\alpha_{k_j} - \alpha|}{\alpha_{k_j}} \|T^g_{\alpha_{k_j}} At^{k_j} - At^{k_j}\| \right\} \\
&= \liminf_{j\to\infty} \left\{ \|At^{k_j} - Ap\| + \frac{|\alpha_{k_j} - \alpha|}{\alpha_{k_j}} \|u^{k_j} - At^{k_j}\| \right\} \\
&= \liminf_{j\to\infty} \|At^{k_j} - Ap\|.
\end{aligned}$$

This is a contradiction. Thus $Ap \in \mathrm{Fix}(T^g_\alpha) = \mathrm{Sol}(Q, g)$. Therefore,

$$Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T). \tag{23}$$

From (22) and (23), we obtain that $p \in \Omega$.

Step 4. *The sequences $\{x^k\}$ and $\{u^k\}$ converge weakly to p and Ap, respectively.*

Otherwise, there exists a subsequence $\{x^{m_i}\}$ of $\{x^k\}$ such that $x^{m_i} \rightharpoonup q \in \Omega$ with $q \neq p$, and again by the Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty} \|x^{m_i} - q\| &< \liminf_{i\to\infty} \|x^{m_i} - p\| \\ &= \liminf_{j\to\infty} \|x^{k_j} - p\| \\ &< \liminf_{j\to\infty} \|x^{k_j} - q\| \\ &= \liminf_{i\to\infty} \|x^{m_i} - q\|.\end{aligned}$$

This is a contradiction. Hence, $\{x^k\}$ converges weakly to p. Together with (18) and (21), we also get $z^k \rightharpoonup p$ and $t^k \rightharpoonup p$, so $At^k \rightharpoonup Ap$. Combining with (13), we conclude that $u^k \rightharpoonup Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)$. Theorem 1 is proved. □

When $S = I_{\mathcal{H}_1}$ and $T = I_{\mathcal{H}_2}$, the problem (SEFPP) reduces to the split equilibrium problem (SEP). In this case, Theorem 1 becomes

Corollary 1 *Suppose that g, f are bifunctions satisfying Assumptions $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. Take $x^1 \in C$; $\{\lambda_k\} \subset [a, b]$, for some $a, b \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$; $0 < \alpha < 1$; $0 < \mu < \frac{1}{\|A\|^2}$; $\{\alpha_k\} \subset (0, +\infty)$ with $\liminf_{k\to\infty} \alpha_k > 0$, and consider the sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, and $\{u^k\}$ defined by*

$$\begin{cases} y^k = \arg\min\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \arg\min\left\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ u^k = T^g_{\alpha_k} A z^k, \\ x^{k+1} = P_C(z^k + \mu A^*(u^k - Az^k)). \end{cases}$$

If $\Omega = \{x^* \in \mathrm{Sol}(C, f) : Ax^* \in \mathrm{Sol}(Q, g)\} \neq \emptyset$, then the sequences $\{x^k\}$ and $\{z^k\}$ converge weakly to an element $p \in \Omega$ and $\{u^k\}$ converges weakly to $Ap \in \mathrm{Sol}(Q, g)$.

The following corollary is immediate from Theorem 1 when $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $g \equiv 0$, and $S = T = A = I_{\mathcal{H}}$.

Corollary 2 [37] *Let f be a bifunction satisfying Assumptions $\mathcal{B}$. Take $x^1 \in C$; $\{\lambda_k\} \subset [a, b]$ for some $a, b \in (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$; and consider the sequences $\{x^k\}$, $\{y^k\}$ generated by*

$$\begin{cases} y^k = \arg\min\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ x^{k+1} = \arg\min\left\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}. \end{cases}$$

If $\mathrm{Sol}(C, f) \neq \emptyset$, then the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to a solution of $\mathrm{EP}(C, f)$.

The following results obtained in [35] can be considered as a special case of Theorem 1 when $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $f \equiv 0$, $S = A = I_{\mathcal{H}}$.

