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Projection Algorithms for Solving Nonmonotone Equilibrium Problems in Hilbert Space

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Abstract. We propose two projection algorithms for solving an equilibrium problem where the bifunction is not required to be satisfied any monotone property. Under assumptions on the continuity, convexity of the bifunction and the nonemptyness of the solution set of the Minty equilibrium problem, we show that the sequences generated by the proposed algorithms converge weakly and strongly to a solution of the primal equilibrium problem respectively.

2010 Mathematics Subject Classification: 90C25; 90C33; 65K10; 65K15

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1 Introduction

Let $\mathbb{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let Ω be an open convex subset in $\mathbb{H}$ containing a nonempty closed convex C, and $f : \Omega \times \Omega \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for every $x \in C$. We consider the following equilibrium problem (shortly EP(C, f)) in the sense of Blum, Muu and Oettli [4, 21], which is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C,$$

and its associated equilibrium problem

$$\text{Find } u \in C \text{ such that } f(y, u) \leq 0, \ \forall y \in C. \tag{1.1}$$

We call problem (1.1) as the Minty equilibrium problem (MEP(C, f) for short) due to M. Castellani and M. Giuli [6]. By S and S_M, we denote the solution set of EP(C, f) and MEP(C, f) respectively. While we denote by '$\to$' the strong convergence and by '$\rightharpoonup$' the weak convergence in the Hilbert space $\mathbb{H}$.

Although problem EP(C, f) has a simple formulation, it includes, as special cases, many important problems in applied mathematics: variational inequality

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problem, optimization problem, fixed point problem, saddle point problem, Nash equilibrium problem in noncooperative game, and others; see, for example, [3, 4, 21, 15], and the references quoted therein. Solution methods for equilibrium problem have been usually extended from those for variational inequality problem [12, 14, 26, 30] and other related problems; see, for instance, [9, 11, 17, 18, 20, 22, 25, 29]. Among them, the extragradient method which was introduced by Korpelevich [19] for solving variational inequality problem and recently extended to equilibrium problem [1, 28] is an important method. However, in our best knowledge, to implement this method, it always requires the solution set S of EP(C, f) is contained in the solution set S_M of MEP(C, f). This condition is guaranteed under the pseudomonotonicity assumption of bifunction f on C, that is, if $x,\ y\ \in C,\ f(x, y)\ \geq\ 0$, then $f(y, x)\ \leq\ 0$. Therefore, if S is not contained in S_M, then the existing extragradient method can not be applied for EP(C, f) directly. For instance, take $C = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in C$, we define bifunction f by the following formula

$$f(x, y) = |x_1 + x_2|(y_1 - x_1 + y_2^2 - x_2^2),$$

it is clear that $S = \{(-1, 0), (t, -t) : t \in [-1, 1]\}$; $S_M = \{(-1, 0)\}$, and $S \not\subset S_M$.

Note that, for each $y \in C$, by setting

$$L(y) = \{u \in C : f(u, y) \geq 0\}, L_M(y) = \{v \in C : f(y, v) \leq 0\},$$

then we can verify that

$$S = \cap_{y\in C} L(y), S_M = \cap_{y\in C} L_M(y).$$

If $f(\cdot, y)$ is upper semicontinuous on C for each $y \in C$, then $L(y)$ is closed for every $y \in C$, hence S is a closed set. While if $f(x, \cdot)$ is lower semicontinuous and quasiconvex on C for each $x \in C$, then $L_M(y)$ is closed and convex for all $y \in C$. Consequently, S_M is a closed and convex set and the Minty equilibrium problem reduces to a so-called convex feasibility problem [2]. If, in addition, the constraint set C is compact or f satisfies some certain coercive conditions [18] then S is nonempty [5, 13]. However, S is not necessary convex as the above example.

Remember that, if $f(\cdot, y)$ is upper semicontinuous on C for each $y \in C$ and $f(x, \cdot)$ is convex for every $x \in C$, then we have $S_M \subset S$; see, for example [16].

In this paper, we propose two projection algorithms for solving the equilibrium problem in a real Hilbert space without pseudomonotonicity assumption of the bifunction, we assume that S_M is nonempty instead. The first algorithm can be considered as an extension of the one introduced by M. Ye and Y. He [33] for solving nonmonotone variational inequality problem in the Euclidean space, and the second one is a combination between the projection algorithm for solving the pseudomonotone equilibrium problem in the finite dimensional space in [10] and hybrid cutting technique proposed by W. Takahashi et al. [27] (see also Y. Censor et al. [7]).

