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Optimization

A Journal of Mathematical Programming and Operations Research

ISSN: 0233-1934 (Print) 1029-4945 (Online) Journal homepage: http://www.tandfonline.com/loi/gopt20

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To cite this article: N. T. T. Huong, J.-C. Yao & N. D. Yen (2016): Connectedness structure of the solution sets of vector variational inequalities, Optimization, DOI: 10.1080/02331934.2016.1172073

To link to this article: http://dx.doi.org/10.1080/02331934.2016.1172073

Published online: 19 Apr 2016.

OPTIMIZATION, 2016
http://dx.doi.org/10.1080/02331934.2016.1172073

Connectedness structure of the solution sets of vector variational inequalities

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ABSTRACT
By a scalarization method and properties of semi-algebraic sets, it is proved that both the Pareto solution set and the weak Pareto solution set of a vector variational inequality, where the constraint set is polyhedral convex and the basic operators are given by polynomial functions, have finitely many connected components. Consequences of the results for vector optimization problems are discussed in details. The results of this paper solve in the affirmative some open questions for the case of general problems without requiring monotonicity of the operators involved.

ARTICLE HISTORY
Received 15 September 2015
Accepted 23 March 2016

KEYWORDS
Vector variational inequality; solution set; scalarization; semi-algebraic set; connectedness structure

1. Introduction

The notion of vector variational inequality (VVI) was introduced by Giannessi in his seminal work.[1] A large collection of related papers is available in [2]. It is well known (see e.g. [3]) that VVI plays an important role in the study of various questions (structure of the solution sets, solution stability, solution sensitivity, etc.) about vector optimization problems. Note also that VVI is one of the most important models of vector equilibrium problems. As an example, we refer to the work by Raciti [4], where the author has studied two different kinds of Wardrop-type vector equilibria related to the vector traffic equilibrium problem and compared their VVI formulations.

Solution sensitivity and topological properties of the solution sets of strongly monotone VVIs with applications to vector optimization problems have been studied in [3,5].

Based on a stability theorem of Robinson [6, Theorem 2] and a scalarization method,[3,7], Yen and Yao [8] established sufficient conditions for the upper semicontinuity of the solution maps of parametric monotone affine vector variational inequalities. As a by-product, the authors obtained new topological properties of the solution sets of those problems and several facts on solution stability and connectedness of the solution sets of convex quadratic vector optimization problems and of linear fractional vector optimization problems (LFVOPs).

Recently, using a scalarization method, Huong et al. [9, Theorems 3.1 and 3.2] have shown that both the Pareto solution set and the weak Pareto solution set of a bicriteria affine VVI, which is not necessarily monotone, have finitely many connected components, provided that a regularity condition is satisfied. An explicit upper bound for the numbers of connected components of the Pareto solution set and the weak Pareto solution set has been given. Applying that result to vector

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Dedicated to Professor Franco Giannessi on the occasion of his 80th birthday and to Professor Diethard Pallaschke on the occasion of his 75th birthday.

optimization problems, the authors have obtained new topological properties of the solution sets of bicriteria LFVOPs and of quadratic vector optimization problems (QVOPs).

The aim of this paper is to clarify the connectedness structure of the solution sets of vector variational inequalities, where the number of criteria can exceed two and the basic operators may be non-affine. We will prove that both the Pareto solution set and the weak Pareto solution set of the problem have finitely many connected components. The results have interesting consequences for vector optimization problems. In particular, it is shown that

(a) The Pareto solution set of any LFVOP has finitely many connected components;
(b) The weak Pareto solution set of any LFVOP has finitely many connected components;
(c) The weak Pareto solution set of any convex vector optimization problem with polynomial criteria under linear constraints has finitely many connected components.

Note that two types of constraint set of a VVI can be considered: a polyhedral convex set (i.e. the solution set of a finite system of linear inequalities), or the solution set of a finite system of convex polynomial inequalities. Herein we will treat only the first case. The second case, which is much more involved and requires using the Slater regularity condition or the Mangasarian-Fromovitz constraint qualification, is left for subsequent investigations.

Our main technique is to reduce the problem under consideration to a question concerning semi-algebraic sets and solve the later by some tools from real algebraic geometry.[10]

The results of this paper provide an affirmative answer to Question 1 in [8, p.66] and also give a partial solution to Question 9.3 in [11, p.267] for the case of general problems without requiring the monotonicity of the operators involved.

The remaining part of this paper consists of three sections. Some definitions, notations and auxiliary results are given in Section 2. Section 3 studies the connectedness structure of the solution sets of vector variational inequalities. The last section applies the obtained results to several fundamental classes of vector optimization problems.

2. Preliminaries

The scalar product of two elements x^1, x^2 and the norm of an element x in an Euclidean space are denoted, respectively, by $\langle x^1, x^2\rangle$ and $\|x\|$. The transpose of a matrix M is denoted by M^T.