Corollary 3 [35] *Let g be a bifunction satisfying Assumptions $\mathcal{A}$. Take $x^1 \in C$; $0 < \mu < 1$; $\{\alpha_k\} \subset (0, +\infty)$ with $\liminf_{k\to\infty} \alpha_k > 0$, and consider the sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, and $\{u^k\}$ defined by*

$$\begin{cases} u^k = T^g_{\alpha_k} x^k, \\ x^{k+1} = (1-\mu)x^k + \mu T u^k. \end{cases}$$

If $\Omega = \{x^* \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)\} \neq \emptyset$, then the sequences $\{x^k\}$ and $\{u^k\}$ converge weakly to an element $p \in \Omega$.

In what follows, we combine the proposed method with a hybrid technique to obtain a strongly convergent algorithm

Theorem 2 (Strong convergence theorem) *Let $x^1 \in C_1 = C$, consider sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, $\{t^k\}$ and $\{u^k\}$ generated by the following process*

$$\begin{cases} y^k = \arg\min\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 :\ y \in C\right\}, \\ z^k = \arg\min\left\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 :\ y \in C\right\}, \\ t^k = (1-\alpha)z^k + \alpha S z^k, \\ u^k = T^g_{\alpha_k} A t^k, \\ s^k = P_C(t^k + \mu A^*(T u^k - A t^k)), \\ C_{k+1} = \{r \in C_k :\ \|s^k - r\| \le \|t^k - r\| \le \|x^k - r\|\}, \\ x^{k+1} = P_{C_{k+1}}(x^1), \quad k \in \mathbb{N}^*, \end{cases} \tag{24}$$

where $0 < \alpha < 1$, $0 < \mu < \frac{1}{\|A\|^2}$, $\{\alpha_k\} \subset (0; \infty)$ with $\liminf_{k\to\infty} \alpha_k > 0$. Then, under assumptions of Theorem 1 and $\Omega = \{x^* \in \mathrm{Sol}(C, f) \cap \mathrm{Fix}(S) : Ax^* \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)\} \neq \emptyset$, the sequences $\{x^k\}$, $\{z^k\}$ converge strongly to an element $p \in \Omega$ and $\{u^k\}$ converges strongly to $Ap \in \mathrm{Sol}(Q, g) \cap \mathrm{Fix}(T)$.

Remark 2 By setting $H^1_k = \{r \in \mathcal{H}_1 : \|s^k - r\| \le \|t^k - r\|\}$, then

$$H^1_k = \{r \in \mathcal{H}_1 : \langle t^k - s^k, r\rangle \le \frac{1}{2}(\|t^k\|^2 - \|v^k\|^2)\},$$

so H^1_k is a halfspace. Similarly, $H^2_k = \{r \in \mathcal{H}_1 : \|t^k - r\| \le \|x^k - r\|\}$ is also a halfspace. Since $C_{k+1} = C_k \cap H^1_k \cap H^2_k$, if $\mathcal{H}_1$ is the Euclidean space $\mathcal{R}^n$ and C is a polyhedra, then by induction C_k are polyhedra for all k. Therefore, the computation of x^{k+1} in (24) is equivalent to find the projection of x^1 onto the polyhedra set C_{k+1} which can be computed efficiently using, for example, strongly convex quadratic programming methods.

Now, let us prove Theorem 2.

Proof We divide the proof of this theorem into several claims.

Claim 1 For all $k \in \mathbb{N}^*$, C_k is a nonempty closed convex set.

Let $x^* \in \Omega$. Then, it follows from (10), (19), and (7) that

$$\begin{aligned}
\|s^k - x^*\|^2 &\le \|t^k - x^*\|^2 - \mu(1-\mu\|A\|^2)\|Tu^k - At^k\|^2 - \mu\|u^k - At^k\|^2 \\
&\le \|z^k - x^*\|^2 - \alpha(1-\alpha)\|Sz^k - z^k\|^2 - \mu(1-\mu\|A\|^2)\|Tu^k - At^k\|^2 \\
&\quad -\mu\|u^k - At^k\|^2 \\
&\le \|x^k - x^*\|^2 - \alpha(1-\alpha)\|Sz^k - z^k\|^2 - \mu(1-\mu\|A\|^2)\|Tu^k - At^k\|^2 \\
&\quad -\mu\|u^k - At^k\|^2. \qquad (25)
\end{aligned}$$

Since $0 < \alpha < 1$, $0 < \mu < \frac{1}{\|A\|^2}$ and (25), we get

$$\|s^k - x^*\| \le \|t^k - x^*\| \le \|z^k - x^*\| \le \|x^k - x^*\| \quad \forall k. \qquad (26)$$

