Vietnam J. Math.
DOI 10.1007/s10013-017-0247-x

p-Adic Admissible Measures Attached to Siegel Modular Forms of Arbitrary Genus

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Received: 28 April 2016 / Accepted: 11 December 2016

Abstract Let p be a prime number and $f \in S_n^l(\Gamma_0(N), \psi)$ be a Siegel cusp eigenform of genus n. We consider the standard zeta function $D^{(Np)}(f, s, \chi)$, which takes algebraic values at critical points after normalization. We construct two h-admissible measures μ^+ and μ^- for certain $h = [4\mathrm{ord}_p(\alpha_0(p))] + 1$ explained in the Main Theorem with the following properties:

(i) For all pairs (s, χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet characters, $s \in \mathbb{Z}$ with $1 \le s \le l - n$, $s \equiv \delta \mod 2$ and for $s = 1$ the character χ^2 is non-trivial, the following equality holds

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p \left(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s, \chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle f_0, f_0 \rangle} \cdot D^{(Np)}(f, s, \overline{\chi\chi^0}) \right),$$

where f_0 is a modular form, associated to f and an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ is fixed.

(ii) For all pairs (s, χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $1 - l + n \le s \le 0$, $s \equiv \delta + 1 \mod 2$ the following equality holds

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p \Bigg(c_\chi^{n(1-s)} A^+(\chi) \cdot E_p^-(1 - s, \chi\chi^0) \frac{\Lambda_\infty^-(s)}{\langle f_0, f_0 \rangle} \times D^{(Np)}(f, 1 - s, \overline{\chi\chi^0}) \Bigg).$$

Here, $\delta = 0$ or 1 according to whether $\chi(-1) = 1$ or $\chi(-1) = -1$ and $\Lambda_\infty(s)$, $A(\chi)$, $E_p(s, \psi)$ are certain elementary factors including Gauss sum, Satake p-parameters, conductor c_χ of Dirichlet character χ etc.

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Published online: 28 March 2017

Keywords Siegel modular forms · Special values · Critical points · Petersson product · Rankin–Selberg method · p-Adic L-functions

Mathematics Subject Classification (2010) 11F67 · 11F46

1 Introduction

The purpose of this note is to give a construction of admissible measures (in the sense of Amice–Vélu) attached to a standard L-function of a Siegel cusp eigenform. For this purpose, we use the theory of p-adic integration in spaces of holomorphic Siegel modular forms (in the sense of Shimura) over $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. Often we simply assume that $A = \mathbb{C}_p$. We study the action of certain differential operators on Siegel Eisenstein distributions with values in the spaces of modular forms. In order to obtain from them numerically valued distributions interpolating critical values attached to standard L-functions of Siegel modular forms, one applies a suitable linear form coming from the Petersson scalar product.

In the previous works, some special cases were treated by Böcherer, Schmidt for arbitrary genus in the ordinary case (Annales Inst. Fourier, 2000, by doubling method), Courtieu, Panchishkin (LNM 1471, 2004, 1990) for even genus in the general h-admissible cases, by Ranking-Selberg method in the form of Andrianov.

In the present work, we give a conceptual explanation of these p-adic properties satisfied the special values of the standard L-function $D^{(Np)}(f, s, \chi)$ where f is a Siegel cusp form of weight l and of arbitrary genus.

2 Non-Archimedean Integration and Admissible Measures

2.1 Non-Archimedean Integration

Let p be a prime number, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ the Tate field, and S a finite set of primes containing p. The set on which our non-Archimedean zeta functions are defined is the $\mathbb{C}_p$-adic analytic Lie group

$$X_S = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_S, \mathbb{C}_p^\times),$$

where Gal_S is the Galois group of the maximal abelian extension of $\mathbb{Q}$-unramified outside S and infinity. Now, we recall the notation of h-admissible measures on Gal_S and properties of their Mellin transform. These Mellin transforms are certain p-adic analytic functions on the $\mathbb{C}_p$-analytic group X_S.

$$\mathrm{Gal}_S = \varprojlim_M (\mathbb{Z}/M\mathbb{Z})^\times = \mathbb{Z}_S^\times,$$

where M runs over integers with support in the set of primes S. The canonical $\mathbb{C}_p$-analytic structure on X_S is obtained by shift from the obvious $\mathbb{C}_p$-analytic structure on the subgroup $\mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times, \mathbb{C}_p^\times\right)$. We regard the elements of finite order $\chi \in X_S^{\mathrm{tors}}$ as Dirichlet character whose conductor c_χ may contain only primes in S, by means of the decomposition

$$\chi : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{Z}_S^\times \longrightarrow \overline{\mathbb{Q}}^\times \longrightarrow \mathbb{C}^\times.$$

The character $\chi \in X_S^{\text{tors}}$ forms a discrete subgroup X_S^{tors}. We shall need also the natural homomorphism

$$x_p : \mathbb{Z}_S^\times \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times, \quad x_p \in X_S,$$

such that all integers $k \in \mathbb{Z}$ can be regarded as characters of the type $x_p^k : y \mapsto y^k$. Let us fix an embedding

$$i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$$

and we shall identify $\mathbb{Q}$ with a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$. Recall that a p-adic measure on $\mathbb{Z}_S^\times$ may be regarded as a bounded $\mathbb{C}_p^\times$-linear form μ on the space $\mathcal{C}\left(\mathbb{Z}_S^\times, \mathbb{C}_p\right)$ of all continuous $\mathbb{C}_p$-valued functions

$$\begin{aligned} \mathcal{C}\left(\mathbb{Z}_S^\times, \mathbb{C}_p\right) &\longrightarrow \mathbb{C}_p \\ \varphi &\longmapsto \mu(\varphi) = \int_{\mathbb{Z}_S^\times} \varphi d\mu, \end{aligned}$$

which is uniquely determined by its restriction to the subspace $\mathcal{C}^1\left(\mathbb{Z}_S^\times, \mathbb{C}_p\right)$ of locally constant function.

The Mellin transform L_μ of μ is a bounded analytic function

$$\begin{aligned} L_\mu : X_S &\longrightarrow \mathbb{C}_p \\ \chi &\longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi d\mu \end{aligned}$$

on X_S, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in X_S^{\text{tors}}$.

2.2 *h*-Admissible Measure

A more delicate notation of an h-admissible measure was introduced by Amice, Vélu, and Višik (see [1, 8]). For $h \in \mathbb{N}^*$, we define $C^h\left(\mathbb{Z}_p^\times, \mathbb{C}_p\right)$ the space of $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^\times$ which are locally polynomials in x_p of degree less than or equal to h. In particular, $C^1\left(\mathbb{Z}_p^\times, \mathbb{C}_p\right)$ is the space of locally constant functions. Let us recall the definition of admissible measures with scalar and vector values.

