10 Stationary Solutions of the Navier–Stokes Equations

Assuming that the forces are independent of time, we are looking for time independent solutions of the Navier–Stokes equations, i.e., a function $u=u(x)$ (and a function $p=p(x)$) which satisfies (1.4), (1.5), and either (1.9) with $\phi=0$, or (1.10):

(10.1) $$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f \quad \text{in } \mathcal{O}\ (=\Omega \text{ or } Q),$$

(10.2) $$\operatorname{div} u=0 \quad \text{in } \mathcal{O},$$

(10.3) $$u=0 \quad \text{on } \Gamma=\partial\Omega \quad \text{if } \mathcal{O}=\Omega,$$

or

(10.4) $$u(x+Le_i)=u(x),\quad i=1,\dots,n,\quad x\in Q \quad \text{if } \mathcal{O}=Q.$$

In the functional setting of § 2, the problem is

(10.5) Given f in H (or V'), to find $u\in V$ which satisfies

(10.6) $$\nu((u,v))+b(u,u,v)=(f,v)\quad \forall\, v\in V,$$

or

(10.7) $$\nu Au+Bu=f \quad \text{in } H \text{ (or } V').$$

10.1. Behavior for $t\to\infty$. The trivial case. We start by recalling briefly the results of existence and uniqueness of solutions for (10.5)–(10.7).

THEOREM 10.1. *We consider the flow in a bounded domain with periodic or zero boundary conditions* ($\mathcal{O}=\Omega$ *or* Q), *and* $n=2$ or 3. *Then:*

i) *For every* f *given in* V' *and* $\nu>0$, *there exists at least one solution of* (10.5)–(10.7).

ii) *If* f *belongs to* H, *all the solutions belong to* $D(A)$.

iii) *Finally, if*

(10.8) $$\nu^2>c_1\|f\|_{V'},$$

where c_1 *is a constant depending only on* $\mathcal{O}$[1], *then the solution of* (10.5)–(10.7) *is unique.*

Proof. We give only the principle of the proof and refer the reader to the literature for further details.

For existence we implement a Galerkin method (cf. (3.41)–(3.45)) and look,

[1] c_1 is the constant in (2.30) when $m_1=m_3=1$, $m_2=0$. Since $|f|\geq\sqrt{\lambda_1}\,\|f\|_{V'}$ if $f\in H$, a sufficient condition for (10.8) is $\nu^2>c_1\sqrt{\lambda_1}\,|f|$, which can be compared to (9.1).

for every $m \in \mathbb{N}$, for an approximate solution u_m,

$$u_m = \sum_{j=1}^{m} \xi_{jm} w_j, \qquad \xi_{jm} \in \mathbb{R}, \tag{10.9}$$

such that

$$\nu((u_m, v)) + b(u_m, u_m, v) = \langle f, v \rangle \tag{10.10}$$

for every v in W_m = the space spanned by $w_1, \ldots, w_m$. Equation (10.10) is also equivalent to

$$\nu A u_m + P_m B u_m = P_m f. \tag{10.11}$$

The existence of a solution u_m of (10.10)–(10.11) follows from the Brouwer fixed point theorem (cf. [RT, Chap. II, § 1] for the details). Taking $v = u_m$ in (10.10) and taking into account (2.34), we get

$$\nu \|u_m\|^2 = \langle f, u_m \rangle \leqq \|f\|_{V'} \|u_m\|, \tag{10.12}$$

and therefore

$$\|u_m\| \leqq \frac{1}{\nu} \|f\|_{V'}. \tag{10.13}$$

We extract from u_m a sequence $u_{m'}$, which converges weakly in V to some limit u, and since the injection of V in H is compact, this convergence holds also in the norm of H:

$$u_{m'} \to u \quad \text{weakly in } V, \quad \text{strongly in } H. \tag{10.14}$$

Passing to the limit in (10.10) with the sequence m', we find that u is a solution of (10.6).