Because $x^* \in C_1$ and (26), we get by induction that $x^* \in C_k$ for all $k \in \mathbb{N}^*$, i.e., $\Omega \subset C_k$. So $C_k \neq \emptyset$ for all k. Define

$$D_k = \{r \in \mathcal{H}_1 : \|s^k - r\| \le \|t^k - r\| \le \|x^k - r\|\}, \quad k \in \mathbb{N}^*.$$

Then, $C_{k+1} = C_k \cap D_k$. Since C_1 and D_k are closed for all k, C_k is closed for all k.

Next, we verify that D_k is convex for all k. Indeed, let $r^1, r^2 \in D_k$ and $\lambda \in [0, 1]$, using Lemma 2, we have

$$\begin{aligned}
\|s^k - (\lambda r^1 + (1-\lambda)r^2)\|^2 &= \|\lambda(s^k - r^1) + (1-\lambda)(s^k - r^2)\|^2 \\
&= \lambda\|s^k - r^1\|^2 + (1-\lambda)\|s^k - r^2\|^2 - \lambda(1-\lambda)\|r^1 - r^2\|^2 \\
&\le \lambda\|t^k - r^1\|^2 + (1-\lambda)\|t^k - r^2\|^2 - \lambda(1-\lambda)\|r^1 - r^2\|^2 \\
&= \|t^k - (\lambda r^1 + (1-\lambda)r^2)\|^2.
\end{aligned}$$

So,

$$\|s^k - (\lambda r^1 + (1-\lambda)r^2)\| \le \|t^k - (\lambda r^1 + (1-\lambda)r^2)\|.$$

Similarly,

$$\|t^k - (\lambda r^1 + (1-\lambda)r^2)\| \le \|x^k - (\lambda r^1 + (1-\lambda)r^2)\|.$$

Thus,

$$\|s^k - (\lambda r^1 + (1-\lambda)r^2)\| \le \|t^k - (\lambda r^1 + (1-\lambda)r^2)\| \le \|x^k - (\lambda r^1 + (1-\lambda)r^2)\|.$$

Therefore,

$$\lambda r^1 + (1-\lambda)r^2 \in D_k.$$

Since C_1 and D_k are convex for all k, C_k is convex for all k.

Claim 2 The sequences $\{x^k\}$, $\{s^k\}$ *are bounded and* $x^k \to p$.

By the definition of x^{k+1}, we have $x^{k+1} \in C_{k+1} \subset C_k$ and $x^k = P_{C_k}(x^1)$. So

$$\|x^k - x^1\| \le \|x^{k+1} - x^1\| \quad \forall k.$$

In addition, $x^{k+1} = P_{C_{k+1}}(x^1)$ and $x^* \in C_{k+1}$. Hence,

$$\|x^{k+1} - x^1\| \le \|x^* - x^1\|.$$

Thus,

$$\|x^k - x^1\| \le \|x^{k+1} - x^1\| \le \|x^* - x^1\| \quad \forall k.$$

Therefore, $\lim_{k\to\infty} \|x^k - x^1\|$ exists and the sequence $\{x^k\}$ is bounded. Hence, by (26), $\{s^k\}$ is also bounded.

Furthermore, for all $m > n$, we have $x^m \in C_m \subset C_n$, $x^n = P_{C_n}(x^1)$. Combining this fact with Lemma 1, we get

$$\|x^m - x^n\|^2 \le \|x^m - x^1\|^2 - \|x^n - x^1\|^2.$$

Since $\lim_{k\to\infty} \|x^k - x^1\|$ exists, the sequence $\{x^k\}$ is a Cauchy sequence. Thus,

$$\lim_{k\to\infty} x^k = p \quad \text{for some } p \in C. \tag{27}$$

Claim 3 $p \in \Omega$, $\lim_{k\to\infty} z^k = p$ and $\lim_{k\to\infty} u^k = Ap$.