Definition 1 An h-admissible measure on $\mathbb{Z}_p^\times$ is a $\mathbb{C}_p$-linear map:

$$\phi : C^h\left(\mathbb{Z}_p^\times, \mathbb{C}_p\right) \longrightarrow \mathbb{C}_p$$

with the following growth condition: for all $t = 0, 1, \ldots, h-1$

$$\left| \int_{a+(p^m)} (x_p - a_p)^t d\phi \right|_p = o(p^{m(h-t)}) \quad \text{for } m \to \infty,$$

where $a_p = x_p(a)$.

We know that each h-admissible measure can be uniquely extended to a linear form on the $\mathbb{C}_p$-space of all locally analytic functions so that one can associate to its Mellin transform

$$\begin{aligned} L_\mu : X_S &\longrightarrow \mathbb{C}_p \\ \chi &\longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi d\mu, \end{aligned}$$

which is a $\mathbb{C}_p$-analytic function X_S of the type $o(\log(x_p)^h)$. Moreover, the measure μ is uniquely determined by the special values of the type $L_\mu(\chi x_p^r)$ with $\chi \in X_S^{\text{tors}}$ and $r = 0, 1, \cdots, h-1$.

3 The Standard L-Functions of Siegel Cusp Eigenform and its Critical Values

For a Siegel modular form $f(z)$ of genus n and weight l, which is an eigenfunction of the Hecke algebra and for each prime number p, one can define the Satake p-parameters of f, denoted by $\alpha_i(p)$ $(i = 0, 1, \ldots, n)$. In this introduction, we assume for simplicity that f is a modular form with respect to the full Siegel modular group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$. The standard zeta function of f is defined by means of the Satake p-parameters as the following Euler product:

$$\mathcal{D}(s, f, \chi) = \prod_p \left\{ \left(1 - \frac{\chi(p)}{p^s}\right) \prod_{i=1}^{n} \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)\alpha_i(p)^{-1}}{p^s}\right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. We introduce the following normalized functions:

$$\begin{aligned} \mathcal{D}^\star(s, f, \chi) &= (2\pi)^{-n(s+l-(n+1)/2}\Gamma((s+\delta)/2) \prod_{j=1}^{n} (\Gamma(s+k-j))\, \mathcal{D}(s, f, \chi), \\ \mathcal{D}^+(s, f, \chi) &= \Gamma((s+\delta)/2)\mathcal{D}^\star(s, f, \chi), \\ \mathcal{D}^-(s, f, \chi) &= \frac{i^\delta \pi^{1/2-s}}{\Gamma((1-s+\delta)/2)}\mathcal{D}^\star(s, f, \chi), \end{aligned}$$

where $\delta = 0$ or 1 according to whether $\chi(-1) = 1$ or $\chi(-1) = -1$ and let

$$f(z) = \sum_{\xi > 0} a(\xi) e_n(\xi z) \in \mathcal{S}_n^k$$

be the Fourier expansion of the Siegel cusp form $f(z)$ of weight l, the sum is extended over all positive definite half integral $n \times n$ matrices, $z \in \mathbb{H}_n$,

$$\mathbb{H}_n = \{z \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t z = z, \mathrm{Im}(z) > 0\}$$

is the Siegel upper half plane of degree n and $e_n(z) = e^{2\pi i \mathrm{tr}(z)}$.

Theorem 1 (Theorem A, [4])

(a) *For all integers s with $1 \le s \le l - \delta - n$, $s \equiv \delta \mod 2$ and Dirichlet character χ such that χ^2 is non-trivial for $s = 1$, we have that:*

$$\langle f, f \rangle^{-1} \mathcal{D}^+(s, f, \chi) \in K = \mathbb{Q}(f, \Lambda_f, \chi),$$

where $K = \mathbb{Q}(f, \Lambda_f, \chi)$ denotes the field generated by Fourier coefficients of f, by the eigenvalues $\Lambda_f(X)$ of the Hecke operator X on f, and by the values of the character χ.

(b) *For all integer s with $1 - l + \delta + n \le s \le 0$, $s \not\equiv \delta \mod 2$, we have that:*

$$\langle f, f \rangle^{-1} \mathcal{D}^-(s, f, \chi) \in K.$$

This theorem was proved by Harris in 1981 for even n and by Böcherer–Schmidt for arbitrary n.

4 Differential Operator

To construct a sequence of modular distributions, we use the action of certain differential operator on Siegel modular forms which was given by Böcherer in [2]. These differential operators are built up from the operators (with $1 \le i, j \le 2n$)

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial z_{ij}}, & i = j, \\ \frac{1}{2}\frac{\partial}{\partial z_{ij}}, & i \ne j \end{cases}$$

which we put together in the symmetric $2n \times 2n$ matrix

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix},$$

where ∂_i are block matrices of size n which correspond to the decomposition

$$\mathfrak{z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

of $\mathbb{H}_{2n}$ into block matrices (with $z_3 = {}^t z_2$). We consider the polynomial $\Delta(r, q)$, $r + q = n$, in the ∂_{ij}, their coefficients being polynomials in the entries of z_2

$$\Delta(p, q) = \sum_{a+b=q} (-1)^b \binom{n}{b} z_2^{[a]} \partial_4^{[a]} \sqcap \left(\left(1_n^{[r]} \sqcap z_2^{[b]} \partial_3^{[b]} \right) \left(\mathrm{Ad}^{[r+b]} \partial_1 \right) \partial_2^{[r+b]} \right).$$

The special multiplication $\sqcap$ was introduced by Freitag in [5] and used by Böcherer in [3, p.1379]. In particular, we have $\Delta(n, 0) = \det(\partial_2)$, $\Delta(0, n) = \det(z_2)\det(\partial)$. We define for any $\alpha \in \mathbb{C}$

$$\mathfrak{D}_{n,\alpha} = \sum_{r+q=n} (-1)^r \binom{n}{r} C_r \left(\alpha - n + \frac{1}{2} \right) \Delta(r, q),$$

where

$$C_q(s) = s\left(s + \frac{1}{2}\right) \cdots \left(s + \frac{q-1}{2}\right) = \frac{\Gamma_q\left(s + \frac{q+1}{2}\right)}{\Gamma_q\left(s + \frac{q-1}{2}\right)}$$

with $\Gamma_q(s) := \pi^{\frac{q(q-1)}{2}} \prod_{i=1}^{q} \Gamma\left(s - \frac{q-1}{2}\right)$, “Gamma function of genus $q \ge 1$”. For $\nu \in \mathbb{N}$, we put

$$\mathfrak{D}_{n,\alpha}^{\nu} = \mathfrak{D}_{n,\alpha+\nu-1} \circ \cdots \circ \mathfrak{D}_{n,\alpha},$$
$$\mathring{\mathfrak{D}}_{n,\alpha}^{\nu} = \left(\mathfrak{D}_{n,\alpha}^{\nu}\right)|_{z_2=0}.$$

