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Semilinear Strongly Degenerate Parabolic Equations with a New Class of Nonlinearities

Dao Trong Quyet[1] **· Le Thi Thuy**[2] **· Nguyen Xuan Tu**[3]

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Abstract We study the existence and long-time behavior of weak solutions in terms of the existence of a global attractor to a class of semilinear strongly degenerate parabolic equations with a new class of nonlinearities containing the exponential ones. The main novelty of our results is that no restriction on the upper growth of the nonlinearities is imposed.

Keywords Degenerate parabolic equation · Δ_λ-Laplace operator · Global attractor · Exponential nonlinearity

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✉ Dao Trong Quyet
dtq100780@gmail.com

Le Thi Thuy
thuylephuong@gmail.com

Nguyen Xuan Tu
nguyenxuantu1982@gmail.com

1 Faculty of Information Technology, Le Quy Don Technical University, 100 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam

2 Department of Mathematics, Electric Power University, 235 Hoang Quoc Viet, Tu Liem, Hanoi, Vietnam

3 Department of Basic Sciences, Food Industrial College, Nguyen Tat Thanh, Viet Tri, Phu Tho, Vietnam

1 Introduction

In this paper, we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} u_t - \Delta_\lambda u + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $u_0, g \in L^2(\Omega)$, the nonlinear term $f(u)$ satisfies certain conditions specified later, and Δ_λ is a strongly degenerate operator of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2(x)\partial_{x_i}),$$

where $\lambda = (\lambda_1, \dots, \lambda_N) : \mathbb{R}^N \to \mathbb{R}^N$ satisfies certain conditions specified below. This operator was introduced by Franchi and Lanconelli in [9] (see also [8]) and recently reconsidered in [12] under an additional assumption that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. Here, the functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes, i.e., $\lambda_i > 0$, $i = 1, \dots, N$ in $\mathbb{R}^N \setminus \prod$,where $\prod = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\}$. As in [12], we assume that λ_i satisfy the following properties:

1. $\lambda_1(x) \equiv 1, \lambda_i(x) = \lambda_i(x_1, \dots, x_{i-1}), i = 2, \dots, N$;
2. For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, \dots, N$, where
$$x^* = (|x_1|, \dots, |x_N|) \text{ if } x = (x_1, \dots, x_N);$$
3. There exists a constant $\rho \geq 0$ such that
$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_i(x) \quad \forall k \in \{1, \dots, i-1\}, i = 2, \dots, N,$$
and for every $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i \geq 0 \ \forall i = 1, \dots, N\}$;
4. There exists a group of dilations $\{\delta_t\}_{t>0}$
$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_t(x) = \delta_t(x_1, \dots, x_N) = (t^{\epsilon_1} x_1, \dots, t^{\epsilon_N} x_N),$$
where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_N$ such that λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$,i.e.,
$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1}\lambda_i(x) \quad \forall x \in \mathbb{R}^N, t > 0, i = 1, \dots, N.$$
This implies that the operator Δ_λ is δ_t-homogeneous of degree two, i.e.,
$$\Delta_\lambda(u(\delta_t(x))) = t^2(\Delta_\lambda u)(\delta_t(x)) \quad \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$,i.e.,

$$Q := \epsilon_1 + \dots + \epsilon_N.$$

The homogeneous dimension Q plays a crucial role, both in the geometry and the functional associated to the operator Δ_λ.

The Δ_λ-Laplace operator contains many degenerate elliptic operators such as the Grushin type operator

$$G_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha > 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, and the strongly degenerate operator of the form

$$P_{\alpha,\beta} = \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ ($N_i \geq 1, i = 1, 2, 3$), α, β are real positive constants, see [19]. We refer the interested reader to [13, Section 2.3] for other examples of Δ_λ-Laplacians. See also [4, 16] for recent results related to elliptic equations involving this operator.