2.1. Vector variational inequalities

Let $K \subset \mathbb{R}^n$ be a nonempty closed convex subset. Given m vector-valued functions $F_i : K \to \mathbb{R}^n$, $i = 1, \dots, m$, we put $F = (F_1, \dots, F_m)$ and

$$F(x)(u) = \big(\langle F_1(x), u\rangle, \dots, \langle F_m(x), u\rangle\big)^T, \quad \forall x \in K,\ \forall u \in \mathbb{R}^n.$$

Denoting the nonnegative orthant of $\mathbb{R}^m$ by $\mathbb{R}^m_+$, we consider the set

$$\Sigma = \left\{\xi = (\xi_1, \dots, \xi_m)^T \in \mathbb{R}^m_+ \, : \, \sum_{i=1}^m \xi_i = 1\right\},$$

whose relative interior is given by

$$\mathrm{ri}\Sigma = \{\xi \in \Sigma \, : \, \xi_i > 0 \text{ for all } i = 1, \dots, m\}.$$

The VVI [1, p.167] defined by F, K and the cone $C := \mathbb{R}^m_+$ is the problem:

$$\text{(VVI)} \qquad \text{Find } x \in K \text{ such that } F(x)(y - x) \nleq_{C\setminus\{0\}} 0, \quad \forall y \in K,$$

where the inequality $v \nleq_{C\setminus\{0\}} 0$ for $v \in \mathbb{R}^m$ means that $-v \notin C \setminus \{0\}$. Following [12], to this problem we associate the next one:

$$(\mathrm{VVI})^w \qquad \text{Find } x \in K \text{ such that } F(x)(y-x) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K,$$

where $\mathrm{int}C$ denotes the interior of C and the inequality $v \nleq_{\mathrm{int}C} 0$ indicates that $-v \notin \mathrm{int}C$. The solution sets of (VVI) and $(\mathrm{VVI})^w$ are abbreviated, respectively, by Sol(VVI) and Sol^w(VVI). The elements of the first set (resp., of the second set) are said to be the *Pareto solutions* (resp., the *weak Pareto solutions*) of (VVI).

For $m = 1$, one has $F = F_1 : K \to \mathbb{R}^n$, hence (VVI) and $(\mathrm{VVI})^w$ coincide with the classical *variational inequality* problem [13, p.13]:

$$(\mathrm{VI}) \qquad \text{Find } x \in K \text{ such that } \langle F(x), y-x\rangle \geq 0, \quad \forall y \in K.$$

Let us denote the solution set of the latter by Sol(VI). For each $\xi \in \Sigma$, consider the variational inequality

$$(\mathrm{VI})_\xi \qquad \text{Find } x \in K \text{ such that } \left\langle \sum_{i=1}^{m} \xi_i F_i(x), y-x \right\rangle \geq 0, \quad \forall y \in K,$$

and denote its solution set by $\mathrm{Sol(VI)}_\xi$. Taking the union of $\mathrm{Sol(VI)}_\xi$ on $\xi \in \mathrm{ri}\Sigma$ (resp., on $\xi \in \Sigma$) we can find a part of Sol(VVI) (resp., the whole set Sol^w(VVI)). Namely, we have the following result.

Theorem 2.1: (See [3] and [7]) *It holds that*

$$\bigcup_{\xi\in\mathrm{ri}\Sigma} Sol(VI)_\xi \subset Sol(VVI) \subset Sol^w(VVI) = \bigcup_{\xi\in\Sigma} Sol(VI)_\xi. \tag{2.1}$$

If K is a polyhedral convex set, i.e. K is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$ (the intersection of an empty family of closed half-spaces is set to be $\mathbb{R}^n$), then

$$\bigcup_{\xi\in\mathrm{ri}\Sigma} Sol(VI)_\xi = Sol(VVI). \tag{2.2}$$

If $\langle F(y) - F(x), y - x\rangle \geq 0$ for all $x, y \in K$, then we say that F is *monotone* on K and (VI) is a monotone variational inequality. If $\mathrm{VI}(F_i, K)$ $(i = 1, \ldots, m)$ are monotone VIs, then (VVI) is said to be a *monotone VVI*. If $F(x) = Mx + q$, where $M \in \mathbb{R}^{n\times n}, q \in \mathbb{R}$ and K is a polyhedral convex set, then (VI) is said to be an *affine variational inequality* (or briefly AVI). One says that (VVI) is an *affine VVI* (or AVVI) if K is a polyhedral convex set and there exist matrices $M_i \in \mathbb{R}^{n\times n}$ and vectors $q_i \in \mathbb{R}^n$ $(i = 1, \ldots, m)$ such that $F_i(x) = M_i x + q_i$ for $i = 1, \ldots, m$ and for all $x \in K$.

2.2. Sets having finitely many connected components

To study the connectedness structure of the sets Sol(VVI) and Sol^w(VVI), we now recall some definitions from general topology and prove an auxiliary result.

Definition 2.2: (See [14]) A topological space X is said to be *connected* if one cannot represent $X = U \cup V$ where $U,\ V$ are nonempty open sets of X with $U \cap V = \emptyset$. A nonempty subset $A \subset X$ of a topological space X is said to be a *connected component* of X if A (equipped with the induced topology) is connected and it is not a proper subset of any connected subset of X.

The next lemma will be useful for our subsequent considerations.