For $T \in \mathbb{C}_{\mathrm{sym}}^{2n,2n}$, we recall a polynomial $\mathfrak{P}_{n,\alpha}^{\nu}(T)$ defined by Böcherer in the entries $t_{ij} (1 \le i \le j \le 2n)$ of T by

$$\mathring{\mathfrak{D}}_{n,\alpha}^{\nu}(e^{\mathrm{tr}(TZ)}) = \mathfrak{P}_{n,\alpha}^{\nu}(T) e^{\mathrm{tr}(T_1 z_1 + T_4 z_4)}, \quad T = \begin{pmatrix} T_1 & T_2 \\ {}^t T_2 & T_4 \end{pmatrix},$$

that is, it represents “action of differential operator on exponential function”. The $\mathfrak{P}_{n,\alpha}^{\nu}$ are homogeneous polynomials of degree $n\nu$. In the ordinary case, Böcherer–Schmidt only need the main term $c_{n,\alpha}^{\nu} \det(T_2)^{\nu}$ with a certain constant $c_{n,\alpha}^{\nu}$. In the present work, we find all terms of this polynomial. For simplicity, we write $P(T_1, T_4, T_2)$ instead of $\mathfrak{P}^{\nu}_{n,\alpha}(T)$. We see that for each $(T_1, T_4, T_2) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$ the following property is satisfied for any $A, B \in \mathrm{GL}(n, \mathbb{R})$.

$$P(AT_1\,{}^tA, BT_4\,{}^tB, AT_2\,{}^tB) = \det(AB)^{\nu} P(T_1, T_4, T_2). \tag{1}$$

Indeed, this polynomial is determined by its values on any non-empty open subset (e.g., the open set consisting of (T_1, T_4, T_2) such that $T_1 > 0$, $T_4 > 0$, and $T_2 \in \mathrm{GL}(n, \mathbb{R})$). For these (T_1, T_4, T_2), we can take the matrices $A, B \in \mathrm{GL}(n, \mathbb{R})$ such that

$$AT_1\,{}^tA = BT_4\,{}^tB = 1_n.$$

We put

$$W_0 = AT_2\,{}^tB.$$

There exist two orthogonal matrices h_1, h_2 such that

$$h_1 W_0 h_2 = D,$$

where D is the diagonal matrix with diagonal elements d_i $(1 \le i \le n)$, $d_i \ne 0$. So by (1) we have

$$P(T_1, T_4, T_2) = \det(h_1 h_2 AB)^{-\nu} P(1_n, 1_n, D).$$

Therefore, $P(T_1, T_4, T_2)$ is determined by its values at $T_1 = T_4 = 1_n$, $T_2 = D$. Then, we prove that $P(1_n, 1_n, D)$ for ν even and $P(1_n, 1_n, D)/(d_1 \ldots d_n)$ for ν odd is a polynomial in elementary symmetric polynomials of $d_1^2, \ldots, d_n^2$. We put

$$\det(x1_n - W_0\,{}^tW_0) = \sum_{j=0}^{n} P_j(W_0) x^j.$$

Then, we can write this polynomial in the following forms. For even ν,

$$P(T_1, T_4, T_2) = (\det T_1 \det T_4)^{\frac{\nu}{2}} \sum_{(e_0,\ldots,e_{n-1}) \ne 0} c(e_0, \ldots, e_{n-1}) \prod_{j=0}^{n-1} P_j(W_0)^{e_j}$$

and for odd ν,

$$P(T_1, T_4, T_2) = \det(T_2)(\det T_1 \det T_4)^{\frac{\nu-1}{2}} \sum_{(e_0,\ldots,e_{n-1}) \ne 0} c(e_0, \ldots, e_{n-1}) \prod_{j=0}^{n-1} P_j(W_0)^{e_j},$$

where $P_j(W_0)$ is the elementary symmetric polynomial of d_i^2.

We denote by Λ_n^+ the subsets of positive definite matrices of size n.

Theorem 2 *Using the relations $l = k + \nu$, $\nu \ge 0$ with l the weight of Siegel modular form f and $T = \begin{pmatrix} L^2 T_1 & T_2 \\ {}^tT_2 & L^2 T_4 \end{pmatrix} \in \Lambda_{2n}^+$, $T_1, T_4 \in \Lambda_n^+$, L fixed positive number, we have that the following expression holds:*

$$\mathfrak{P}^{\nu}_{n,k}(T) = \det(L^4 T_1 T_4)^{\frac{\nu}{2}} \sum_{|M| \le \frac{\nu}{2}} C_M(k) Q_M(L^{-2}D) \quad \textit{if } \nu \textit{ is even}$$

and if ν is odd

$$\mathfrak{P}^{\nu}_{n,k}(T) = \det(T_2) \det(L^4 T_1 T_4)^{\frac{\nu-1}{2}} \sum_{|M| \le \frac{\nu-1}{2}} C_M(k) Q_M(L^{-2}D),$$

where M runs over the set of $(e_0, \dots, e_{n-1}) \neq 0$ such that $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$, $C_M(k)$ is a polynomial of variable k of degree $|M|$, and $Q_M(L^{-2}D)$ is a homogeneous polynomial of variables $L^{-2}d_i^2$, $i = 1, \dots, n$ of degree $|M|$.

Proof We consider the matrix $T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix}$ with $T \in \Lambda_{2n}^+$, $T_1, T_4 \in \Lambda_n^+$, L a fixed positive number. We can take $\hat{A} = L^{-1}A$ and $\hat{B} = L^{-1}B \in GL(n, \mathbb{R})$ with matrices A, B defined as above such that

$$\begin{aligned} \hat{A}T_1\,{}^t\hat{A} &= 1_n, \\ \hat{B}T_4\,{}^t\hat{B} &= 1_n. \end{aligned}$$

We put

$$\hat{W}_0 = \hat{A}T_2\,{}^t\hat{B}.$$

Since we assumed that $\det(T_2) \neq 0$, there exist two orthogonal matrices h_1, h_2 such that

$$h_1\hat{W}_0h_2 = \hat{D}.$$

Compare with previous statement we have

$$\begin{aligned} \hat{W}_0 &= L^{-2}W_0, \\ \hat{D} &= L^{-2}D. \end{aligned}$$

Then, we have that $P(T_1, T_4, T_2)$ is determined by $P(1_n, 1_n, \hat{D})$ where $\hat{D}$ is a diagonal matrix with diagonal elements $L^{-2}d_i^2$, $i = 1, 2, \dots, n$. We put

$$\det(x1_n - \hat{D}^2) = \sum_{j=0}^{n} P_j(\hat{D})x^j.$$

Then similarly as the above statement, we can write the polynomial in this case in the following forms:
If ν is even, then

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(L^4T_1T_4)^{\nu/2} \sum_{(e_0,\dots,e_{n-1})\neq 0} c(e_0, \dots, e_{n-1}) \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j}.$$

If ν is odd, then

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(\hat{D}) \det(L^4T_1T_4)^{\nu/2} \sum_{(e_0,\dots,e_{n-1})\neq 0} c(e_0, \dots, e_{n-1}) \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j},$$

where $P_j(\hat{D})$ is a polynomial in elementary symmetric polynomials of $L^{-2}d_i^2$, $i = 1, \dots n$.