In the last years, the existence and long-time behavior in terms of existence of global attractors of solutions to semilinear parabolic equations involving the above degenerate operators have been studied extensively by a number of authors. Up to now, there are two main kinds of nonlinearities that have been considered. The first one is the class of nonlinearities that is locally Lipschitzian continuous and satisfies a Sobolev growth condition

$$|f(u) - f(v)| \leq C(1 + |u|^\rho + |v|^\rho)|u - v|, \quad 0 \leq \rho < \frac{4}{Q-2},$$

and some suitable dissipative conditions; see [2, 13–15, 20]. The second one is the class of nonlinearities that satisfies a polynomial growth

$$C_1|u|^p - C_0 \leq f(u)u \leq C_2|u|^p + C_0 \quad \text{for some } p \geq 2,$$
$$f'(u) \geq -\ell,$$

see [3, 6, 18, 20]. See also some related results in the case of unbounded domains [1, 5], the more delicate case due to the lack of compactness of the Sobolev type embeddings.

Note that for both above classes of nonlinearities, some restriction on the upper growth of the nonlinearity is imposed and an exponential nonlinearity, for example, $f(u) = e^u$, does not hold. In this paper, we try to remove this restriction and we were able to prove the existence of weak solutions and existence of global attractors for a very large class of nonlinearities that particularly covers both above classes and even exponential nonlinearities. This is the main novelty of our paper.

To study problem (1), we assume that the initial datum $u_0 \in L^2(\Omega)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \geq -\ell, \tag{2}$$
$$f(u)u \geq -\mu u^2 - C_1, \tag{3}$$

where C_1, ℓ are two positive constants, $0 < \mu < \gamma_1$ with $\gamma_1 > 0$ is the first eigenvalue of the operator $-\Delta_\lambda$ in Ω with the homogeneous Dirichlet boundary condition, and $F(u) = \int_0^u f(s)ds$ is a primitive of f;

(G) $g \in L^2(\Omega)$.

It follows from (2) that $0 \leq \int_0^u (f'(s)s + \ell s)ds$, and therefore by integrating by parts, we obtain

$$F(u) \leq f(u)u + \ell\frac{u^2}{2} \quad \text{for all } u \in \mathbb{R}. \tag{4}$$

To study problem (1), we use the weighted Sobolev space $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$ defined as the completion of $C_0^1(\Omega)$ in the norm

$$\|u\|_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} := \left(\int_\Omega |\nabla_\lambda u|^2 dx\right)^{1/2}.$$

This is a Hilbert space with respect to the following scalar product

$$((u, v))_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} = \int_\Omega \nabla_\lambda u \cdot \nabla_\lambda v dx.$$

We also use the Hilbert space $D(\Delta_\lambda)$ defined as the domain of the operator $-\Delta_\lambda$ with the homogeneous Dirichlet boundary condition

$$D(\Delta_\lambda) = \left\{ u \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) | \Delta_\lambda u \in L^2(\Omega) \right\},$$

with the graph norm

$$\|u\|_{D(\Delta_\lambda)} := \left(\int_\Omega |\Delta_\lambda u|^2 dx \right)^{1/2}.$$

By the result in [12], we know that the embedding $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Using this embedding and the definition of $D(\Delta_\lambda)$, we will show in Lemma 4 below that the embedding $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$ is also compact. These compact embeddings will play an important role for our investigation.

Let $\gamma_1 > 0$ be the first eigenvalue of the operator $-\Delta_\lambda$ in Ω with homogeneous Dirichlet boundary conditions. Then

$$\gamma_1 = \inf \left\{ \frac{\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}}{\|u\|^2_{L^2(\Omega)}} : u \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \setminus \{0\} \right\}.$$

Therefore,

$$\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} \geq \gamma_1 \|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega). \tag{5}$$

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions by utilizing the compactness method and weak convergence techniques in Orlicz spaces [11]. In Section 3, we prove the existence of global attractors for the semigroup generated by the problem in various spaces. The main novelty of the paper is that the nonlinearity can grow exponentially.