The paper is organized as follows. The next section contains some preliminaries on the metric projection and equilibrium problems. The third section is devoted to presentation of a projection algorithm for $\mathrm{EP}(C, f)$ and its weak convergence. A strong convergence algorithm for EP(C, f) is presented in section 4. The last section, is devoted to present an application of the proposed algorithm for Nash-Cournot equilibrium models of electricity markets and its implementation.

2 Preliminaries

In the rest of this paper, by P_C we denote the metric projection operator on C, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \leq \|y - x\|, \ \forall y \in C,$$

and $d(., C)$ stands for the distance function to C, i.e.,

$$d(x, C) = \inf\{\|x - y\| : y \in C\}.$$

For example, if $H = \{y \in \mathbb{H} : \langle w, y - y^0\rangle \leq 0\}$ for some $w, y^0 \in \mathbb{H}$, then

$$d(x, H) = \begin{cases} \frac{|\langle w, x - y^0\rangle|}{\|w\|} & \text{if } x \notin H \\ 0 & \text{if } x \in H \end{cases}$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1 *Suppose that C is a nonempty closed convex subset in $\mathbb{H}$. Then*

(a) $P_C(x)$ is singleton and well defined for every x;

(b) $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;

(c) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.

Definition 2.1 *A bifunction $\varphi : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x^k\}$, $\{y^k\}$ are two sequences in C converging weakly to x and y respectively, then $\varphi(x^k, y^k)$ converges to $\varphi(x, y)$.*

In the sequel, we need the following blanket assumptions

(A1) $f(x, .)$ is convex on Ω for every $x \in C$;

(A2) f is jointly weakly continuous on $\Omega \times \Omega$.

For each $z,\ x \in C$, by $\partial_2 f(z,x)$ we denote the subgradient of the convex function $f(z,.)$ at x, i.e.,

$$\partial_2 f(z,x) := \{w \in \mathbb{H} : f(z,y) \geq f(z,x) + \langle w, y-x\rangle,\ \forall y \in C\}.$$

In particular,

$$\partial_2 f(z,z) = \{w \in \mathbb{H} : f(z,y) \geq \langle w, y-z\rangle,\ \forall y \in C\}.$$

The next lemma can be considered as an infinite-dimensional version of Theorem 24.5 in [24]

Lemma 2.2 *[31, Proposition 4.3] Let $f : \Omega \times \Omega \to \mathbb{R}$ be a function satisfying conditions $(A1)$ and $(A2)$. Let $\bar{x}, \bar{y} \in \Omega$ and $\{x^k\}$, $\{y^k\}$ be two sequences in Ω converging weakly to $\bar{x}, \bar{y}$, respectively. Then, for any $\epsilon > 0$, there exist $\eta > 0$ and $k_\epsilon \in \mathbb{N}$ such that*

$$\partial_2 f(x^k, y^k) \subset \partial_2 f(\bar{x}, \bar{y}) + \frac{\epsilon}{\eta} B,$$

for every $k \geq k_\epsilon$, where B denotes the closed unit ball in $\mathbb{H}$.

Lemma 2.3 *[20] Under assumptions $(A1)$ and $(A2)$, a point $x^* \in C$ is a solution of EP(C, f) if and only if it is a solution to the equilibrium problem:*

$$\text{Find } x^* \in C : f(x^*, y) + \frac{\rho}{2}\|y - x^*\|^2 \geq 0,\ \forall y \in C. \qquad (AEP)$$

Lemma 2.4 *[32] Let C be a nonempty closed convex subset of $\mathbb{H}$. Let $\{x^k\}$ be a sequence in $\mathbb{H}$ and $u \in \mathbb{H}$. If any weak limit point of $\{x^k\}$ belongs to C and*

$$\|x^k - u\| \leq \|u - P_C(u)\|,\ \forall k.$$

Then $x^k \to P_C(u)$.