From the definitions of C_{k+1} and x^{k+1}, we have

$$\|s^k - x^{k+1}\| \le \|t^k - x^{k+1}\| \le \|x^k - x^{k+1}\|.$$

Thus,

$$\begin{aligned} \|s^k - x^k\| &\le \|s^k - x^{k+1}\| + \|x^{k+1} - x^k\| \\ &\le \|x^k - x^{k+1}\| + \|x^k - x^{k+1}\| \\ &= 2\|x^k - x^{k+1}\| \end{aligned} \tag{28}$$

and

$$\begin{aligned} \|t^k - x^k\| &\le \|t^k - x^{k+1}\| + \|x^{k+1} - x^k\| \\ &\le \|x^k - x^{k+1}\| + \|x^k - x^{k+1}\| \\ &= 2\|x^k - x^{k+1}\|. \end{aligned} \tag{29}$$

Combining with (27), we deduce from (28) and (29) that

$$\lim_{k\to\infty} \|s^k - x^k\| = \lim_{k\to\infty} \|t^k - x^k\| = 0. \tag{30}$$

From (25), Claim 2, and (30), we can write

$$\begin{aligned} &\alpha(1-\alpha)\|Sz^k - z^k\|^2 + \mu(1-\mu\|A\|^2)\|Tu^k - At^k\|^2 + \mu\|u^k - At^k\|^2 \\ &\quad \le \|x^k - x^*\|^2 - \|s^k - x^*\|^2 \\ &\quad = (\|x^k - x^*\| + \|s^k - x^*\|)(\|x^k - x^*\| - \|s^k - x^*\|) \\ &\quad \le \|x^k - s^k\|(\|x^k - x^*\| + \|s^k - x^*\|) \to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{31}$$

From $0 < \alpha < 1$ and $0 < \mu < \frac{1}{\|A\|^2}$, we get from (31) that

$$\begin{aligned} \lim_{k\to\infty} \|Tu^k - At^k\| &= \lim_{k\to\infty} \|u^k - At^k\| = 0, \\ \lim_{k\to\infty} \|Sz^k - z^k\| &= 0. \end{aligned} \tag{32}$$

So (32) and the inequality

$$\|Tu^k - u^k\| \le \|Tu^k - At^k\| + \|u^k - At^k\|,$$

imply that

$$\lim_{k\to\infty} \|Tu^k - u^k\| = 0. \tag{33}$$

On the other hand, from (17), (18), (21), and $\lim_{k\to\infty} x^k = p$, we have

$$\lim_{k\to\infty} y^k = p, \quad \lim_{k\to\infty} z^k = p, \quad \lim_{k\to\infty} t^k = p. \tag{34}$$

Consequently, it follows from (32) and (34) that

$$\begin{aligned}\|Sp-p\| &\le \|Sp-Sz^k\|+\|Sz^k-z^k\|+\|z^k-p\|\\ &\le \|p-z^k\|+\|Sz^k-z^k\|+\|z^k-p\|\\ &= 2\|z^k-p\|+\|Sz^k-z^k\|\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

So, $Sp=p$, i.e., $p\in \text{Fix}(S)$.

On the other side, we obtain from Lemma 7 that

$$\lambda_k[f(x^k,y)-f(x^k,y^k)]\ge \langle y^k-x^k, y^k-y\rangle \quad \forall y\in C.$$

Letting $k\to\infty$, using the joint weak continuity of f, $\lim_{k\to\infty} x^k=p$ and (34), we get in the limit that

$$f(p,y)\ge 0 \quad \forall y\in C.$$

It is immediate that $p\in \text{Sol}(C,f)$. Consequently,

$$p\in \text{Sol}(C,f)\cap \text{Fix}(S). \tag{35}$$

From (34), it follows that $\lim_{k\to\infty} At^k=Ap$. Combining this fact with (32), one has

$$\lim_{k\to\infty} u^k = Ap. \tag{36}$$

From (33), (36), one can deduce

$$\begin{aligned}\|TAp-Ap\| &\le \|TAp-Tu^k\|+\|Tu^k-u^k\|+\|u^k-Ap\|\\ &\le \|Ap-u^k\|+\|Tu^k-u^k\|+\|u^k-Ap\|\\ &= 2\|u^k-Ap\|+\|Tu^k-u^k\|\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

Hence, $TAp=Ap$, i.e., $Ap\in\text{Fix}(T)$.