2 Existence and Uniqueness of Weak Solutions

Definition 1 A function u is called a weak solution of problem (1) on $(0, T)$ if $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \overset{\circ}{W}{}^{1,2}_\lambda(\Omega))$, $f(u) \in L^1(Q_T)$, $u(0) = u_0$, $\frac{du}{dt} \in L^2(0, T; (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*) + L^1(Q_T)$, and

$$\frac{du}{dt} - \Delta_\lambda u + f(u) = g \quad \text{in } L^2(0, T; (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*) + L^1(Q_T)$$

or equivalently,

$$\left\langle \frac{du}{dt} - \Delta_\lambda u + f(u), w \right\rangle = \langle g, w \rangle$$

for all test functions $w \in W := \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \cap L^\infty(\Omega)$ and for a.e. $t \in (0, T)$. Here, $\langle \cdot, \cdot \rangle$ denotes the dual bracket between W and its dual W^*, and $(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*$ is the dual space of $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$.

Theorem 1 *Assume* (**F**)–(**G**) *hold. Then for any* $u_0 \in L^2(\Omega)$ *and* $T > 0$ *given, problem* (1) *has a unique weak solution* u *on the interval* $(0, T)$*. Moreover, the mapping* $u_0 \mapsto u(t)$ *is continuous on* $L^2(\Omega)$*, that is, the solutions depend continuously on the initial data.*

Proof i) Existence. We will prove the existence of a weak solution by using the compactness method. The main difference compared with the proofs in [3, 20] is that the nonlinear term $f(u)$ only belongs to $L^1(Q_T)$ due to no restriction imposed on its upper growth. This introduces some essential difficulties when establishing a priori estimates and passing to the limit for the nonlinear term.

Let $\{u_n\}$ be the Galerkin appropriate solutions. We will establish some a priori estimates for u_n. We have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \int_\Omega f(u_n)u_n dx = \int_\Omega g u_n dx. \tag{6}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} - \mu\|u_n(t)\|^2_{L^2(\Omega)} - C_1|\Omega|\\ &\quad\le \frac{1}{2\varepsilon}\|g\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|u_n(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the inequality (5), we get

$$\begin{aligned}&\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \varepsilon\|u_n(t)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + (2\gamma_1 - 2\mu - \varepsilon\gamma_1 - \varepsilon)\|u_n(t)\|^2_{L^2(\Omega)}\\ &\quad\le \frac{1}{\varepsilon}\|g\|^2_{L^2(\Omega)} + 2C_1|\Omega|,\end{aligned}$$

where $\varepsilon > 0$ is small enough so that $2\gamma_1 - 2\mu - \varepsilon\gamma_1 - \varepsilon > 0$. Integrating from 0 to t, $0 \le t \le T$, we get

$$\begin{aligned}&\|u_n(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t \|u_n(s)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} ds + (2\gamma_1 - 2\mu - \varepsilon\gamma_1 - \varepsilon)\int_0^t \|u_n(s)\|^2_{L^2(\Omega)} ds\\ &\quad\le \frac{1}{\varepsilon}\|g\|^2_{L^2(\Omega)}T + 2C_1|\Omega|T + \|u(0)\|^2_{L^2(\Omega)}.\end{aligned}$$

This inequality yields

$$\begin{aligned}&\{u_n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\ &\{u_n\} \text{ is bounded in } L^2(0,T;\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)).\end{aligned}$$

Due to the boundedness of $\{u_n\}$ in $L^2(0,T;\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega))$, it is easy to check that $\{\Delta_\lambda u_n\}$ is bounded in $L^2(0,T;(\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega))^*)$. From the above results, we can assume that (up to a subsequence)

$$\begin{aligned}u_n &\rightharpoonup u \text{ in } L^2(0,T;\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)),\\ u_n &\rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega)),\\ \Delta_\lambda u_n &\rightharpoonup \Delta_\lambda u \text{ in } L^2(0,T;(\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega))^*).\end{aligned}$$

On the other hand, using the Cauchy inequality in (6), we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \int_\Omega f(u_n)u_n dx \le \frac{1}{2\gamma_1}\|g\|^2_{L^2(\Omega)} + \frac{\gamma_1}{2}\|u_n\|^2_{L^2(\Omega)}.$$