We denote for each $M = (e_0, \dots, e_{n-1}) \neq 0$ the polynomial

$$Q_M(L^{-2}D) = \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j}.$$

It is easy to see that $Q_M(L^{-2}D)$ is a homogeneous polynomial of variables $L^{-2}d_i^2$, $i = 1, 2, \dots, n$ of degree $|M|$ with $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$.

Otherwise, we know that the coefficients $C_M(k)$ is a polynomial of variable k, $k = n + j$ degree $|M|$. Therefore, we have the expression for this polynomial. □

5 Integral Representation for Standard Zeta Function

First, we recall briefly the definition of Eisenstein series. For a Dirichlet character ψ mod M, $M > 1$, a weight $k \in \mathbb{N}$ with $\psi(-1) = (-1)^k$ and a complex parameter s with $\mathrm{Re}(s) \gg 0$, we define an Eisenstein series

$$\hat{\mathbb{F}}_n^k(Z, M, \psi, s) \quad \text{and} \quad \mathbb{F}_n^k(Z, M, \psi, s) = \det(Y)^s \hat{\mathbb{F}}_n^k(Z, M, \psi, s)$$

of degree n (with $Z = X + iY \in \mathbb{H}_n$) by

$$\hat{\mathbb{F}}_n^k(Z, M, \psi, s) = \sum_{(C,D)} \psi(\det(C)) \det(CZ + D)^{-k} |\det(CZ + D)|^{-2s}.$$

Here, (C, D) runs over all "non-associated coprime symmetric pairs" with $\det C$ coprime to M. This series converges for $k + 2\mathrm{Re}(s) > n + 1$.

Let q be a prime number, $q \nmid N$. We denote

$$\Delta = \Delta_q^n(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}}^+ \cap \mathrm{GL}_{2n}(\mathbb{Z}[q^{-1}]) \mid \nu(\gamma)^{\pm} \in \mathbb{Z}[q^{-1}], c \equiv 0_n \mod N \right\},$$

a subgroup in $\mathbb{Q}^+$ containing $\Gamma = \Gamma_0^n(N)$. The Hecke algebra over $\mathbb{Q}$ is denoted by $\mathcal{L} = \mathcal{L}_q^n(N) = \mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta)$ and defined as a $\mathbb{Q}$-linear space generated by the double cosets $(g) = (\Gamma g \Gamma)$, $g \in \Delta$ of the group Δ.

For each $j, 1 \leq j \leq n$ let us denote by W_j an automorphism of the algebra $\mathbb{Q}[X_0^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ defined by the rule:

$$X_0 \mapsto X_0 X_j, \quad X_j \mapsto X_j^{-1}, \quad X_i \mapsto X_i, \quad 1 \leq i \leq n, i \neq j.$$

Then, the automorphisms W_j and the permutation group S_n of the variables $X_i (1 \leq i \leq n)$ generate together the Weil group $W = W_n$ and there is the Satake isomorphism:

$$\mathrm{Sat} : \mathcal{L} \to \mathbb{Q}\left[X_0^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}\right]^{W_n}.$$

For any commutative $\mathbb{Q}$-algebra A, the group W_n acts on the set $(A^\times)^{n+1}$, therefore any homomorphism of $\mathbb{Q}$-algebra $\lambda : \mathcal{L} \to A$ can be identified with some element

$$(\alpha_0, \alpha_1, \ldots, \alpha_n) \in [(A^\times)^{n+1}],$$

which is defined up to the action of W_n.

Definition 2 The Satake p-parameters associated to the eigenform f of genus n and weight l are the elements of the $(n+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in [(A^\times)^{n+1}]^{W_n}$ which is the image of the map $f \longmapsto \lambda_f(s)$ under the isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathbb{C}) \cong [(A^\times)^{n+1}]$, which is defined up to the action of W_n.

Let f_0 be an element of $S_n^l(\Gamma_0(Np), \bar{\varphi})$. We assume that f_0 is an eigenform for the Hecke algebras

$$\otimes_{q \nmid Np} \mathcal{L}_{Np,q}^{\circ} \quad \text{and} \quad \otimes_{q | Np} \mathcal{L}_{Np,q}^{\circ}$$

and also an eigenform of $U(L)$ for all L:

$$f_0 | U(L) = \alpha(L) f_0.$$

Proposition 1 *Let* χ *be a Dirichlet character* mod N, $N^2|M$, $l = k + \nu$, $\nu \geq 0$ *and* $\varphi = \psi\bar{\chi}$. *There exists a two variables function* $g(*, *)$ *in* $C^\infty M_n^l(\Gamma_0(M), \varphi)$ *such that for any* $h \in S_n^l(\Gamma_0(N^2 p), \bar{\varphi})$, *the following product*

$$\left\langle\left\langle f_0|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g(*, *)|^z \mathfrak{K}\right\rangle^w_{\Gamma^0(N^2p)}, h\right\rangle^z_{\Gamma^0(N^2p)}$$

equals

$$\Omega_{l,\nu}(0)(RN)^{n(2k+\nu-n-1)}(N^2S)^{\frac{n(n+1)-nl}{2}} \times \chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SL^4}{R^2}\right) D^{(Np,\frac{p}{R_0})}(f_0, k-n, \bar{\chi})\left\langle f_0|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2p \end{pmatrix}, h\right\rangle_{\Gamma^0(N^2p)}$$

with

$$\Omega_{l,\nu}(s) = (-1)^{\frac{nl}{2}} 2^{1+\frac{n(n+1)}{2}-2ns} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n(l+s-\frac{n}{2})\Gamma_n\left(l+s-\frac{n+1}{2}\right)}{\Gamma_n(k+s)\Gamma_n\left(k+s-\frac{n}{2}\right)}$$

and $\alpha(L)$ *is defined by* $f_0|U(L) = \alpha(L) f_0$.