Lemma 2.3: *Let Ω be a subset of a topological space X with the closure denoted by $\overline{\Omega}$. If Ω has k connected components, then any subset $A \subset X$ with the property $\Omega \subset A \subset \overline{\Omega}$ can have at most k connected components.*

Proof: Suppose that $\Omega \subset A \subset \overline{\Omega}$ and Ω has k connected components, denoted by $\Omega_i,\ i = 1, \ldots, k$. It is easy to show that $\overline{\Omega} = \bigcup_{i=1}^{k} \overline{\Omega}_i$, where $\overline{\Omega}_i$ stands for the closure of Ω_i in the topology of X. On one hand, by the inclusion $A \subset \bigcup_{i=1}^{k} \overline{\Omega}_i$ we have

$$A = \bigcup_{i=1}^{k} (\overline{\Omega}_i \cap A). \tag{2.3}$$

On the other hand, since $\bigcup_{i=1}^{k} \Omega_i \subset A$, we have

$$\Omega_i = \Omega_i \cap A \subset \overline{\Omega}_i \cap A \subset \overline{\Omega}_i \tag{2.4}$$

for $i = 1, \ldots, k$. Applying a remark after [14, Theorem 20] (see also [15, p.188]), which says that if $B \subset C \subset \overline{B}$ and B is connected then C is also connected, from (2.4) and the connectedness of Ω_i we can assert that $\overline{\Omega}_i \cap A$ is connected for all $i = 1, \ldots, k$. Thus, (2.3) shows that A can have at most k connected components, and completes the proof. □

2.3. Semi-algebraic sets

The proofs of the main results of this paper rely on several results on the connectedness structure of semi-algebraic sets.

We now present some knowledge about semi-algebraic sets, which can be found in [10, Chapter 1]. An *ordering* of a field $\mathcal{F}$ is a total order relation, denoted by $\leq$, satisfying two properties:

(i) For any $x, y, z \in \mathcal{F}$, if $x \leq y$ then $x + z \leq y + z$;
(ii) For any $x, y \in \mathcal{F}$, if $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

An *ordered field* $(\mathcal{F}, \leq)$ is a field $\mathcal{F}$ equipped with an ordering $\leq$.

A field which can be ordered is called a *real field*. Typical examples of a real field are, for example, $\mathcal{F} = \mathbb{R}$ with the natural ordering of real numbers, $\mathcal{F} = \mathbb{Q}$, and

$$\mathcal{F} = \mathbb{Q}[\sqrt{2}] := \left\{\gamma = \alpha + \beta\sqrt{2}\ :\ \alpha \in \mathbb{Q},\ \beta \in \mathbb{Q}\right\}$$

with the ordering induced from that of $\mathbb{R}$.

A *real closed field* $\mathcal{F}$ is a real field that has no nontrivial real algebraic extension $\mathcal{F}_1 \supset \mathcal{F}$, $\mathcal{F}_1 \neq \mathcal{F}$. Typical examples of a real closed field are $\mathcal{F} = \mathbb{R}$ and $\mathcal{F} = \mathbb{R}_{\text{alg}}$ (with the ordering induced from the natural ordering of $\mathbb{R}$), where $\mathbb{R}_{\text{alg}}$ is the set of the real numbers algebraic over $\mathbb{Q}$. Hence, $\gamma \in \mathbb{R}_{\text{alg}}$ if and only if there exists a nonzero polynomial $p(x)$ with coefficients from $\mathbb{Q}$ such that $p(\gamma) = 0$.

From now on, although the major part of the theory in [10, Chapter 2] *works for semi-algebraic subsets of R^n with R being any real closed field, we will consider only the case $R = \mathbb{R}$.*

The ring of polynomials in the variables $x_1, x_2, \ldots, x_n$ with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[x_1, x_2, \ldots, x_n]$.

Definition 2.4: (See [10, Definition 2.1.4]) A *semi-algebraic subset* of $\mathbb{R}^n$ is a subset of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \left\{x \in \mathbb{R}^n\ :\ f_{i,j}(x) *_{i,j} 0\right\},$$

where $f_{i,j} \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ and $*_{i,j}$ is either $<$ or $=$, for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, r_i$, with s and r_i being arbitrary natural numbers.

By this definition, every semi-algebraic subset of $\mathbb{R}^n$ can be represented as a finite union of sets of the form:

$$\left\{x \in \mathbb{R}^n \,:\, f_1(x) = \cdots = f_\ell(x) = 0,\ g_1(x) < 0, \ldots, g_m(x) < 0\right\},$$

where ℓ and m are natural numbers, $f_1, f_2, \ldots, f_\ell$, $g_1, g_2, \ldots, g_m$ are in $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Open balls, closed balls, spheres and unions of finitely many of those sets are some typical examples of semi-algebraic subsets in $\mathbb{R}^n$. Semi-algebraic subsets of $\mathbb{R}$ are exactly the finite unions of points and open intervals (bounded or unbounded).

As concerning the problem (VVI), we will be able to prove that: *If $F_1, \ldots, F_m$ are polynomial functions and K is a polyhedral convex set, then Sol(VVI) and Sol^w(VVI) are semi-algebraic subsets of $\mathbb{R}^n$.*

In what follows, $\mathbb{R}^n$ will be considered with its Euclidean topology. It is well known that polynomials are continuous with respect to the Euclidean topology.

By induction, the following useful result can be derived from [10, Theorem 2.2.1].

Theorem 2.5: *Let S be a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the natural projection on the space of the first n coordinates, i.e.*

$$\Phi\left(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\right) = \left(x_1, x_2, \ldots, x_n\right)^T$$

for every $x = \left(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\right)^T \in \mathbb{R}^n \times \mathbb{R}^m$. Then $\Phi(S)$ is a semi-algebraic subset of $\mathbb{R}^n$.