Noting that $\|u_n\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} \geq \gamma_1 \|u_n\|^2_{L^2(\Omega)}$, integrating this inequality from 0 to T, we have

$$\int_0^T \|u_n\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} ds + 2\int_{Q_T} f(u_n)u_n dxdt \leq \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma_1}\|g\|^2_{L^2(\Omega)} T.$$

Hence,

$$\int_{Q_T} f(u_n)u_n dxdt \leq C. \tag{7}$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(Q_T)$. Putting $h(s) = f(s) - f(0) + \gamma s$, where $\gamma > \ell$ and noting that $h(s)s = (f(s) - f(0))s + \gamma s^2 = f'(c)s^2 + \gamma s^2 \geq (\gamma - \ell)s^2 \geq 0$ for all $s \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{Q_T} |h(u_n)|dxdt &\leq \int_{Q_T \cap \{|u_n|>1\}} |h(u_n)u_n|dxdt + \int_{Q_T \cap \{|u_n|\leq 1\}} |h(u_n)|dxdt \\
&\leq \int_{Q_T} h(u_n)u_n dxdt + \sup_{|s|\leq 1} |h(s)||Q_T| \\
&\leq \int_{Q_T} f(u_n)u_n dxdt + |f(0)|\|u_n\|_{L^1(Q_T)} + \gamma\|u_n\|^2_{L^2(Q_T)} \\
&\quad + \sup_{|s|\leq 1} |h(s)||Q_T| \\
&\leq C,
\end{aligned}$$

where we have used (7) and the boundedness of $\{u_n\}$ in $L^\infty(0, T; L^2(\Omega))$. Hence, it implies that $\{h(u_n)\}$, and therefore, $\{f(u_n)\}$ is bounded in $L^1(Q_T)$. Since

$$\frac{du_n}{dt} = \Delta_\lambda u_n - f(u_n) + g,$$

we deduce that $\{\frac{du_n}{dt}\}$ is bounded in $L^2(0, T; (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*) + L^1(Q_T)$, and therefore in $L^1(0, T; (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^* + L^1(\Omega))$. Because $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \subset\subset L^2(\Omega) \subset (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^* + L^1(\Omega)$, by the Aubin–Lions–Simon compactness lemma (see [7]), we have that $\{u_n\}$ is compact in $L^2(0, T; L^2(\Omega))$. Hence, we may assume, up to a subsequence, that $u_n \to u$ a.e. in Q_T. Applying Lemma 6.1 in [10], we obtain that $h(u) \in L^1(Q_T)$ and for all test function $\xi \in C_0^\infty([0, T]; \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \cap L^\infty(\Omega))$,

$$\int_{Q_T} h(u_n)\xi dxdt \to \int_{Q_T} h(u)\xi dxdt.$$

Hence, $f(u) \in L^1(Q_T)$ and

$$\int_{Q_T} f(u_n)\xi dxdt \to \int_{Q_T} f(u)\xi dxdt \quad \text{for all } \xi \in C_0^\infty([0, T]; \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \cap L^\infty(\Omega)).$$

Thus, u satisfies (6). Repeating the arguments in [3], we get $u(0) = u_0$ and this implies that u is a weak solution to problem (1).

ii) Uniqueness and continuous dependence on the initial data. Let u and v be two weak solutions of (1) with initial data $u_0, v_0 \in L^2(\Omega)$. Putting $w = u - v$, we have

$$\begin{cases} w_t - \Delta_\lambda w + \widetilde{f}(u) - \widetilde{f}(v) - \ell w = 0, \\ w(0) = u_0 - v_0, \end{cases} \tag{8}$$

where $\widetilde{f}(s) = f(s) + \ell s$. Here, because $w(t)$ does not belong to $W := \overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega) \cap L^{\infty}(\Omega)$, we cannot choose $w(t)$ as a test function as in [3]. Consequently, the proof will be more involved.