To prove ii), we note that if $u \in V$, then $Bu \in V_{-1/2}$ (instead of V') because of (2.36) (applied with $m_1 = 1$, $m_2 = 0$, $m_3 = \frac{1}{2}$). Hence $u = \nu^{-1} A^{-1}(f - Bu)$ is in $V_{3/2}$. Applying again Lemma 2.1 (with $m_1 = \frac{3}{2}$, $m_2 = \frac{1}{2}$, $m_3 = 0$), we conclude now that $Bu \in H$, and thus u is in $D(A)$.

We can provide useful a priori estimates for the norm of u in V and in $D(A)$. Setting $v = u$ in (10.6) we obtain (compare to (10.12)–(10.13))

$$\nu \|u\|^2 = \langle f, u \rangle \leqq \|f\|_{V'} \|u\|, \tag{10.15}$$

$$\|u\| \leqq \frac{1}{\nu} \|f\|_{V'} \quad \left(\leqq \frac{1}{\nu\sqrt{\lambda_1}} |f| \text{ if } f \in H \right). \tag{10.16}$$

For the norm in $D(A)$ we infer from (10.7) that

$$\begin{aligned}
\nu |Au| &\leqq |f| + |Bu| \leqq |f| + c_2 \|u\|^{3/2} |Au|^{1/2} && \text{(by the first inequality (2.32))} \\
&\leqq |f| + \frac{\nu}{2} |Au| + \frac{c_2^2}{2\nu} \|u\|^3 && \text{(by the Schwarz inequality)} \\
&\leqq |f| + \frac{\nu}{2} |Au| + \frac{c_2^2}{2\nu^4 \lambda_1^{3/2}} |f|^3 && \text{(with (10.6)).}
\end{aligned}$$

Finally

$$|Au| \leqq \frac{2}{\nu}|f| + \frac{c_2^2}{\nu^5 \lambda_1^{3/2}}|f|^3. \tag{10.17}$$

For the uniqueness result iii), let us assume that u_1 and u_2 are two solutions of (10.6):

$$\nu((u_1, v)) + b(u_1, u_1, v) = \langle f, v \rangle,$$
$$\nu((u_2, v)) + b(u_2, u_2, v) = \langle f, v \rangle.$$

Setting $v = u_1 - u_2$, we obtain by subtracting the second relation from the first one:

$$\begin{aligned} \nu \|u_1 - u_2\|^2 &= -b(u_1, u_1, u_1 - u_2) + b(u_2, u_2, u_1 - u_2) \\ &= -b(u_1 - u_2, u_2, u_1 - u_2) \qquad \text{(with (2.34))} \\ &\leqq c_1 \|u_1 - u_2\|^2 \|u_2\| \qquad \text{(using (2.30) with } m_1 = m_3 = 1, m_2 = 0) \\ &\leqq \frac{c_1}{\nu} \|f\|_{V'} \|u_1 - u_2\|^2 \qquad \text{(with (10.16) applied to } u_2). \end{aligned}$$

Therefore

$$\left(\nu - \frac{c_1}{\nu}\|f\|_{V'}\right)\|u_1 - u_2\|^2 \leqq 0$$

and $u_1 - u_2 = 0$ if (10.8) holds. □

Concerning the behavior for $t \to \infty$ of the solutions of the time-dependent Navier–Stokes equations, the easy case (which corresponds to point a) in § 9) is the following:

THEOREM 10.2. *We consider the flow in a bounded domain with periodic or zero boundary conditions* ($\mathcal{O} = \Omega$ *or* Q) *and* $n = 2$ *or* 3. *We are given* $f \in H$, $\nu > 0$ *and we assume that*

$$\nu\left(\frac{\lambda_1}{c_2'}\right)^{3/4} > \left(\frac{2}{\nu}|f| + \frac{c_2^2 |f|^3}{\nu^5 \lambda_1^{3/2}}\right) \tag{10.18}$$

where c_2', c_2 *depend only on* $\mathcal{O}$.