In addition, we obtain from Lemma 6, $\lim_{k\to\infty} At^k=Ap$, and (32) that

$$\begin{aligned}\|T^g_\alpha Ap-Ap\| &\le \|T^g_\alpha Ap-T^g_{\alpha_k}At^k\|+\|T^g_{\alpha_k}At^k-At^k\|+\|At^k-Ap\|\\ &= \|T^g_\alpha Ap-T^g_{\alpha_k}At^k\|+\|u^k-At^k\|+\|At^k-Ap\|\\ &\le \|At^k-Ap\|+\frac{|\alpha_k-\alpha|}{\alpha_k}\|T^g_{\alpha_k}At^k-At^k\|+\|u^k-At^k\|+\|At^k-Ap\|\\ &= 2\|At^k-Ap\|+\frac{|\alpha_k-\alpha|}{\alpha_k}\|u^k-At^k\|+\|u^k-At^k\|\to 0 \quad \text{as } k\to\infty,\end{aligned}$$

which implies that $T^g_\alpha Ap=Ap$, i.e., $Ap\in \text{Fix}(T^g_\alpha)=\text{Sol}(Q,g)$. So,

$$Ap\in \text{Sol}(Q,g)\cap\text{Fix}(T). \tag{37}$$

From (35) and (37), we get $p\in\Omega$. The proof is complete. □

The following corollary is immediate from Theorem 2 when $S=I_{\mathcal{H}_1}$ and $T=I_{\mathcal{H}_2}$.

Corollary 4 *Let $g: Q\times Q\to\mathbb{R}$ be a bifunction satisfying Assumptions $\mathcal{A}$ and $f: C\times C\to\mathbb{R}$ be a bifunction satisfying Assumptions $\mathcal{B}$. Let $A:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded* *linear operator with its adjoint* A^*. *Choose* $x^1 \in C$ *and* $C_1 = C$. *Consider the sequences* $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ *and* $\{u^k\}$ *generated by the following iteration*

$$\begin{cases} y^k = \arg\min \left\{ \lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C \right\}, \\ z^k = \arg\min \left\{ \lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C \right\}, \\ u^k = T^g_{\alpha_k} A z^k, \\ s^k = P_C(z^k + \mu A^*(u^k - Az^k)), \\ C_{k+1} = \{r \in C_k : \|s^k - r\| \le \|z^k - r\| \le \|x^k - r\|\}, \\ x^{k+1} = P_{C_{k+1}}(x^1), \end{cases}$$

where $0 < \alpha < 1$, $0 < \mu < \frac{1}{\|A\|^2}$, $\{\alpha_k\} \subset (0; \infty)$ *with* $\liminf_{k\to\infty} \alpha_k > 0$. *Suppose that* $\Omega = \{x^* \in \mathrm{Sol}(C, f) : Ax^* \in \mathrm{Sol}(Q, g)\} \neq \emptyset$, *then the sequences* $\{x^k\}$ *and* $\{z^k\}$ *converge strongly to an element* $p \in \Omega$ *and* $\{u^k\}$ *converges strongly to* $Ap \in \mathrm{Sol}(Q, g)$.

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $g = f = 0$, and $A = T = I_{\mathcal{H}}$. Then, we get the following result.

Corollary 5 [34] *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$, $x^1 \in C$ *and* $C_1 = C$. *Define a sequence* $\{x^k\}$ *as follows*

$$\begin{cases} t^k = (1-\alpha)x^k + \alpha S x^k, \\ C_{k+1} = \{r \in C_k : \|t^k - r\| \le \|x^k - r\|\}, \\ x^{k+1} = P_{C_{k+1}}(x^1), \end{cases}$$

where $0 < \alpha < 1$. *Then, the sequence* $\{x^k\}$ *converges strongly to an element* $p \in \mathrm{Fix}(S)$.

4 Conclusion

We have proposed two algorithms for solving a split equilibrium and fixed problem (SEFPP) in real Hilbert spaces, when the bifunctions are pseudomonotone and monotone, respectively, and the fixed point mappings are nonexpansive. Then, we have proved that the iterative sequences generated by the algorithms converge weakly and strongly to a solution of this problem, respectively. Some special cases of (SEFPP) have also been considered.