We proceed furthermore with the concept of semi-algebraically connected subset.

Definition 2.6: (See [10, Definition 2.4.2]) A semi-algebraic subset S of $\mathbb{R}^n$ is *semi-algebraically connected* if for every pair of disjoint semi-algebraic sets F_1 and F_2, which are closed in S and satisfy $F_1 \cup F_2 = S$, one has $F_1 = S$ or $F_2 = S$.

From definitions, any connected semi-algebraic subset is semi-algebraically connected. By [10, Theorem 2.4.5], which is recalled in Theorem 2.9 below, the converse is also true. In other words, for semi-algebraic subsets of $\mathbb{R}^n$, the concepts of semi-algebraical connectedness and (topological) connectedness are equivalent.

Example 2.7: The set $A := \{(x_1, x_2)^T \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 \le 1,\ x_1 - x_2 < 0\}$ is semi-algebraically connected. The set $B := \{(x_1, x_2)^T \in \mathbb{R}^2 \,:\, x_1 \neq 0\}$ is not semi-algebraically connected.

The next theorem clearly describes the connectedness structure of semi-algebraic subsets of $\mathbb{R}^n$. The major fact is that sets of this type have finitely many connected components.

Theorem 2.8: (See [10, Theorem 2.4.4]) *Every semi-algebraic subset S of $\mathbb{R}^n$ is the disjoint union of a finite number of nonempty semi-algebraically connected semi-algebraic sets $C_1, C_2, \ldots, C_k$ which are both closed and open in S. The sets $C_1, C_2, \ldots, C_k$ are called the semi-algebraically connected components of S.*

We finish this section with a result about the relationship between semi-algebraically connected sets and connected sets in $\mathbb{R}^n$.

Theorem 2.9: (See [10, Theorem 2.4.5]) *A semi-algebraic subset S of $\mathbb{R}^n$ is semi-algebraically connected if and only if it is connected. Every semi-algebraic set has a finite number of connected components, which are semi-algebraic.*

3. Connectedness structure of the solution sets

We will study the connectedness structure of the solution sets of vector variational inequalities of the form (VVI) under the following blanket assumptions:

(a1) All the components of F_i, $i = 1, \ldots, m$, are polynomial functions in the variables $x_1, \ldots, x_n$, i.e. for every $i \in \{1, \ldots, m\}$ one has $F_i = (F_{i1}, \ldots, F_{in})$ with $F_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$ for all $j = 1, \ldots, n$;

(a2) K is a polyhedral convex set, i.e. there exist a natural number $p \geq 1$, $A = (a_{ij}) \in \mathbb{R}^{p \times n}$, and $b = (b_i) \in \mathbb{R}^p$ such that $K = \{x \in \mathbb{R}^n : Ax \leq b\}$.

Our main result can be formulated as follows.

Theorem 3.1: *If the assumptions* (a1) *and* (a2) *are satisfied, then*

(i) *the weak Pareto solution set* $\mathrm{Sol}^w(\mathrm{VVI})$ *is a semi-algebraic subset of* $\mathbb{R}^n$ (*so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$), *and*

(ii) *the Pareto solution set Sol(VVI) is a semi-algebraic subset of* $\mathbb{R}^n$ (*so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$).

Proof:

(i) To every index set $\alpha \subset I$ with $I := \{1, \ldots, p\}$, we associate the *pseudo-face*

$$\mathcal{F}_\alpha := \left\{ x \in \mathbb{R}^n \ : \ \sum_{j=1}^n a_{ij}x_j = b_i \ \forall i \in \alpha, \ \sum_{j=1}^n a_{ij}x_j < b_i \ \forall i \notin \alpha \right\}$$

of K, where a_{ij} is the element in the ith row and the jth column of A, and b_i denotes the ith component of b. By Theorem 2.1, we have

$$\mathrm{Sol}^w(\mathrm{VVI}) = \bigcup_{\xi \in \Sigma} \mathrm{Sol}(\mathrm{VI})_\xi \tag{3.1}$$

with $(\mathrm{VI})_\xi$ denoting the variational inequality

$$\text{Find } x \in K \ \text{ such that } \ \langle F(x, \xi), y - x \rangle \geq 0, \quad \forall y \in K,$$

where $F(x, \xi) := \sum_{i=1}^m \xi_i F_i(x)$ for every $\xi = (\xi_1, \ldots, \xi_m)^T \in \Sigma$. Denote the *normal cone* to the convex set K at $x \in \mathbb{R}^n$ by $N_K(x)$ and recall that

$$N_K(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq 0 \ \forall y \in K\}$$

if $x \in K$, and $N_K(x) = \emptyset$ if $x \notin K$. Using the notations $F(x, \xi)$ and $N_K(x)$, we can rewrite the inclusion $x \in \mathrm{Sol}(\mathrm{VI})_\xi$ equivalently as

$$F(x, \xi) \in -N_K(x). \tag{3.2}$$

We have $K = \bigcup_{\alpha \subset I} \mathcal{F}_\alpha$, $\mathcal{F}_\alpha \cap \mathcal{F}_{\widetilde{\alpha}} = \emptyset$ if $\alpha \neq \widetilde{\alpha}$ and, therefore,

$$\mathrm{Sol}^w(\mathrm{VVI}) = \bigcup_{\alpha \subset I} \left[\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha \right]. \tag{3.3}$$

Since a finite union of semi-algebraic subsets of $\mathbb{R}^n$ is again a semi-algebraic subset of $\mathbb{R}^n$, by (3.3) and by Theorem 2.9 we see that the proof of our assertion will be completed if we can establish the following result.