Then the solution of (10.7) (*denoted* u_∞) *is unique. If* $u(\cdot)$ *is any weak solution*[2] *of Problem* 2.1 *with* $u_0 \in H$ *arbitrary and* $f(t) \equiv f$ *for all* t, *then*

$$u(t) \to u_\infty \quad \text{in } H \quad \text{as } t \to \infty. \tag{10.19}$$

Proof. Let $w(t) = u(t) - u_\infty$. We have, by differences,

$$\frac{dw(t)}{dt} + \nu A w(t) + Bu(t) - Bu_\infty = 0,$$

[2] If $n = 2$, $u(\cdot)$ is unique and is a strong solution (at least for $t > 0$). If $n = 3$, $u(\cdot)$ is not necessarily unique, and we must assume that $u(\cdot)$ satisfies the energy inequality (see Remark 3.2).

and, taking the scalar product with $w(t)$,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2+b(u(t),u(t),w(t))-b(u_\infty,u_\infty,w(t))=0^3. \tag{10.20}$$

Hence, with (2.34),

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2=-b(w(t),u_\infty,w(t)).$$

Using (2.30) with $m_1=\frac{1}{2}$, $m_2=1$, $m_3=0$, we can majorize the right-hand side of this equality by

$$c_1\,|w(t)|_{1/2}\,|w(t)|\,|Au_\infty|,$$

which, because of (2.20), is less than or equal to

$$c_1'\,|w(t)|^{3/2}\,\|w(t)\|^{1/2}\,|Au_\infty|;$$

with (3.10) this is bounded by

$$\frac{\nu}{2}\|w(t)\|^2+\frac{c_2'}{\nu^{1/3}}|w(t)|^2\,|Au_\infty|^{4/3}$$

where $c_2'=3(c_1'/2)^{4/3}$. Therefore,

$$\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2\leqq\frac{c_2'}{\nu^{1/3}}|w(t)|^2\,|Au_\infty|^{4/3}, \tag{10.21}$$

$$\frac{d}{dt}|w(t)|^2+\left(\nu\lambda_1-\frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3}\right)|w(t)|^2\leqq 0. \tag{10.22}$$

If

$$\bar{\nu}=\nu\lambda_1-\frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3}>0, \tag{10.23}$$

then (10.22) shows that $|w(t)|$ decays exponentially towards 0 when $t\to\infty$:

$$|w(t)|\leqq|w(0)|e^{-\bar{\nu}t},$$

$w(0)=u_0-u_\infty$. Using the estimation (10.17) for u_∞, we obtain a sufficient condition for (10.23), which is exactly (10.18).

If we replace $u(t)$ by another stationary solution u_∞^* of (10.7) in the computations leading to (10.22), we obtain instead of (10.22)

$$\nu\,|u_\infty^*-u_\infty|^2\leqq 0.$$

Thus (10–23) and (10.18) ensure that $u_\infty^*=u_\infty$; i.e., they are sufficient conditions for uniqueness of a stationary solution, like (10.8).

The proof is complete. □

[3] The proof that $\langle w'(t), w(t)\rangle=\frac{1}{2}(d/dt)|w(t)|^2$ is not totally easy when $n=2$, and relies on [RT, Chap. III, Lemma 1.2]. If $n=3$ we do not even have an equality in (10.20), but an inequality $\leqq$, which is sufficient for our purposes: a technically similar situation arises in [RT, Chap. III, § 3.6].