Lemma 2.5 *Under assumptions (A1) and (A2), if $\{z^k\} \subset C$ is a sequence such that $\{z^k\}$ converges strongly to $\bar{z}$ and the sequence $\{w^k\}$, with $w^k \in \partial_2 f(z^k, z^k)$, converges weakly to $\bar{w}$, then $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$.*

Proof. Let $w^k \in \partial_2 f(z^k, z^k)$. Then

$$f(z^k, y) \geq f(z^k, z^k) + \langle w^k, y - z^k\rangle = \langle w^k, y - z^k\rangle,\ \forall y \in C.$$

Taking the limit as $k \to \infty$ on both sides of the above inequality, by the weak continuity of f(., y) with respect to the first argument, we obtain

$$\begin{aligned} f(\bar{z}, y) \geq \limsup_{k\to\infty} f(z^k, y) &\geq \lim_{k\to\infty} \langle w^k, y - z^k\rangle \\ &= \lim_{k\to\infty} \langle w^k, y - \bar{z}\rangle + \lim_{k\to\infty} \langle w^k, \bar{z} - z^k\rangle \\ &= \langle \bar{w}, y - \bar{z}\rangle,\ \forall y \in C, \end{aligned}$$

which, together with $f(\bar{z}, \bar{z}) = 0$, implies that $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$. $\square$

Lemma 2.6 *Suppose the bifunction f satisfies the assumptions (A1), (A2). If $\{x^k\} \subset C$ is bounded, $\rho > 0$, and $\{y^k\}$ is a sequence such that*

$$y^k = \arg\min \Big\{ f(x^k, y) + \frac{\rho}{2}\|y - x^k\|^2 : \ y \in C \Big\},$$

then $\{y^k\}$ is bounded.

Proof. Firstly, we show that if $\{x^k\}$ weakly converges to x^*, then $\{y^k\}$ is bounded. Indeed,

$$y^k = \arg\min \Big\{ f(x^k, y) + \frac{\rho}{2}\|y - x^k\|^2 : \ y \in C \Big\},$$

and

$$f(x^k, x^k) + \frac{\rho}{2}\|x^k - x^k\|^2 = 0,$$

therefore

$$f(x^k, y^k) + \frac{\rho}{2}\|y^k - x^k\|^2 \leq 0, \ \forall k.$$

In addition, for all $w^k \in \partial_2 f(x^k, x^k)$ we have

$$f(x^k, y^k) + \frac{\rho}{2}\|y^k - x^k\|^2 \geq \langle w^k, y^k - x^k \rangle + \frac{\rho}{2}\|y^k - x^k\|^2.$$

This implies $-\|w^k\|\|y^k - x^k\| + \frac{\rho}{2}\|y^k - x^k\|^2 \leq 0$. Hence,

$$\|y^k - x^k\| \leq \frac{2}{\rho}\|w^k\|, \ \forall k.$$

Because $\{x^k\}$ converges weakly to x^* and $w^k \in \partial_2 f(x^k, x^k)$, by Lemma 2.2, the sequence $\{w^k\}$ is bounded, combining with the boundedness of $\{x^k\}$, we get $\{y^k\}$ is also bounded.

Now we prove the Lemma 2.6. Suppose contradict that $\{y^k\}$ is unbounded, i.e., there exists an subsequence $\{y^{k_i}\} \subseteq \{y^k\}$ such that $\lim_{i\to\infty} \|y^{k_i}\| = +\infty$. By the boundedness of $\{x^k\}$, it implies $\{x^{k_i}\}$ is also bounded, without loss of generality, we may assume that $\{x^{k_i}\}$ converges weakly to some x^*. By the same argument as above, we obtain $\{y^{k_i}\}$ is bounded, which contradicts. Therefore $\{y^k\}$ is bounded. $\square$

3 Weak convergence algorithm for EP(C, f)

Let $F : C \to \mathbb{H}$ be an operator, by setting $f(x, y) = \langle F(x), y - x \rangle$, the equilibrium problem EP(C, f) becomes the following variational inequality problem VIP(C, F):

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0, \ \forall y \in C.$$

In this case, the MEP(C, F) reduces to the Minty variational inequality MEP(C, F):

$$\text{Find } x^* \in C \text{ such that } \langle F(y), x^* - y\rangle \leq 0, \ \forall y \in C.$$

To find a solution of nonmonotone variational inequality problem in the Euclidean space $\mathbb{R}^n$, M. Ye and Y. He proposed to use the following double projection algorithm (see [33, Algorithm 2.1]):