We use some ideas in [11]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function

$$B_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \le k, \\ -k & \text{if } s < -k. \end{cases}$$

Consider the corresponding Nemytskii mapping $\widehat{B_k} : W \to W$ defined as follows

$$\widehat{B_k}(w)(x) = B_k(w(x)) \quad \text{for all } x \in \Omega.$$

By Lemma 2.3 in [11], we have that $\|\widehat{B_k}(w) - w\|_W \to 0$ as $k \to \infty$. Now multiplying the first equation in (8) by $\widehat{B_k}(w)$, then integrating over $\Omega \times (\varepsilon, t)$, where $t \in (0, T)$, we get

$$\begin{aligned}&\int_{\varepsilon}^{t}\int_{\Omega}\frac{d}{ds}\left(w(s)\widehat{B_k}(w)(s)\right)dxds - \int_{\varepsilon}^{t}\int_{\Omega} w\frac{d}{ds}\left(\widehat{B_k}(w)(s)\right)dxds\\ &+\frac{1}{2}\int_{\varepsilon}^{t}\int_{\{x\in\Omega:|w(x,s)|\le k\}}|\nabla_{\lambda}w|^2dxds\\ &+\int_{\varepsilon}^{t}\int_{\Omega}(\widetilde{f}(u)-\widetilde{f}(v))\widehat{B_k}(w)dxds - \ell\int_{\varepsilon}^{t}\int_{\Omega} w\widehat{B_k}(w)dxds = 0.\end{aligned}$$

Noting that $w\frac{d}{dt}(\widehat{B_k}(w)) = \frac{1}{2}\frac{d}{dt}((\widehat{B_k}(w))^2)$, we have

$$\begin{aligned}&\int_{\Omega} w(t)\widehat{B_k}(w)(t)dx - \frac{1}{2}\left\|\widehat{B_k}(w)(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\varepsilon}^{t}\int_{\{x\in\Omega:|w(x,s)|\le k\}}|\nabla_{\lambda}w|^2dxds\\ &+\int_{\varepsilon}^{t}\int_{\Omega}\widetilde{f}'(\xi)w\widehat{B_k}(w)dxds\\ &=\int_{\Omega} w(\varepsilon)\widehat{B_k}(w)(\varepsilon)dx - \frac{1}{2}\left\|\widehat{B_k}(w)(\varepsilon)\right\|^2_{L^2(\Omega)} + \ell\int_{\varepsilon}^{t}\int_{\Omega} w\widehat{B_k}(w)dxds.\end{aligned}$$

Note that $\widetilde{f}'(s) \ge 0$ and $sB_k(s) \ge 0$ for all $s \in \mathbb{R}$, by letting $\varepsilon \to 0$ and $k \to \infty$ in the above equality, we obtain

$$\|w(t)\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)} + 2\ell\int_0^t \|w(s)\|^2_{L^2(\Omega)}ds.$$

Hence, by the Gronwall inequality of integral form, we get

$$\|w(t)\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)}e^{2\ell t} \le \|w(0)\|^2_{L^2(\Omega)}e^{2\ell T} \quad \text{for all } t \in [0, T].$$

Note that $w \in C([0, T]; L^2(\Omega))$, in particular, we get the uniqueness if $w(0) = 0$. □

3 Existence of a Global Attractor

By Theorem 1, we can define a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ associated to problem (1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique weak solution of (1) with the initial datum u_0. We will prove that the semigroup $S(t)$ has a global attractor $\mathcal{A}$ in the space $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$.

For brevity, in the following lemmas, we give some formal calculation, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [17].

Lemma 1 *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $L^2(\Omega)$.