Proof The existence of function $g(z, w)$ could be seen in [3]. □

6 Main Theorem

We fix a Siegel modular form $f \in S_n^l(\Gamma_0(N), \bar{\varphi})$ which is an eigenform of the "good" Hecke algebra $\otimes_{q \nmid Np} \mathcal{L}^\circ_{Np,q}$. Also, we fix a prime p which does not divide N. We consider f_0, an eigenform which is defined as in Section 5. We could see the relation between the eigenforms f and f_0, especially their Satake parameters at p in [3, Proposition 9.1]. We denote by $\beta_1, \ldots, \beta_n$ the Satake p-parameters of f_0. We recall here some notations that we shall use in the statement of our main theorem. For an arbitrary Dirichlet character ψ,we introduce the modified p-Euler factor

$$E_p(s, \psi) := \prod_{j=1}^{n} \frac{\left(1 - \psi(p)\beta_j^{-1} p^{s-1}\right)}{\left(1 - \bar{\psi}(p)\beta_j^{-1} p^{-s}\right)},$$
$$\chi = \chi^0 \cdot \chi_1, \quad \chi \text{ is a Dirichlet character modulo } RN.$$

To formulate our result, let

$$\Lambda^+_\infty(s) := \frac{(2i)^s \cdot \Gamma(s)}{(2\pi i)^s} \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s+l-j),$$
$$\Lambda^-_\infty(s) := (2i)^s \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(1-s+l-j),$$

where $\Gamma_{\mathbb{C}}(s) := 2 \cdot (2\pi)^{-s}\Gamma(s)$. Furthermore, for any character χ of p-power conductor c_χ, we let

$$\begin{aligned} A^-(\chi) &:= c_\chi^{nl-\frac{n(n+1)}{2}} \alpha_0(c_\chi)^{-2} \cdot (\chi^0([p, c_\chi]) \cdot \chi(-1)G(\chi)), \\ A^+(\chi) &:= (\overline{\chi^0\varphi})_o(c_\chi) \cdot \frac{A^-(\chi)}{\chi(-1)G(\chi)}, \end{aligned}$$

where $[a, b]$ denotes the least common multiple of the integers a, b and $G(\chi)$ is the usual Gauss sum. Finally, we let

$$\begin{aligned} E_p^+ &:= (1 - (\overline{\varphi\chi\chi^0})_0(p)p^{t-1}) \cdot E_p(s, \chi\chi^0), \\ E_p^- &:= E_p(p, \chi\chi^0). \end{aligned}$$

Theorem 3 (Main theorem) *For each prime number p, there exist two p-adic admissible measures μ^+, μ^- on $\mathbb{Z}_p^\times$ with values in $\mathbb{C}_p$ verifying the following properties:*

(i) *For all pairs (s, χ) such that $\chi \in X_p^{\text{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $1 \le s \le l - n$, $s \equiv \delta \mod 2$ and for $s = 1$ the character χ^2 is non-trivial, the following equality holds:*

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p\left(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s, \chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, s, \overline{\chi\chi^0})\right).$$

(ii) *For all pairs (s, χ) such that $\chi \in X_p^{\text{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $1 - l + n \le s \le 0$, $s \equiv \delta + 1 \mod 2$ the following equality holds:*

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p\left(c_\chi^{n(1-s)} A^+(\chi) \cdot E_p^-(1-s, \chi\chi^0) \frac{\Lambda_\infty^-(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, 1-s, \overline{\chi\chi^0})\right).$$

(iii) *If $\mathrm{ord}_p(\alpha_0(p)) = 0$ (i.e., f is p-ordinary), then the measures in* (i) *and* (ii) *are bounded.*

(iv) *In the general case (but assuming that $\alpha_0(p) \neq 0$) with $x \in \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ the holomorphic functions*

$$\begin{aligned} \mathcal{D}^+(x) &= \int x d\mu^+, \\ \mathcal{D}^-(x) &= \int x d\mu^- \end{aligned}$$

belong to type $o(\log(x_p)^h)$ where $h = [4\mathrm{ord}_p(\alpha_0(p))] + 1$. Furthermore, they can be represented as the Mellin transforms of certain h-admissible measures.

(v) *If $h \le k - m - 1$, then the functions $\mathcal{D}^\pm$ are uniquely determined by the above conditions* (i) *and* (ii).

7 Fourier Expansion of $\mathcal{H}$-Functions

Fourier expansion of $\mathcal{H}$-functions was investigated carefully by Bocherer–Schmidt in [3, Section 5]. We consider the Fourier expansion of $\mathcal{H}$-functions and $\mathcal{H}'$ at two values of s, namely

$$s_0 := 0 \quad \text{and} \quad s_1 := \frac{m+1}{2} - k = \frac{1}{2} - t, \quad k = n + t, t \ge 1.$$

We denote by Λ_n the set of all half-integral symmetric matrices of size n and by Λ_n^*, Λ_n^+ the subsets of matrices of maximal rank and of positive definite matrices respectively. The Fourier expansion of $\mathcal{H}_{L,\chi}^{(t)}$ in the case $\chi \neq 1$ at $s = 0$ is

$$\begin{aligned}
&A_{2n}^k (2\pi i)^{n\nu} R^{\frac{n(n-1)}{2}} (N^2 p)^{-ln} G(\chi_1)^n \chi^0(R)^n \chi_1(N)^n \\
&\times \sum_{T_1\in\Lambda_n^+} \sum_{T_4\in\Lambda_n^+} \sum_{2T_2\in\mathbb{Z}^{(n,n)},\, T=\begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda_{2n}^+} \mathfrak{P}_{n,k}^{\nu}(T) G_n(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2)) \\
&\times \sum_{G\in\mathrm{GL}(2n,\mathbb{Z})\backslash\mathbb{D}(T)} (\varphi\chi)^2(\det G)\det(2T[G^{-1}])^{k-\frac{2n+1}{2}} L(k-n, \epsilon_{T[G^{-1}]}\varphi\chi) \\
&\times \sum_{\substack{b\mid \det(2T[G^{-1}]) \\ b>0}} (\varphi\chi^0)(b)\cdot b^{-k}\cdot d(b, \mathfrak{T}[G^{-1}]),
\end{aligned}$$

where the quadratic character ϵ_T is defined as follows:

$$\epsilon_T(*) := \left(\frac{(-1)^n \det(2T)}{*}\right)$$

and $D(T) = \{G \in M_n(\mathbb{Z}^* | T[G^{-1}] \in \Lambda_n\}$, $b|\det(2T)$. For any $b|\det(2T)$, there exists an integer $d(b, T)$ such that

$$\prod_q B_q^m(q^{-s}, T) = \sum_{\substack{b\mid \det(2T) \\ b>0}} \psi(b) b^{-s} d(b, T),$$

where $B_q^m(q^{-s}, T)$ is a polynomial in $\mathbb{Z}[x]$ of degree $\leq m-1$ satisfying certain conditions.