Remark 10.1. Under the assumptions of Theorem 10.2, consider the linear operator $\mathscr{A}$ from $D(A)$ into H defined by

$$(\mathscr{A}\phi, \psi) = \nu(A\phi, \psi) + b(\phi, u_\infty, \psi) + b(u_\infty, \phi, \psi) \quad \forall \phi, \psi \in D(A),$$

whose adjoint $\mathscr{A}^*$ from $D(A)$ into H is given by

$$(\mathscr{A}^*\phi, \psi) = \nu(A\phi, \psi) + b(\psi, u_\infty, \phi) + b(u_\infty, \psi, \phi) \quad \forall \phi, \psi \in D(A).$$

Then with the same notation as in the proof of Theorem 10.2, we have

$$\frac{dw(t)}{dt} + \mathscr{A}w(t) + Bw(t) = 0,$$

and (10.20) is the same as

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu(\mathscr{A}w(t), w(t)) = 0$$

or

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\left(\frac{\mathscr{A} + \mathscr{A}^*}{2} w(t), w(t)\right) = 0.$$

The operator $((\mathscr{A} + \mathscr{A}^*)/2)^{-1}$ is selfadjoint and compact from H into itself, and the conclusions of Theorem 10.2 will still hold if we replace (10.18) by the conditions that the *eigenvalues of* $(\mathscr{A} + \mathscr{A}^*)/2$ *are* >0.

10.2. An abstract theorem on stationary solutions. In this section we derive an abstract theorem on the structure of the set of solutions of a general equation

$$N(u) = f; \tag{10.24}$$

this theorem will then be applied to (10.7).

Nonlinear Fredholm operators. If X and Y are two real Banach spaces, a linear continuous operator L from X into Y is called a Fredholm operator if

i) $\dim \ker L < \infty$,

ii) range L is closed,

iii) coker $L = Y/_{\text{range } L}$ has finite dimension.

In such a case the *index* of L is the integer

$$i(L) = \dim \ker L - \dim \operatorname{coker} L. \tag{10.25}$$

For instance, if $L = L_1 + L_2$ where L_1 is compact from X into Y and L_2 is an isomorphism (resp. is surjective and $\dim \ker L_2 = q$), then L is Fredholm of index 0 (resp. of index q). For the properties of Fredholm operators, see for instance R. Palais [1], S. Smale [1].

Now let ω be a connected open set of X, and N a nonlinear operator from ω into Y; N is a nonlinear Fredholm map if N is of class $\mathscr{C}^1$ and its differential $N'(u)$ is a Fredholm operator from X into Y, at every point $u \in \omega$. In this case it follows from the properties of Fredholm operators that the index of $N'(u)$ is independent of u; we define the index of N as the number $i(N'(u))$.

Let N be a $\mathscr{C}^1$ mapping from an open set ω of X into Y, X, Y being again two real Banach spaces. We recall that $u \in X$ is called a regular point of N if $N'(u)$ is onto, and a singular point of N otherwise. The image given by N of the set of singular points of N constitutes the set of singular values of N. Its complement in Y constitutes the set of regular values of N. Thus a *regular value of* N is a point $f \in Y$ which does not belong to the image $N(\omega)$, or such that $N'(u)$ is onto at every point u in the preimage $N^{-1}(f)$.

Finally we recall that a mapping N of the preceding type is *proper* if the preimage $N^{-1}(K)$ of any compact set K of Y is compact in X.

We will make use of the following infinite dimensional version of Sard's theorem due to S. Smale [1] (see also K. Geba [1]).

THEOREM. *Let X and Y be two real Banach spaces and ω a connected open set of X. If $N: \omega \to Y$ is a proper $\mathscr{C}^k$ Fredholm map with $k > \max(\text{index } N, 0)$, then the set of regular values of N is a dense open set of Y.*

We deduce easily from this theorem:

THEOREM 10.3. *Let X and Y be two real Banach spaces and ω a connected open set of X, and let $N: \omega \to Y$ be a proper $\mathscr{C}^k$ Fredholm map, $k \geqq 1$, of index* 0.

Then there exists a dense open set ω_1 in Y and, for every $f \in \omega_1$, $N^{-1}(f)$ is a finite set.

If index $N = q > 0$ and $k \geqq q$, then there exists a dense open set ω_1 in Y and, for every $f \in \omega_1$, $N^{-1}(f)$ is empty or is a manifold in ω of class $\mathscr{C}^k$ and dimension q.