Claim 1 For every index set $\alpha \subset I$, the intersection $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic subset of $\mathbb{R}^n$.

By the Farkas lemma [16, Corollary 22.3.1], we have

$$N_K(x) = \mathrm{pos}\{a_{i.}^T : i \in \alpha\} \tag{3.4}$$

for every $x \in \mathcal{F}_\alpha$, where $a_{i.} := (a_{i1}, \dots, a_{in})$ denotes the ith row of A and

$$\mathrm{pos}\{z_1, \dots, z_k\} := \{\lambda_1 z_1 + \dots + \lambda_k z_k \,:\, \lambda_i \geq 0,\ i = 1, \dots, k\}$$

is the convex cone generated by vectors $z_i \in \mathbb{R}^n$, $i = 1, \dots, k$. Due to formulas (3.1), (3.2) and (3.4),

$$\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \bigcup_{\xi \in \Sigma} \left\{ x \in \mathcal{F}_\alpha \,:\, F(x,\xi) \in -\mathrm{pos}\left\{ a_{i.}^T \,:\, i \in \alpha \right\} \right\}. \tag{3.5}$$

Since $\mathrm{pos}\left\{a_{i.}^T \,:\, i \in \alpha\right\}$ is a convex polyhedral cone, there exists a matrix $C_\alpha = \left(c_{ij}\right) \in \mathbb{R}^{n_\alpha \times n}$, where $n_\alpha \in \mathbb{N}$, such that

$$\mathrm{pos}\left\{ a_{i.}^T \,:\, i \in \alpha \right\} = \left\{ y \in \mathbb{R}^n \,:\, C_\alpha y \geq 0 \right\}. \tag{3.6}$$

By (3.5) and (3.6),

$$\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \bigcup_{\xi \in \Sigma} \{ x \in \mathcal{F}_\alpha \,:\, C_\alpha F(x,\xi) \leq 0 \}. \tag{3.7}$$

The inequality on the right-hand-side of (3.7) can be rewritten as

$$C_\alpha \left(\sum_{i=1}^m \xi_i F_i(x) \right) \leq 0,$$

which is the following system of n_α polynomial inequalities

$$\sum_{j=1}^n \sum_{i=1}^m c_{kj} \xi_i F_{ij}(x) \leq 0, \quad k = 1, \dots, n_\alpha. \tag{3.8}$$

Note that expression $\sum_{j=1}^n \sum_{i=1}^m c_{kj} \xi_i F_{ij}(x)$ on the right-hand-side of (3.8) is a polynomial in the variables $x_1, \dots, x_n, \xi_1, \dots, \xi_m$. Consider the set

$$\Omega_\alpha := \{ (x,\xi) \in \mathcal{F}_\alpha \times \Sigma \,:\, C_\alpha F(x,\xi) \leq 0 \}. \tag{3.9}$$

By (3.8), we have

$$\begin{aligned} \Omega_\alpha = \Bigg\{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, & \sum_{j=1}^n a_{ij} x_j = b_i,\ i \in \alpha, \\ & \sum_{j=1}^n a_{ij} x_j < b_i,\ i \notin \alpha, \\ & \sum_{j=1}^n \sum_{i=1}^m c_{kj} \xi_i F_{ij}(x) \leq 0,\ k = 1, \dots, n_\alpha, \\ & \sum_{i=1}^m \xi_i = 1,\ \xi_i \geq 0,\ i = 1, \dots, m \Bigg\}. \end{aligned}$$

Denote by $|\alpha|$ the number of elements of α and observe that Ω_α is determined by $|\alpha| + 1$ polynomial equations, $n_\alpha + m$ polynomial inequalities, and $p - |\alpha|$ strict polynomial inequalities of the variables $(x,\xi) = (x_1, \dots x_n, \xi_1, \dots, \xi_m) \in \mathbb{R}^{n+m}$. Hence Ω_α is a semi-algebraic set.

From (3.7), it follows that $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \Phi(\Omega_\alpha)$, where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection on the space of the first n coordinates. According to Theorem 2.5, $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic set. This proves Claim 1.

We have thus shown that the weak Pareto solution set $\mathrm{Sol}^w(\mathrm{VVI})$ is a semi-algebraic subset of $\mathbb{R}^n$. Then, thanks to Theorem 2.9, $\mathrm{Sol}^w(\mathrm{VVI})$ has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$.

(ii) Since K is a polyhedral convex set, the representation (2.2) for Sol(VVI) is valid. Combining this with the formula $K = \bigcup_{\alpha \subset I} \mathcal{F}_\alpha$ we get

$$\mathrm{Sol}(\mathrm{VVI}) = \bigcup_{\alpha \subset I} \left[\mathrm{Sol}(\mathrm{VVI}) \cap \mathcal{F}_\alpha\right]. \tag{3.10}$$

Due to (3.10), our assertion will be proved if we can establish the following

Claim 2 For every index set $\alpha \subset I$, the intersection $\mathrm{Sol}(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic subset of $\mathbb{R}^n$.