Proof Multiplying the first equation in (1) by u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \int_\Omega f(u)u dx = \int_\Omega g u dx. \tag{9}$$

Using inequalities (3), (5), and the Cauchy inequality, we arrive at

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + (\gamma_1 - \mu)\|u\|^2_{L^2(\Omega)} \leq 2C_1|\Omega| + \frac{1}{\gamma_1 - \mu}\|g\|^2_{L^2(\Omega)}.$$

Hence, thanks to the Gronwall inequality, we obtain

$$\|u\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} e^{-(\gamma_1-\mu)t} + R_1,$$

where $R_1 = R_1(\gamma_1, \mu, |\Omega|, \|g\|_{L^2(\Omega)})$. Hence, choosing $\rho_1 = 2R_1$, we have

$$\|u\|^2_{L^2(\Omega)} \leq \rho_1 \quad \text{for all } t \geq T_1 = T_1(\|u_0\|_{L^2(\Omega)}). \tag{10}$$

This completes the proof. □

Lemma 2 *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$.

Proof Multiplying the first equation in (1) by $-\Delta_\lambda u$ and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \|\Delta_\lambda u\|^2_{L^2(\Omega)} &= -\int_\Omega f'(u)|\nabla_\lambda u|^2 dx - \int_\Omega g\Delta_\lambda u dx \\ &\leq \ell\|u\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \frac{1}{2}\|g\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta_\lambda u\|^2_{L^2(\Omega)}.\end{aligned}$$

In particular,

$$\frac{d}{dt}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} \leq 2\ell\|u\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \|g\|^2_{L^2(\Omega)}. \tag{11}$$

On the other hand, integrating (9) from t to $t+1$ and using (3), we have

$$\begin{aligned}&\int_t^{t+1} \|u(s)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} ds + \frac{1}{2}\|u(t+1)\|^2_{L^2(\Omega)} \\ &\leq (\mu+1)\int_t^{t+1} \|u(s)\|^2_{L^2(\Omega)} ds + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + C_1|\Omega| + \frac{1}{4}\|g\|^2_{L^2(\Omega)} \\ &\leq \rho_2 = \rho_2(\gamma_1, \mu, |\Omega|, \|g\|_{L^2(\Omega)})\end{aligned} \tag{12}$$

for all $t \geq T_1 = T_1(\|u_0\|_{L^2(\Omega)})$, where we have used the Cauchy inequality and estimate (10). By the uniform Gronwall inequality, from (11) and (12), we deduce that

$$\|u(t)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} \leq \rho_2 \quad \text{for all } t \geq T_2 = T_1 + 1. \tag{13}$$

This completes the proof. □

Lemma 3 *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $D(\Delta_\lambda)$.*

Proof By differentiating (1) in time, we get

$$u_{tt} - \Delta_\lambda u_t + f'(u)u_t = 0.$$

Taking the inner product of this equality with u_t in $L^2(\Omega)$ and using (2), in particular, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} \leq \ell\|u_t\|^2_{L^2(\Omega)}. \tag{14}$$

Multiplying the first equation in (1) by u_t, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx\right) = -\|u_t\|^2_{L^2(\Omega)} \leq 0. \tag{15}$$

On the other hand, integrating (9) from t to $t+1$ and using (10), we have

$$\int_t^{t+1}\left[\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega f(u)udx - \int_\Omega gudx\right]ds \leq \|u(t)\|^2_{L^2(\Omega)} \leq \rho_1 \quad \forall t \geq T_1.$$

Using the inequality (4), we deduce that

$$\begin{aligned}
&\int_t^{t+1}\left[\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega f(u)udx - \int_\Omega gudx\right]ds \\
&\geq \int_t^{t+1}\left[\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega F(u)dx - \frac{\ell}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega gudx\right]ds \\
&\geq \int_t^{t+1}\left[\frac{1}{2}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx\right]ds - \frac{\ell}{2}\rho_1 \quad \text{for all } t \geq T_1,
\end{aligned}$$

where we have used the inequality (10). Hence,

$$\int_t^{t+1}\left[\frac{1}{2}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx\right]ds \leq \left(1 + \frac{\ell}{2}\right)\rho_1 \quad \forall t \geq T_1. \tag{16}$$

By the uniform Gronwall inequality, from (15) and (16), we deduce that

$$\frac{1}{2}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx \leq \rho_3 \quad \text{for all } t \geq T_2 = T_1 + 1. \tag{17}$$