At $s = s_1$, we look at the Fourier expansion of $\mathcal{H}'^{(t)}_{L,\chi}(z, w)$ with $\chi \neq 1$

$$\begin{aligned}
\mathcal{H}'^{(t)}_{L,\chi}(z, w) \;=\; &\mathcal{L}(k+2s, \varphi\chi)\mathring{\mathfrak{D}}_{n,k}^{\nu}\left(\mathbb{F}_{2n}^k\left(--, R^2N^2\frac{p}{R_0}, \varphi, s\right)^{(\chi)}\right) \\
&|^z U(L^2)|^w U(L^2)|_l^z \begin{pmatrix} 1 & 0 \\ 0 & N^2p\end{pmatrix} |_l^w \begin{pmatrix} 1 & 0 \\ 0 & N^2p\end{pmatrix}.
\end{aligned}$$

The Fourier expansion of $\mathcal{H}'^{(t)}_{L,\chi}(z, w)$ at $s = s_1$ is

$$\begin{aligned}
&B_{2n}^k (2\pi i)^{n\nu} R^{\frac{n(n-1)}{2}} (N^2 p)^{-ln} G(\chi_1)^n \chi^0(R)^n \chi_1(N)^n \\
&\times \sum_{T_1\in\Lambda_n^+} \sum_{T_4\in\Lambda_n^+} \sum_{2T_2\in\mathbb{Z}^{(n,n)},\, T=\begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda_{2n}^+} \mathfrak{P}_{n,k}^{\nu}(T) G_n(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2)) \\
&\times \sum_{G\in\mathrm{GL}(2n,\mathbb{Z})\backslash\mathbb{D}(T)} (\varphi\chi)^2(\det G)\det(2T[G^{-1}])^{2t-1} L(k-n, \epsilon_{T[G^{-1}]}\varphi\chi) \\
&\times \sum_{\substack{b\mid \det(2T[G^{-1}]) \\ b>0}} (\varphi\chi^0)(b)\cdot b^{-k}\cdot d(b, \mathfrak{T}[G^{-1}]).
\end{aligned}$$

The Fourier expansion of $\mathcal{H}_{L,\chi}^{(t)}$ at $s=0$ is given by

$$\begin{aligned}
&A_{2n}^{k}(2\pi i)^{n\nu}R^{\frac{n(n-1)}{2}}(N^2S)^{-ln}G(\chi_1')^n\chi^0(R')^n\chi_1'(N)^n\\
&\times\sum_{T_1\in\Lambda_n^+}\sum_{T_4\in\Lambda_n^+}\sum_{2T_2\in\mathbb{Z}^{(n,n)},T=\begin{pmatrix}L^2T_1 & T_2\\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda_{2n}^+}\mathfrak{P}_{n,k}^{\nu}(T)G_n(2T_2,N,\chi^0)\bar{\chi}_1'(\det(2T_2))\\
&\times\sum_{G\in\mathrm{GL}(2n,\mathbb{Z})\backslash\mathbb{D}(T)}(\varphi\chi')^2(\det G)\det(2T[G^{-1}])^{k-\frac{2n+1}{2}}L(k-n,\epsilon_{T[G^{-1}]}\varphi\chi')\\
&\times\sum_{\substack{b\mid\det(2T[G^{-1}])\\ b>0}}(\varphi\chi')(b)\cdot b^{-k}\cdot d(b,\mathfrak{T}[G^{-1}]),
\end{aligned}$$

where

χ' is a Dirichlet character mod $R'N$ and
$\chi'=\chi^0\cdot\chi_1'$ where χ^0 is a Dirichlet character mod N,
χ_1' is a primitive Dirichlet character mod R'.

The Fourier expansion of $\mathcal{H'}_{L,\chi}^{(t)}(z,w)$ at $s=0$ is

$$\begin{aligned}
&B_{2n}^{k}(2\pi i)^{n\nu}R'^{\frac{n(n-1)}{2}}(N^2p)^{-ln}G(\chi_1')^n\chi^0(R')^n\chi_1'(N)^n\\
&\times\sum_{T_1\in\Lambda_n^+}\sum_{T_4\in\Lambda_n^+}\sum_{2T_2\in\mathbb{Z}^{(n,n)},T=\begin{pmatrix}L^2T_1 & T_2\\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda_{2n}^+}\mathfrak{P}_{n,k}^{\nu}(T)G_n(2T_2,N,\chi^0)\bar{\chi}_1'(\det(2T_2))\\
&\times\sum_{G\in\mathrm{GL}(2n,\mathbb{Z})\backslash\mathbb{D}(T)}(\varphi\chi')^2(\det G)\det|G|^{2t-1}L(1-t,\epsilon_{T[G^{-1}]}\varphi\chi)\\
&\times\sum_{\substack{b\mid\det(2T[G^{-1}])\\ b>0}}(\varphi\chi')(b)\cdot b^{-k}\cdot d(b,\mathfrak{T}[G^{-1}]).
\end{aligned}$$

8 Algebraic Linear Form

For a modular form $g(z,w)$ of genus n and weight l,which is a function of z (or w), we consider the following $\mathbb{C}$-valued function (see [3, Section 9]):

$$\mathcal{F}(g)=\frac{\left\langle\left\langle f_0|_l\begin{pmatrix}0&-1\\1&0\end{pmatrix},g|^z\mathfrak{K}\right\rangle^w_{\Gamma_0(N^2p)},f_0|_l\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\right\rangle^z_{\Gamma_0(N^2p)}}{\langle f_0,f_0\rangle^2_{\Gamma_0(N^2p)}},$$

where $(f\mid\mathfrak{K})(z)=f(-\bar{z})$.

We want to know the action of the linear form $\mathcal{F}$ on $\mathcal{H}_{L,\chi}^{(t)}(z,w)$ and $\mathcal{H'}_{L,\chi}^{(t)}(z,w)$. First, we define the p-Euler factor

$$E_p(s,\psi)=\prod_{j=1}^{n}\frac{(1-\psi(p)\beta_j^{-1}p^{s-1})}{(1-\bar{\psi}(p)\beta_jp^{-s})},$$

where ψ is an arbitrary Dirichlet character and $\beta_1, \ldots, \beta_n$ are Satake p-parameters of eigenform f_0 which is a modular form associated to f. For simplicity, we now write χ for χ_1. Then computing $\mathcal{F}(\mathcal{H}_{L,\chi}^{(t)}(z, w))$, we have