Proof. We just take $\omega_1 =$ the set of regular values of N which is dense and open by Smale's theorem. For every $f \in \omega_1$, the set $N^{-1}(f)$ is compact since N is proper. If index $(N) = 0$, then for every $f \in \omega_1$ and $u \in N^{-1}(f)$, $N'(u)$ is onto (by definition of ω_1) and is one-to-one since

$$\dim \ker N'(u) = \dim \operatorname{coker} N'(u) = 0.$$

Thus $N'(u)$ is an isomorphism, and by the implicit function theorem, u is an isolated solution of $N(v) = f$. We conclude that $N^{-1}(f)$ is compact and made of isolated points: this set is discrete.

If index $(N) = q \leqq k$, for every $f \in \omega_1$, and every $u \in N^{-1}(f)$, $N'(u)$ is onto and the dimension of its kernel is q: it follows that $N^{-1}(f)$ is a manifold of dimension q, of class $\mathscr{C}^k$ like N. □

Applications of this theorem to the stationary Navier–Stokes equation (and to other equations) will be given below.

10.3. Application to the Navier–Stokes equations. We are going to show that Theorem 10.3 applies to the stationary Navier–Stokes equation (10.7) in the following manner:

$$X = D(A), \quad Y = H, \quad N(u) = \nu Au + Bu. \tag{10.26}$$

It follows from (2.36) that $B(\cdot, \cdot)$ is continuous from $D(A) \times D(A)$ and even $V_{3/2} \times V_{3/2}$ into H and N makes sense as a mapping from $D(A)$ into H.

LEMMA 10.1. *$N: D(A) \to H$ is proper.*

Proof. Let K denote a compact set of H. Since K is bounded in H, it follows from the a priori estimation (10.17) that $N^{-1}(K)$ is bounded in $D(A)$ and thus compact in $V_{3/2}$. As observed before, $B(\cdot, \cdot)$ is continuous from $V_{3/2} \times V_{3/2}$ into H, and thus $B(N^{-1}(K))$ is compact in H.

We conclude that the set $N^{-1}(K)$ is included in

$$\nu^{-1}A^{-1}(K - B(N^{-1}(K))),$$

which is relatively compact in $D(A)$, and the result follows. □

We have the following *generic* properties of the set of stationary solutions to the Navier–Stokes equations.

THEOREM 10.4. *We consider the stationary Navier–Stokes equations in a bounded domain (Ω or Q) with periodic or zero boundary conditions, and $n = 2$ or 3.*

Then, for every $\nu > 0$, there exists a dense open set $\mathcal{O}_\nu \subset H$ such that for every $f \in \mathcal{O}_\nu$, the set of solution of (10.5)–(10.7) *is finite and odd in number.*

On every connected component of $\mathcal{O}_\nu$, the number of solutions is constant, and each solution is a $\mathscr{C}^\infty$ function of f.

Proof. i) We apply Theorem 10.3 with the choice of X, Y, N indicated in (10.26) (and $\omega = X$). It is clear that N is a $\mathscr{C}^\infty$ mapping from $D(A)$ into H and that

$$N'(u) \cdot v = \nu A v + B(u, v) + B(v, u) \quad \forall\, u, v \in D(A). \tag{10.27}$$

It follows from (3.20) that for every $u \in D(A)$, the linear mappings

$$v \mapsto B(u, v), \qquad v \mapsto B(v, u)$$

are continuous from V into H and they are therefore compact from $D(A)$ into H. Since A is an isomorphism from $D(A)$ onto H, it follows from the properties of Fredholm operators (recalled in § 10.2) that $N'(u)$ is a Fredholm operator of index 0.