Let C_α, n_α, Ω_α and Φ be defined as above. Instead of (3.7), now we have

$$\mathrm{Sol}(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \bigcup_{\xi \in \mathrm{ri}\Sigma} \{x \in \mathcal{F}_\alpha \, : \, C_\alpha F(x,\xi) \le 0\}. \tag{3.11}$$

Let

$$\widetilde{\Omega}_\alpha := \{(x,\xi) \in \mathcal{F}_\alpha \times \mathrm{ri}\Sigma \, : \, C_\alpha F(x,\xi) \le 0\}. \tag{3.12}$$

The formula

$$\begin{aligned}\widetilde{\Omega}_\alpha = \Bigg\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \, : \, & \sum_{j=1}^n a_{ij}x_j = b_i, \ i \in \alpha, \\ & \sum_{j=1}^n a_{ij}x_j < b_i, \ i \notin \alpha, \\ & \sum_{j=1}^n \sum_{i=1}^m c_{kj}\xi_i F_{ij}(x) \le 0, \ k = 1, \dots, n_\alpha, \\ & \sum_{i=1}^m \xi_i = 1, \ \xi_i > 0, \ i = 1, \dots, m\Bigg\}\end{aligned}$$

shows that $\widetilde{\Omega}_\alpha$ is the solution set of a system of $|\alpha| + 1$ polynomial equations, n_α polynomial inequalities, and $p - |\alpha| + m$ strict polynomial inequalities in the variables $(x,\xi) = (x_1, \dots x_n, \xi_1, \dots, \xi_m) \in \mathbb{R}^{n+m}$. Hence $\widetilde{\Omega}_\alpha$ is a semi-algebraic set.

By (3.11), $\mathrm{Sol}(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \Phi(\widetilde{\Omega}_\alpha)$. Thus, according to Theorem 2.5, $\mathrm{Sol}(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic set. This proves Claim 2.

We have shown that Sol(VVI) is a semi-algebraic subset. So, by Theorem 2.9, Sol(VVI) has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$. □

Remark 3.2: It is clear that if $F_i(x) = M_i(x) + q_i$, where M_i is an $n \times n$ matrix and $q_i \in \mathbb{R}^n$, for $i = 1, \dots, m$, then each component $F_{ij}(x)$ of the functions F_i, $i = 1, \dots, m$, is a polynomial function in the variables $x_1, \dots, x_n$. Therefore, Theorem 3.1 solves in the affirmative Question 1 of [8, p.66] about the connectedness structure of the solution sets of affine vector variational inequalities, without requiring the positive semidefiniteness of the matrices M_i. Moreover, it assures that each connected component of the solution set under consideration is a semi-algebraic subset. Note that, by [8, Theorems 4.1 and 4.2], if the AVVI under consideration is monotone and if the solution set in question is disconnected, then each of its connected components is unbounded. The later result cannot be obtained by tools of algebraic geometry.

Remark 3.3: In [9], by a different approach using fractional functions and some properties of determinant, it has been proved that both the Pareto solution set and the weak Pareto solution set of an AVVI have finitely many connected components, provided that $m = 2$ and a regularity condition is satisfied. So, Theorem 3.1 encompasses the results of [9].

The problem of finding an upper bound for the numbers of connected components of $\mathrm{Sol}^w(\mathrm{VVI})$ and $\mathrm{Sol}(\mathrm{VVI})$ requires further investigations. In the case $m = 2$, an explicit upper bound for the numbers of connected components of $\mathrm{Sol}^w(\mathrm{VVI})$ and $\mathrm{Sol}(\mathrm{VVI})$ is given in [9] under a regularity condition. This result gives a partial solution to Question 2 of [8].

4. Applications to vector optimization problems under linear constraints

LFVOPs and QVOPs are two fundamental classes of vector optimization problems. Both classes contain linear vector optimization problems as an important subclass. We will apply Theorem 3.1 to establish some facts about the connectedness structure of the solution sets in LFVOPs. Moreover, a property of the stationary point set of polynomial vector optimization problems is obtained in this section and applied to convex QVOPs.

First, we consider a general vector optimization problem and recall some solution concepts which will be addressed later on.

4.1. Some solution concepts in vector optimization

Let $K \subset \mathbb{R}^n$ be a nonempty closed convex subset, $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$ a continuously differentiable function defined on an open set $\Omega \subset \mathbb{R}^n$ containing K. The vector minimization problem with the constraint set K and the vector objective function f is written formally as follows:

$$\text{(VP)} \qquad \text{Minimize } f(x) \quad \text{subject to} \quad x \in K.$$

Definition 4.1: A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathbb{R}^m_+ \setminus \{0\}\big) = \emptyset$.

Definition 4.2: One says that $x \in K$ is a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathrm{int}\,\mathbb{R}^m_+\big) = \emptyset$.

Put $F_i(x) = \nabla f_i(x)$, with $\nabla f_i(x)$ denoting the gradient of f_i at x. For any $\xi \in \Sigma$, where Σ is as in Section 2, we consider the parametric variational inequality $(\mathrm{VI})_\xi$, which now becomes

$$\text{Find } x \in K \text{ such that } \left\langle \sum_{i=1}^m \xi_i \nabla f_i(x), y - x \right\rangle \geq 0 \quad \forall y \in K. \tag{4.1}$$

According to [3, Theorem 3.1(i)], if x is a weakly efficient solution of (VP), then there exists $\xi \in \Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$. The converse is true if all the functions f_i are convex; see [3, Theorem 3.1(ii)].