$$\begin{aligned}
&\mathcal{F}(\mathcal{H}_{L,\chi}^{(t)}(z, w)) \\
&= \frac{\left\langle\left\langle f_0|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{H}_{L,\chi}^{(t)}(z, w)|^z\mathfrak{K}\right\rangle^w_{\Gamma_0(N^2p)}, f_0|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2p \end{pmatrix}\right\rangle^z_{\Gamma_0(N^2p)}}{\langle f_0, f_0\rangle^2_{\Gamma_0(N^2p)}} \\
&= \langle f_0, f_0\rangle^{-2}_{\Gamma_0(N^2p)} \cdot \Omega_{l,\nu}(0)(RN)^{n(2k+\nu-n-1)}(N^2p)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl} \\
&\quad \times\alpha_0\left(\frac{pL^4}{R^2}\right) D^{(NS,\frac{p}{R_0})}(f_0, k-n, \bar{\chi})\left\langle f_0 \,|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2p \end{pmatrix}, f_0 \,|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2p \end{pmatrix}\right\rangle^z_{\Gamma^0(N^2p)} \\
&= \langle f_0, f_0\rangle^{-1} \cdot \Omega_{l,\nu}(0) \cdot (N^2p)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\chi\chi^0(-1)^n(-1)^{ln} \\
&\quad \times(Nc_\chi)^{n(l+t-1)}\alpha_0\left(pL^4c_\chi^{-2}\right) \cdot E_p(t, \chi^0\chi) \cdot \overline{\chi^0}\left(\frac{p}{(p, c_\chi)}\right)^n \cdot D^{(Np)}(f, t, \overline{\chi^0\chi}),
\end{aligned}$$

for any character χ, whose conductor c_χ is a power of p. Similarly, we have

$$\begin{aligned}
\mathcal{F}(\mathcal{H'}_{L,\chi}^{(t)}) = &\langle f_0, f_0\rangle^{-1} \cdot \Omega_{l,\nu}(s_1)\frac{p_{s_1}(k)}{d_{s_1}(k)} \cdot (N^2p)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\chi\chi^0(-1)^n(-1)^{ln} \\
&\times(Nc_\chi)^{n(l-t)}\alpha_0\left(pL^4c_\chi^{-2}\right) \cdot E_p(1-t, \chi^0\chi) \cdot \overline{\chi^0}\left(\frac{p}{(p, c_\chi)}\right)^n \\
&\times D^{(Np)}(f, 1-t, \overline{\chi^0\chi}).
\end{aligned}$$

$\mathcal{F}(\mathcal{H}_{L,\chi})$, $\mathcal{F}(\mathcal{H}'_{L,\chi})$ depend only on L by the factor $\alpha_0(L^4)$.

9 Distribution in Siegel Modular Forms

Consider the $\mathbb{C}_p$-linear forms $\mathcal{D}^\pm : C^{l-n}(\mathbb{Z}_S^\times) \to \mathbb{C}_p$ defined on the local monomials x_p^j for $j = 0, 1, \ldots, l-n-1$ by

$$\begin{aligned}
\int_{a+(L)} x_p^j d\mathcal{D}^+ &:= \int_{a+(L)} d\mathcal{D}^+_{j+1,L}, \\
\int_{a+(L)} x_p^j d\mathcal{D}^- &:= \int_{a+(L)} d\mathcal{D}^-_{-j,L},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}^+_{j+1,L} &= \mathcal{F}\left(\mathcal{H}_{L,\chi}^{(j+1)}\right), \\
\mathcal{D}^-_{-j,L} &= \mathcal{F}\left(\mathcal{H'}_{L,\chi}^{(-j)}\right).
\end{aligned}$$

Then, we have

$$\begin{aligned}\int_{a+(L)}(x_p-a_p)^r d\mathcal{D}^+ &= \sum_{j=0}^{r}\binom{t}{j}(-a)^{r-j}\int_{a+(L)} d\mathcal{D}^+_{j+1,L}\\ &= \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)\mathcal{D}^+_{j+1,L}(\chi)\\ &= \gamma(L)\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)\mathcal{F}\left(\mathcal{H}^{(j+1)}_{L,\chi}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{a+(L)}(x_p-a_p)^r d\mathcal{D}^- &= \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\int_{a+(L)} d\mathcal{D}^{c-}_{-j,L}\\ &= \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)\mathcal{D}^-_{-j,L}(\chi)\\ &= \gamma'(L)\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)\mathcal{F}\left(\mathcal{H}'^{(-j)}_{L,\chi}\right).\end{aligned}$$

We denote

$$\begin{aligned}A^+ &:= \gamma(L)\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)v^+(L, j+1, \chi),\\ A^- &:= \gamma'(L)\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)v^-(L, -j, \chi).\end{aligned}$$

Then, we need to prove that A^+, A^- also verify the growth conditions

$$\begin{aligned}A^+ &= o\left(|L|_p^{r-h}\right),\\ A^- &= o\left(|L|_p^{r-h}\right).\end{aligned}$$

We state here the main congruences

$$\left|\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)v^+(L, j+1, \chi)\right|_p \le C\cdot p^{-\nu r}$$

and

$$\left|\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)v^-(L, -j, \chi)\right|_p \le C\cdot p^{-\nu r}$$

with $r = 0, 1, \ldots, l-n-1$, $L = p^\nu$.

$$A^+ = \gamma(L)\sum_{j=0}^{t}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)v^+(L, j+1, \chi),$$

where

$$v^+(L, j+1, \chi) = A_{2n}^k (2\pi i)^{n\nu} \cdot R^{\frac{n(n-1)}{2}} (N^2 S)^{-ln} G(\chi_1)^n \chi^0(R)^n \chi_1(N^n)$$
$$\times \sum_{T_1 \in \Lambda_n^+} \sum_{T_4 \in \Lambda_n^+} \sum_{2T_2 \in \mathbb{Z}^{(n,n)}} \mathfrak{P}_{n,k}^{\nu}(T) G_n(2T_2, N, \chi^0) \bar{\chi}_1(\det(2T_2))$$
$$\times \sum_{G \in \mathrm{GL}(2n,\mathbb{Z})\backslash D(T)} (\varphi\chi)^2 (\det G) \det(2T[G^{-1}])^{k-\frac{2n+1}{2}} \cdot L(k-n, \epsilon_{\det(T[G^{-1}])}\varphi\chi)$$
$$\times \sum_{b|\det T[G^{-1}]} (\varphi\chi)(b) b^{-k} d(b, T[G^{-1}]).$$

10 Proof of the Main Theorem

(i) We denote $\omega = \bar{\epsilon}_{T[G^{-1}]}$. For $s \in \mathbb{Z}$, $s > 0$, we use the Mazur measure and the functional equation of L-functions associated to Dirichlet characters

$$i_p \left(\int_{\mathbb{Z}_p^\times} \chi x_p^s d\mu^+(\omega) \right) = \frac{C_{\omega\bar{\chi}}}{G(\omega\bar{\chi})} \times \prod_q \left(\frac{1 - \chi\bar{\omega}(q) q^{s-1}}{1 - \chi\bar{\omega}(q) q^{-s}} \right) L_{M_0}^+(s, \bar{\chi}\omega),$$

and for $s \in \mathbb{Z}$, $s \leq 0$,

$$i_p \left(\int_{\mathbb{Z}_p^\times} \chi x_p^s d\mu^-(\omega) \right) = L_{M_0}^+(s, \bar{\chi}\omega),$$

where

$$L_{M_0}^+(s, \bar{\chi}\omega) = L_{\bar{M}}(s, \bar{\chi}\omega) 2i^\delta \frac{\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s},$$
$$L_{M_0}^-(s, \bar{\chi}\omega) = L_{\bar{M}}(s, \bar{\chi}\omega)$$

are normalized Dirichlet L-functions with $\delta \in \{0, 1\}$ and $\bar{\chi}\omega(-1) = (-1)^\delta$. The function $G(\omega\bar{\chi})$ denotes the Gauss sum of the Dirichlet character $\omega\bar{\chi}$. The functions satisfy the functional equation

$$L_{M_0}^-(1-s, \bar{\chi}\omega) = \prod_{q \in S \backslash S(\chi)} \left(\frac{1 - \chi\bar{\omega}(q) q^{s-1}}{1 - \chi\bar{\omega}(q) q^{-s}} \right) L_{M_0}^+(s, \bar{\chi}\omega).$$