We have shown in Lemma 10.1 that N is proper: all the assumptions of Theorem 10.3 are satisfied. Setting $\mathcal{O}_\nu$ = the set of regular values of $N = N_\nu$, we conclude that $\mathcal{O}_\nu$ is open and dense in H and, for every $f \in \mathcal{O}_\nu$, $N^{-1}(f)$, which is the set of solutions of (10.7), is finite.

ii) Let $(\mathcal{O}_i)_{i \in I}$ be the connected components of $\mathcal{O}_\nu$ (which are open), and let f_0, f_1 be two points of $\mathcal{O}_i$ for some i. Let $u_0 \in N^{-1}(f_0)$. There exists a continuous curve

$$s \in [0, 1] \mapsto f(s) \in \mathcal{O}_i, \qquad f(0) = f_0, \quad f(1) = f_1,$$

and the implicit function theorem shows the existence and uniqueness of a continuous curve $s \mapsto u(s)$ with

$$N(u(s)) = f(s), \qquad u(0) = u_0.$$

Since $f(s)$ is a regular value of N, for all $s \in [0, 1]$, $u(s)$ is defined for every s, $0 \leq s \leq 1$, and therefore $u(1) \in N^{-1}(f_1)$. Such a curve $\{s \mapsto u(s)\}$ can be constructed, starting from any $u_k \in N^{-1}(f_0)$. Two different curves cannot reach the same point $u_* \in N^{-1}(f_1)$ and cannot intersect at all, since this would not be consistent with the implicit function theorem around u_* or around the intersection point. Hence there are at least as many points in $N^{-1}(f_1)$ as in $N^{-1}(f_0)$. By symmetry the number of points is the same.

It is clear that each solution $u_k = u_k(f)$ is a $\mathscr{C}^\infty$ function of f on every $\mathcal{O}$.

iii) It remains to show that the number of solutions is odd. This is an easy application of the Leray–Schauder degree theory.

For fixed $\nu > 0$ and $f \in \mathcal{O}_\nu$, we rewrite (1.7) in the form

$$T_\nu(u) = \nu u + A^{-1}Bu = A^{-1}f = g, \tag{10.28}$$

$u, g \in D(A)$. By (10.17), every solution u_λ of $T_\nu(u_\lambda) = \lambda g$, $0 \leqq \lambda \leqq 1$, satisfies

$$|Au_\lambda| < R, \qquad R = 1 + \frac{2}{\nu}|f| + \frac{c_2^2}{\nu^5 \lambda_1^{3/2}}|f|^3.$$

Therefore the Leray–Schauder degree $d(T_\nu, \lambda g, B_R)$ is well defined, with B_R the ball of $D(A)$ of radius R. Also, when λg is a regular value of T_ν, i.e., λf is a regular value of N, the set $T_\nu^{-1}(\lambda g)$ is discrete $= \{u_1, \dots, u_k\}$, and $d(T_\nu, \lambda g, B_R) = \sum_{j=1}^k i(u_j)$ where $i(u_j) =$ index u_j.

It follows from Theorem 10.1 that there exists $\lambda_* \in [0, 1]$, and for $0 \leqq \lambda \leqq \lambda_*$, $N^{-1}(\lambda f)$ contains only one point u_λ. By arguments similar to that used in the proof of Theorem 10.1, one can show that $N'(u_\lambda)$ is an isomorphism, and hence for these values of λ, $d(T_\nu, \lambda g, B_R) = \pm 1$. By the homotopy invariance property of degree, $d(T_\nu, g, B_R) = \pm 1$ and consequently k must be an odd number.

The proof of Theorem 10.4 is complete. □

Remark 10.2.

i) The set $\mathcal{O}$ is actually unbounded in H; cf. C. Foias–R. Temam [13].

ii) Similar generic results have been proved for the flow in a bounded domain with a nonhomogeneous boundary condition (i.e., $\phi \neq 0$ in (1.9)): generic finiteness with respect to f for ϕ fixed, with respect to ϕ for f fixed and with respect to the pair f, ϕ; see C. Foias–R. Temam [3], [4], J. C. Saut–R. Temam [2], and for the case of time periodic solutions, J. C. Saut–R. Temam [2], R. Temam [7].

iii) When Ω is unbounded, we lack a compactness theorem for the Sobolev spaces $H^m(\Omega)$ (and lack the Fredholm property). We do not know whether results similar to that in Theorem 10.4 are valid in that case; cf. D. Serre [1] where a line of stationary solutions of Navier–Stokes equations is constructed for an unbounded domain Ω.