Definition 4.3: If $x \in K$ and there is $\xi \in \Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then we call x a *stationary point* of (VP).

It is well known [3, Theorem 3.1(iii)] that if all the functions f_i are convex and if there is $\xi \in \mathrm{ri}\Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then x is an efficient solution of (VP). This sufficient optimality condition is a motivation to consider the following concept.

Definition 4.4: If $x \in K$ and there is $\xi \in \mathrm{ri}\Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then we call x a *proper stationary point* of (VP).

The efficient solution set, the weakly efficient solution set, the stationary point set and the proper stationary point set of (VP) are, respectively, abbreviated by Sol(VP), $\mathrm{Sol}^w(\mathrm{VP})$, Stat(VP), and Pr(VP).

From the above discussion and definitions, we have

$$\bigcup_{\xi\in\mathrm{ri}\Sigma} \mathrm{Sol(VI)}_\xi = \mathrm{Pr(VP)} \subset \mathrm{Sol(VP)} \subset \mathrm{Sol}^w\mathrm{(VP)} \subset \mathrm{Stat(VP)} = \bigcup_{\xi\in\Sigma} \mathrm{Sol(VI)}_\xi, \tag{4.2}$$

where the first inclusion is valid if all the functions f_i are convex. In addition, under fulfillment of the latter, the third inclusion in (4.2) becomes equality.

4.2. Linear fractional vector optimization

We now present some basic information about LFVOPs. More details can be found in [17,18] and [19, Chapter 8]. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, be linear fractional functions of the form

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i},$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let $K \subset \mathbb{R}^n$ be satisfying assumption (a2). Suppose that $b_i^T x + \beta_i > 0$ for all $i \in \{1, \cdots, m\}$ and $x \in K$. Put $f(x) = (f_1(x), \ldots, f_m(x))$,

$$\Omega := \left\{x \in \mathbb{R}^n \ : \ b_i^T x + \beta_i > 0, \ \forall i = 1, \cdots, m\right\},$$

and observe that Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. Consider the *LFVOP*

$$(VP_1) \quad \text{Minimize } f(x) \ \textit{subject to } x \in K.$$

LFVOPs have been studied intensively during the last four decades, see [8,11,17,18,20–26] and the references therein.

The efficient solution set and the weakly efficient solution set of (VP_1) are denoted by $\mathrm{Sol(VP_1)}$ and $\mathrm{Sol}^w(\mathrm{VP}_1)$, respectively. According to [26], $x \in \mathrm{Sol(VP_1)}$ if and only if there exists $\xi = (\xi_1, \ldots, \xi_m) \in \mathrm{ri}\Sigma$ such that

$$\left\langle \sum_{i=1}^m \xi_i \left[\left(b_i^T x + \beta_i\right) a_i - \left(a_i^T x + \alpha_i\right) b_i \right], y - x \right\rangle \geq 0, \quad \forall y \in K. \tag{4.3}$$

Similarly, $x \in \mathrm{Sol}^w(\mathrm{VP}_1)$ if and only if there exists $\xi = (\xi_1, \ldots, \xi_m) \in \Sigma$ such that (4.3) holds.

Condition (4.3) can be rewritten in the form of a parametric affine variational inequality as follows:

$$(VI)'_\xi \quad \langle M(\xi)x + q(\xi), y - x\rangle \geq 0, \quad \forall y \in K,$$

with

$$M(\xi) := \sum_{i=1}^n \xi_i M_i, \quad q(\xi) := \sum_{i=1}^n \xi_i q_i,$$

where

$$M_i = a_i b_i^T - b_i a_i^T, \quad q_i = \beta_i a_i - \alpha_i b_i \quad (i = 1, \ldots, m). \tag{4.4}$$

It is well known [18] that $\mathrm{(VI)}'_\xi$ is a monotone AVI for every $\xi \in \Sigma$. Denote by $\Psi(\xi)$ the solution set of $\mathrm{(VI)}'_\xi$ and consider the multifunction $\Psi : \Sigma \rightrightarrows \mathbb{R}^n$, $\xi \mapsto \Psi(\xi)$. According to the recalled necessary and sufficient optimality conditions recalled above for (VP_1), we have

$$\mathrm{Sol(VP_1)} = \bigcup_{\xi\in\mathrm{ri}\Sigma} \Psi(\xi) = \Psi(\mathrm{ri}\Sigma), \tag{4.5}$$

and

$$\mathrm{Sol}^w(\mathrm{VP}_1) = \bigcup_{\xi\in\Sigma} \Psi(\xi) = \Psi(\Sigma). \tag{4.6}$$

By formulas (4.5), (4.6), and Theorem 2.1, Sol(VP$_1$) (resp., Sol^w(VP$_1$)) coincides with the Pareto solution set (resp., the weak Pareto solution set) of the monotone AVVI defined by K and the affine functions $F_i(x) = M_i x + q_i$, $i = 1, \ldots, m$. Therefore, from Theorem 3.1 we can obtain the following result.