Integrating (15) from t to $t+1$ and using (17), we obtain

$$\int_t^{t+1}\|u_t\|^2_{L^2(\Omega)}ds \leq \rho_3 \quad \text{for all } t \geq T_2. \tag{18}$$

Combining (14) with (18) and using the uniform Gronwall inequality, we have

$$\|u_t\|^2_{L^2(\Omega)} \leq \rho_3 \quad \text{for all } t \geq T_3 = T_2 + 1. \tag{19}$$

On the other hand, multiplying the first equation in (1) by $-\Delta_\lambda u$, using (2) and the Cauchy inequality, we obtain

$$\begin{aligned}
\|\Delta_\lambda u\|^2_{L^2(\Omega)} &= \int_\Omega u_t\Delta_\lambda udx - \int_\Omega f'(u)|\nabla_\lambda u|^2dx - \int_\Omega g\Delta_\lambda udx \\
&\leq \ell\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta_\lambda u\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using estimates (13) and (19), we arrive at

$$\|\Delta_\lambda u(t)\|^2_{L^2(\Omega)} \leq \rho_4 \quad \text{for all } t \geq T_3.$$

This completes the proof. □

We now prove the following important lemma.

Lemma 4 *The embedding* $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$ *is compact.*

Proof First, for any $u \in D(\Delta_\lambda)$, we have

$$\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} = \int_\Omega |\nabla_\lambda u|^2 dx = -\int_\Omega u \cdot \Delta_\lambda u dx \le \|u\|_{L^2(\Omega)} \|\Delta_\lambda u\|_{L^2(\Omega)}. \tag{20}$$

Next, we will prove that for any $\varepsilon > 0$, there exists $C(\varepsilon)$ such that

$$\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} \le \varepsilon \|\Delta_\lambda u\|^2_{L^2(\Omega)} + C(\varepsilon)\|u\|^2_{L^1(\Omega)} \tag{21}$$

for all $u \in D(\Delta_\lambda)$. Indeed, since $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \subset\subset L^2(\Omega) \subset L^1(\Omega)$, by the Ehrling lemma, we have for any $\eta > 0$,

$$\|u\|_{L^2(\Omega)} \le \eta \|u\|_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + C_1(\eta)\|u\|_{L^1(\Omega)}.$$

Substituting this inequality into (20), we obtain

$$\begin{aligned}\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} &\le \|\Delta_\lambda u\|_{L^2(\Omega)} \left(\eta \|u\|_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + C_1(\eta)\|u\|_{L^1(\Omega)} \right) \\ &\le \eta \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \eta \|\Delta_\lambda u\|^2_{L^2(\Omega)} + C_2(\eta)\|u\|^2_{L^1(\Omega)},\end{aligned}$$

where we have used the Cauchy inequality. Hence, we obtain (21) for suitable choosing of η.

We now prove the compactness of the embedding $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$. Let $\{u_n\}$ be a bounded sequence in $D(\Delta_\lambda)$. Then there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u$ in $D(\Delta_\lambda)$. Using (21), we have

$$\|u_{n_k} - u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} \le \varepsilon \|\Delta_\lambda u_{n_k} - \Delta_\lambda u\|^2_{L^2(\Omega)} + C(\varepsilon)\|u_{n_k} - u\|^2_{L^1(\Omega)}.$$

Since $D(\Delta_\lambda) \subset \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \subset\subset L^1(\Omega)$ and the boundedness of the sequence $\{u_{n_k} - u\}$ in $D(\Delta_\lambda)$, we conclude that $u_{n_k} \to u$ in $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$, up to a subsequence if necessary. This completes the proof. □

As a direct consequence of Lemma 3 and the compactness of the embedding $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$, we get the main result of this section.

Theorem 2 *Suppose* **(F)**–**(G)** *hold. Then the semigroup* $S(t)$ *generated by problem* (1) *has a compact global attractor* $\mathcal{A}$ *in the space* $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$.

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