We now present another application of Theorem 10.3, with an operator of index 1, leading to a generic result in bifurcation. We denote by $S(f, \nu) \subset D(A)$ the set of solutions of (1.5)–(1.7) and

$$S(f) = \bigcup_{\nu > 0} S(f, \nu). \tag{10.29}$$

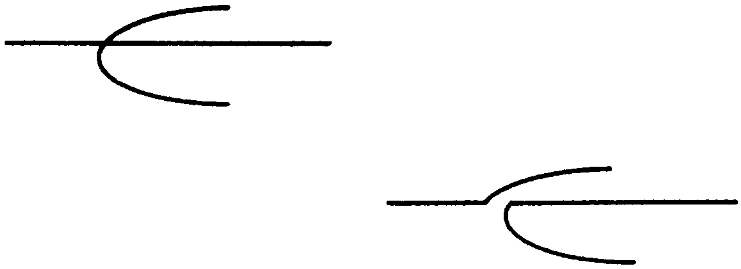

FIG. 10.1

THEOREM 10.5. *Under the same hypotheses as in Theorem* 10.4, *there exists a* G_δ*-dense set* $\mathcal{O} \subset H$, *such that for every* $f \in \mathcal{O}$, *the set* $S(f)$ *defined in* (10.29) *is a* $\mathcal{C}^\infty$ *manifold of dimension* 1.

Proof. We apply Theorem 10.3 with $X = D(A) \times \mathbb{R}$, $Y = H$, $\omega = \omega_m = D(A) \times (1/m, \infty) \subset X$, $m \in \mathbb{N}$, $N(u, \nu) = \nu Au + Bu$, for all $(u, \nu) \in X$. It is clear that N is $\mathcal{C}^\infty$ from ω into Y and

$$N'(u, \nu) \cdot (v, \mu) = \nu Av + B(u, v) + B(v, u) + \mu Au.$$

For $(u, \nu) \in \omega$, $N'(u, \nu)$ is the sum of the operator

$$(v, \mu) \rightarrow B(u, v) + B(v, u) + \mu Au,$$

which is compact[4], and the operator

$$(v, \mu) \rightarrow \nu Av,$$

which is onto and has a kernel of dimension 1. Thus $N'(u, \nu)$ is a Fredholm operator of index 1 and N is a nonlinear Fredholm mapping of index 1.

The proof of Lemma 10.1 and (10.17) shows that N is proper on $D(A) \times (\nu_0, \infty)$, for all $\nu_0 > 0$ and in particular $\nu_0 = 1/m$. Hence Theorem 10.3 shows that there exists an open dense set $\mathcal{O}_m \subset H$, and for every $f \in \mathcal{O}_m$, $N_m^{-1}(f)$ is a manifold of dimension 1, where N_m is the restriction of N to ω_m. We set $\mathcal{O} = \bigcap_{m \geq 1} \mathcal{O}_m$, which is a dense G_δ set in H, and, for every $f \in \mathcal{O}$, $S(f) = \bigcup_{m \geq 1} N_m^{-1}(f)$ is a manifold of dimension 1.

Remark 10.3. i) By the uniqueness result in Theorem 10.1, $S(f)$ contains an infinite branch corresponding to the large values of ν.

ii) Since $S(f)$ is a $\mathcal{C}^\infty$ manifold of dimension 1, it is made of the union of curves which *cannot intersect.* Hence the usual bifurcation picture (Fig. 10.1) is nongeneric and is a schematization (perfectly legitimate of course!) of generic situations of the type shown in Fig. 10.2.

Remark 10.4. Other properties of the set $S(f, \nu)$ are given in C. Foias–R. Temam [3], [4] and J. C. Saut–R. Temam [2]. In particular, for every $\nu, f, S(f, \nu)$ is a real compact analytic set of finite dimension.