Acknowledgments The authors would like to thank the referees very much for their constructive comments and suggestions, especially on the presenting and the structure of the early version of their paper which helped them very much in revising the paper. Their thanks would be addressed to Prof. Le Dung Muu and Prof. Pham Ky Anh for the guidance and discussion. The first author is supported in part by NAFOSTED, under the project 101.01-2014-24, and a grant from Le Quy Don Technical University.

References

1. Anh, P.N.: A hybrid extragradient method extended to fixed point problems and equilibrium problems. Optimization **62**, 271–283 (2013)
2. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**, 367–426 (1996)

3. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. Eur. J. Oper. Res. **227**, 1–11 (2013)
4. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
5. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
6. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
7. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
8. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algor. **59**, 301–323 (2012)
9. Censor, Y., Segal, A.: The split common fixed point problem for directed operators. J. Convex Anal. **16**, 587–600 (2009)
10. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algor. **8**, 221–239 (1994)
11. Combettes, P.L., Hirstoaga, A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
12. Deepho, J., Kumam, W., Kumam, P.: A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems. J. Math. Model. Algor. **13**, 405–423 (2014)
13. Dinh, B.V., Muu, L.D.: A projection algorithm for solving pseudomonotone equilibrium problems and its application to a class of bilevel equilibria. Optimization **64**, 559–575 (2015)
14. Dinh, B.V., Muu, L.D.: On penalty and gap function methods for bilevel equilibrium problems. J. Appl. Math. **2011**, 646452 (2011)
15. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
16. He, Z.: The split equilibrium problem and its convergence algorithms. J. Inequal. Appl. **2012**, 162 (2012)
17. Hieu, D.V., Muu, L.D., Anh, P.K.: Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings. Numer. Algor. **73**, 197–217 (2016)
18. Iiduka, H., Yamada, I.: A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping. SIAM J. Optim. **19**, 1881–1893 (2009)
19. Iiduka, H., Yamada, I.: A subgradient-type method for the equilibrium problem over the fixed point set and its applications. Optimization **58**, 251–261 (2009)
20. Khatibzadeh, H., Mohebbi, V., Ranjbar, S.: Convergence analysis of the proximal point algorithm for pseudo-monotone equilibrium problems. Optim. Methods Softw. **30**, 1146–1163 (2015)
21. Kraikaew, R., Saejung, S.: On split common fixed point problems. J. Math. Anal. Appl. **415**, 513–524 (2014)
22. Korpelevich, G.M.: An extragradient method for finding saddle points and other problems. Matecon **12**, 747–756 (1976)
23. Maingé, P.-E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. SIAM J. Control Optim. **47**, 1499–1515 (2008)
24. Maingé, P.-E.: Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints. Eur. J. Oper. Res. **205**, 501–506 (2010)
25. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
26. Mastroeni, G.: On auxiliary principle for equilibrium problems. In: Daniele, P., Giannessi, F., Maugeri, A. (eds.) Equilibrium Problems and Variational Models, pp. 289–298. Kluwer Academic Publishers, Dordrecht (2003)
27. Moudafi, A.: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275–283 (2011)
28. Moudafi, A.: The split common fixed-point problem for demicontractive mappings. Inverse Probl. **26**, 055007 (2010)
29. Moudafi, A.: Proximal point algorithm extended to equilibrium problems. J. Nat. Geom. **15**, 91–100 (1999)
30. Muu, L.D., Quoc, T.D.: Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-Cournot equilibrium model. J. Optim. Theory Appl. **142**, 185–204 (2009)
31. Muu, L.D., Oettli, W.: Convergence of an adaptive penalty scheme for finding constrained equilibria. Nonlinear Anal. TMA **18**, 1159–1166 (1992)

32. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
33. Tada, A., Takahashi, W.: Weak and strong convergence theorems for nonexpansive mapping and equilibrium problem. J. Optim. Theory Appl. **133**, 359–370 (2007)
34. Takahashi, W., Takeuchi, Y., Kubota, R.: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **341**, 276–286 (2008)
35. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert space. J. Math. Anal. Appl. **331**, 506–515 (2007)
36. Tam, N.N., Yao, J.C., Yen, N.D.: Solution methods for pseudomonotone variational inequalities. J. Optim. Theory Appl. **138**, 253–273 (2008)
37. Tran, D.Q., Muu, L.D., Nguyen, V.H.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)