Theorem 4.5: *It holds that*

(i) *the Pareto solution set Sol(VP$_1$) is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$), and*
(ii) *the weak Pareto solution set Sol^w(VP$_1$) is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$).*

We have just shown that both the Pareto solution set and the weak Pareto solution set of (VP$_1$) are semi-algebraic subsets and have finitely many connected components. By the first-order necessary and sufficient conditions for the efficiency and weak efficiency in LFVOPs recalled above, Hoa et al. [22] proved that, for any natural number $m \geq 1$, there exists a LFVOP with m objective criteria, where the sets Sol(VP$_1$) and Sol^w(VP$_1$) coincide and have exactly m connected components. The problem of finding an upper estimate for the number of connected components of Sol(VP$_1$) has been solved in [22] for the case $m = 2$.

4.3. Polynomial vector optimization

If $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are polynomial functions and $K \subset \mathbb{R}^n$ is a polyhedral convex set then (VP), now denoted by (VP$_2$), is called a *polynomial vector optimization problem*.

We denote the efficient solution set, the weakly efficient solution set, the stationary point set, and the proper stationary point set of (VP$_2$), respectively, by Sol(VP$_2$), Sol^w(VP$_2$), Stat(VP$_2$), and Pr(VP$_2$).

Theorem 4.6: *The following assertions hold:*

(i) *The set Stat(VP$_2$) (resp., the set Pr(VP$_2$)) is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$);*
(ii) *If all the functions f_i are convex, then Sol^w(VP$_2$) is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$);*
(iii) *If all the functions f_i are convex and the set Pr(VP$_2$) is dense in Sol(VP$_2$), then Sol(VP$_2$) has a finite number of connected components.*

Proof:

(i) Since f_i, $i = 1, \ldots, m$, are polynomial functions, f is continuously differentiable on $\mathbb{R}^n$. By (4.2) we have

$$\mathrm{Stat}(\mathrm{VP}_2) = \bigcup_{\xi\in\Sigma} \mathrm{Sol}(\mathrm{VI})_\xi, \quad \mathrm{Pr}(\mathrm{VP}_2) = \bigcup_{\xi\in\mathrm{ri}\,\Sigma} \mathrm{Sol}(\mathrm{VI})_\xi. \tag{4.7}$$

Combining (4.7) with Theorem 2.1 and taking into account Theorem 3.1, we get the desired properties.

(ii) Since all the components of f are convex polynomial functions, we have

$$\mathrm{Sol}^w(\mathrm{VP}_2) = \mathrm{Stat}(\mathrm{VP}_2) = \bigcup_{\xi\in\Sigma} \mathrm{Sol}(\mathrm{VI})_\xi.$$

Hence the assertion follows from (i).

(iii) Since all the functions f_i are convex and the set $\mathrm{Pr}(\mathrm{VP}_2)$ is dense in $\mathrm{Sol}(\mathrm{VP}_2)$, we have

$$\mathrm{Pr}(\mathrm{VP}_2) \subset \mathrm{Sol}(\mathrm{VP}_2) \subset \overline{\mathrm{Pr}}(\mathrm{VP}_2), \tag{4.8}$$

where the first inclusion is a special case of the first inclusion in (4.2), and $\overline{\mathrm{Pr}}(\mathrm{VP}_2)$ is the closure of $\mathrm{Pr}(\mathrm{VP}_2)$ in the Euclidean topology of $\mathbb{R}^n$. By (i), we see that $\mathrm{Pr}(\mathrm{VP}_2)$ has finitely many connected components. Now, from (4.8) and Lemma 2.3 it follows that $\mathrm{Sol}(\mathrm{VP}_2)$ has a finite number of connected components.

The proof is complete. □

If K is a polyhedral convex set and $f_i(x) = \frac{1}{2}x^T M_i x + q_i^T x$, $i = 1, \ldots, m$, where $M_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, are symmetric matrices, and q_i for $i = 1, \ldots, m$ are vectors in $\mathbb{R}^n$, then (VP) is called a *QVOP* and denoted by (VP_3).

Clearly, a QVOP is a polynomial vector optimization problem. Hence, Theorem 4.6 implies the next result on the connectedness structure of the stationary point set $\mathrm{Stat}(\mathrm{VP}_3)$ and that of the weakly efficient solution set $\mathrm{Sol}^w(\mathrm{VP}_3)$.

Corollary 4.7: *The following properties hold:*

(i) *The set* $\mathrm{Stat}(\mathrm{VP}_3)$ *is a semi-algebraic subset of* $\mathbb{R}^n$ *(so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$*);*

(ii) *If all the matrices* $M_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, *are positive semidefinite, then* $\mathrm{Sol}^w(VP_3)$ *is a semi-algebraic subset of* $\mathbb{R}^n$ *(so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$*).*

We have shown that the stationary point set of a QVOP and the weakly efficient solution set of a convex QVOP have finitely many connected components. Note that, by [8, Theorem 4.1], if the weakly efficient solution set of a convex QVOP is disconnected, then each of its connected components is unbounded.

Acknowledgements

We thank Mr. Vu Xuan Truong for useful discussions on connectedness structure of semi-algebraic sets and the two anonymous referees for many valuable suggestions, which have helped us to significantly improve the paper presentation.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The authors were supported, respectively, by National Foundation for Science & Technology Development (Vietnam) [grant number 101.01-2014.39], China Medical University (Taichung Taiwan) and the Institute of Mathematics (VAST, Hanoi, Vietnam). The first author was also supported in part by RFIT@LQDTU Faculty of Information Technology of Le Quy Don University (Vietnam).

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