[4] Same proof essentially as in Theorem 10.4.

10.4. Counterexamples. A natural question concerning Theorem 10.4 is whether $\mathcal{O}_\nu$ is the whole space H or just a subset. Since Theorem 10.4 is a straightforward consequence of Theorem 10.3, this question has to be raised at the level of Theorem 10.3. Unfortunately, the following examples show that we cannot answer this question at the level of generality of the abstract Theorem 10.3, since for one of the two examples presented $\mathcal{O}_\nu = H$, while for the second one $\mathcal{O}_\nu \neq H$.

Example 1. The first example is the one-dimensional Burgers equation which has been sometimes considered in the past as a model for the Navier–Stokes equations: consider a given $\nu > 0$ and a function $f:[0, 1] \to \mathbb{R}$ which satisfies

$$-\nu \frac{d^2u}{dx^2} + u\frac{du}{dx} = f \quad \text{on } (0, 1), \tag{10.30}$$

$$u(0) = u(1) = 0. \tag{10.31}$$

For the functional setting we take $H = L^2(0, 1)$, $V = H_0^1(0, 1)$, $D(A) = H_0^1(0, 1) \cap H^2(0, 1)$, $Au = -d^2u/dx^2$, for all $u \in D(A)$,

$$B(u, v) = u\frac{dv}{dx}, \quad \forall\, u, v \in V, \quad Bu = B(u, u).$$

Then, given f in H (or V'), the problem is to find $u \in D(A)$ (or V) which satisfies

$$\nu Au + Bu = f. \tag{10.32}$$

We can apply Theorem 10.3 with $X = D(A)$, $Y = H$, $N(u) = \nu Au + Bu$. The mapping N is obviously $\mathscr{C}^\infty$ and

$$N'(u) \cdot v = \nu Av + B(u, v) + B(v, u);$$

$N'(u)$, as the sum of an isomorphism and a compact operator, is a Fredholm operator of index 0. Now we claim that every $u \in D(A)$ is a regular point, so that $\omega_1 = H$.

In order to prove that u is a regular point, we have to show that the kernel of $N'(u)$ is 0 (which is equivalent to proving that $N'(u)$ is onto, as index $N'(u) = 0$). Let v belong to $N'(u)$; v satisfies

$$-\nu\frac{d^2v}{dx^2} + u\frac{dv}{dx} + v\frac{du}{dx} = 0 \quad \text{on } (0, 1), \qquad v(0) = v(1) = 0.$$

By integration $-\nu v' + uv = \text{constant} = a$, and by a second integration taking into account the boundary conditions we find that $v = 0$.

Since the solution is unique if $f = 0$ ($u = 0$), we conclude that

(10.33) (10.30)–(10.31) possesses a unique solution $\forall\, \nu$, $\forall\, f$.

Example 2. The second example is due to Gh. Minea [1] and corresponds to a space H of finite dimension; actually $H = \mathbb{R}^3$.

Theorem 10.3 is applied with $X = Y = \mathbb{R}^3$, $N(u) = \nu Au + Bu$ for all $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. The linear operator A is just the identity, and the nonlinear (quadratic) operator B is defined by $Bu = (\delta(u_2^2 + u_3^2), -\delta u_1 u_2, -\delta u_1 u_3)$ and possesses the orthogonality property $Bu \cdot u = 0$. The equation $N(u) = f$ reads

$$\nu u_1 + \delta(u_2^2 + u_3^2) = f_1,$$
$$\nu u_2 - \delta u_1 u_2 = f_2,$$
$$\nu u_3 - \delta u_1 u_3 = f_3.$$

It is elementary to solve this equation explicitly. There are one or three solutions if $|f_2| + |f_3| \neq 0$. In the "nongeneric case", $f_2 = f_3 = 0$, we get either one solution, or one solution and a whole circle of solutions: $\omega_1 \neq Y$ in this case.