Otherwise, we can write the factor $\sum_{b|\det T[G^{-1}]} (\varphi\chi)(b) b^{-k} d(b, T[G^{-1}])$ as the finite linear combination with integer coefficients $b_i \in \mathbb{Z}$:

$$\sum_{b|\det T[G^{-1}]} (\varphi\chi)(b) b^{-k} d(b, T[G^{-1}]) = \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \chi(\alpha_i) \alpha_i^k \delta_{\alpha_i}$$
$$= \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \int_{\mathbb{Z}_p^\times} \chi(x) x^k \delta_{\alpha_i},$$

where δ_{α_i} is the Dirac measure at the point $\alpha_i \in \mathbb{Z}_p^\times$. We define the measure μ_{T_2} by

$$\int_{\mathbb{Z}_p^\times} \chi(x) x^s d\mu_{T_2} = \int_{\mathbb{Z}_p^\times} \chi(x) x^k \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \delta_{\alpha_i}.$$

We have the following measures obtained by convolution

$$\mu^{+}(T_2,\omega) = \mu^{+}(\omega) * \mu_{T_2},$$
$$\mu^{-}(T_2,\omega) = \mu^{-}(\omega) * \mu_{T_2}.$$

As in the previous section, we know that to prove μ^{+} is an h-admissible measure, we have to prove $A^{\pm} = o\left(|L|_p^{r-h}\right)$.

Actually, we have the factor

$$\gamma(L) = \langle f_0, f_0\rangle \cdot \Omega_{l,\nu}^{-1}(0) \cdot (N^2 p)^{\frac{nl}{2}-\frac{n(n+1)}{2}}(-1)^{-n}(-1)^{-ln}N^{n(1-j-l)}\alpha_0(pL^4)^{-1}.$$

We easily see that

$$\gamma(L) \equiv 0 \mod |L|_p^{[-4\mathrm{ord}_p(\alpha_0(p))]-1}.$$

On the other hand, we use interpolation for the polynomial

$$C_M(j) = \sum_{i=0}^{|M|}\mu_i \cdot \frac{(j+i+1)!}{(j+1)!}.$$

Therefore,

$$B^{+} = \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \bmod L}\chi^{-1}(a)\sum_{i=0}^{|M|}\mu_i \cdot \frac{(j+i+1)!}{(j+1)!}$$
$$\times G_n(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2))\int_{\mathbb{Z}_p^{\times}}\chi x^{j+1}d\mu^{+}(T_2,\omega).$$

Here, we fix T_1 and T_4 and study only the dependence on T_2. Then

$$B^{+} = \int_{x\equiv a \mod L}\sum_{i=0}^{|M|}\mu_i\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{(j+i+1)!}{(j+1)!}x^{j+1}d\mu^{+}(T_2,\omega)$$
$$= \int_{x\equiv a \mod L}\sum_{i=0}^{|M|}\mu_i x^{-1}\cdot\frac{\partial^i}{\partial x^i}(x^{i+1}(x-a)^r)d\mu^{+}(T_2,\omega).$$

Then, $(x-a)^{|M|} \equiv 0 \mod L^{|M|}$ gives the congruence

$$B^{+} \equiv 0 \mod L^{r-i} \equiv 0 \mod L^{r-|M|}.$$

On the other hand, $Q_M(L^{-2}D)$ is a homogeneous polynomial of L^{-2} with degree $|M|$. So $Q_M(L^{-2}D) \equiv 0 \mod L^{-2|M|}$. Thus

$$A^{+} \equiv 0 \mod L^{-h}\cdot L^{2n\nu}L^{r-|M|}\cdot L^{-2|M|} \equiv 0 \mod L^{r-h-3|M|+2n\nu}.$$

Therefore,

$$A^{+} \equiv 0 \mod L^{r-h}.$$

This congruence says that with h as above, μ^{+} is an h-admissible measure.

(ii) The proof for the negative case is actually similar to the proof for the positive case in (i).

(iii) The assertion (iii) (i.e., the ordinary case) which was proved by Panchishkin (see [7]) with even genus and Böcherer–Schmidt with arbitrary genus (see [3]), also follows easily from the main congruence.

(iv) In the general case $h > k-n-1$, the integer s runs over $\{0, 1, \ldots, h-1\}$ and one can extend the values of our functions $\mathcal{D}^{\pm}\left(\chi x_p^s\right)$ by the equality $\mathcal{D}^{\pm}\left(\chi x_p^s\right) = 0$ (for all

$\chi \in X_S^{\text{tors}}$ of a conductor divisible by all the prime divisors of Np) for $s > l-n-1-\nu$ (but keeping the same values for $0 \le s \le l-n-1-\nu$). The verification of the h-admissibility goes without change in this situation. Also, one obtains again the h-admissible measures $\mu^{\pm}$ with $h = [4\text{ord}_p(\alpha_0(p))] + 1$. The functions $\mathcal{D}^{\pm}$ therefore coincide with the Mellin transforms of these h-admissible measures. We also can find the proof of (iv) in [1, 6, 8].

(v) Finally, if $h \le l-n-1$, then the conditions in (i) and (ii) uniquely determine the analytic functions $\mathcal{D}^{\pm}$ of type $o(\log(x_p^h))$ by their values following a general property of admissible measures (see [1, 8]). In the case $h > l-n-1$, there exist many analytic functions $\mathcal{D}^{\pm}$ verifying the condition in (i) and (ii) which depend on the choice of analytic continuation (interpolation) for the values $\mathcal{D}^{\pm}(\chi x_p^s)$ if $s > l-n-1-\nu$. But one shows in Theorem 3 that there exists at least one such continuation (for example, the one which was described in the proof of (iv)).

Acknowledgments The author would like to thank his adviser, Professor Alexei Panchishkin, for all assistance during the 3 years of research at the Institute Fourier. This paper is based on the author's PhD thesis in the University Grenoble-1, 2014. The author would like to thank the anonymous referees of this manuscript whose remarks and corrections improved significantly the exposition.

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