Algorithm 1
Initialization. Choose $x^0 \in C$, choose parameters $\eta \in (0, 1)$, and $\rho \in (0, 1)$.
Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:
Step 1. Compute $y^k = P_C(x^k - F(x^k))$, $r(x^k) = x^k - y^k$.
If $r(x^k) = 0$, stop. Otherwise, go to Step 2.
Step 2. Find m_k as the smallest nonnegative integer number m satisfying

$$\langle F(x^k) - F(x^k - \eta^m r(x^k)), r(x^k)\rangle \leq \rho\|r(x^k)\|^2,$$

and compute $z^k = x^k - \eta_k r(x^K)$, where $\eta_k = \eta_k^m$.
Step 3. Compute $x^{k+1} = P_{C\cap\tilde{H}_k}(x^k)$, where $\tilde{H}_k = \cap_{j=0}^{j=k} H_j$, with

$$H_j = \{x \ \in \mathbb{R}^n : \langle F(z^j), x - z^j\rangle \leq 0\},$$

and go to Step 1 with k is replaced by $k + 1$.

They proved that if F is continuous on $\mathbb{R}^n$ then the sequence $\{x^k\}$ generated by Algorithm 1 converges to a solution of VIP(C, F) provided that the solution set of MVIP(C, F) is nonempty.

Note that Algorithm 1 can be considered as a modification of the one introduced by M.V. Solodov and B.F. Svaiter for solving pseudomonotone variational inequality problems [26]. Motivated by these works and recent paper [10], we now present the first projection algorithm for solving nonmonotone equilibrium problem in Hilbert space.

Algorithm 2
Initialization. Pick $x^0 \in C$, choose parameters $\eta \in (0, 1)$, $\rho > 0$ and $C_0 = C$.
Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min\left\{f(x^k, y) + \frac{\rho}{2}\|y - x^k\|^2 : \ y \in C\right\} \qquad CP(x^k)$$

to obtain its unique solution y^k.

If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m satisfying

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k, \\ w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}), \\ \langle w^{k,m}, x^k - y^k \rangle \geq \frac{\rho}{2}\|y^k - x^k\|^2. \end{cases} \tag{3.1}$$

Step 3. Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$, $w^k = w^{k,m_k}$. Take

$$H_k = \{x \ \in \mathbb{H} : \langle w^k, x - z^k \rangle \leq 0\}, \ C_{k+1} = C_k \cap H_k. \tag{3.2}$$

Step 4. Compute $x^{k+1} = P_{C_{k+1}}(x^k)$, and go to Step 1 with k is replaced by $k+1$.

Remark 3.1

- *If $y^k = x^k$ then x^k is a solution to EP(C, f).*
- $w^k \neq 0, \ \forall k$.

Now we are going to analyze the validity and convergence of the algorithm.

Lemma 3.1 *If the solution set S_M of the Minty equilibrium problem is nonempty. Then the sequence $\{x^k\}$ generated by Algorithm 1 is well defined in the sense that, at each iteration k, there exists an integer number $m > 0$ satisfying the inequality in (3.1) for every $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$, C_k is nonempty closed convex, and*

$$\langle w^k, x^k - z^k \rangle \geq \frac{\eta_k \rho}{2}\|x^k - y^k\|^2. \tag{3.3}$$

Proof. Firstly, we prove that at each iteration k there exists a positive integer m_0 such that

$$\langle w^{k,m_0}, x^k - y^k \rangle \geq \frac{\rho}{2}\|y^k - x^k\|^2, \ \forall \ w^{k,m_0} \in \partial_2 f(z^{k,m_0}, z^{k,m_0}).$$

Indeed, suppose by contradiction that, for every positive integer m and $z^{k,m} = (1-\eta^m)x^k + \eta^m y^k$ there exists $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ such that

$$\langle w^{k,m}, x^k - y^k \rangle < \frac{\rho}{2}\|y^k - x^k\|^2.$$

Since $z^{k,m}$ converges strongly to x^k as $m \to \infty$, by Lemma 2.2, the sequence $\{w^{k,m}\}_{m=1}^{\infty}$ is bounded. Thus we may assume that $w^{k,m}$ converges weakly to $\bar{w}$

for some $\bar{w}$. Taking the limit as $m \to \infty$, from $z^{k,m} \to x^k$ and $w^{k,m} \rightharpoonup \bar{w}$, by Lemma 2.5, it follows that $\bar{w} \in \partial_2 f(x^k, x^k)$ and

$$\langle \bar{w}, x^k - y^k \rangle \leq \frac{\rho}{2}\|y^k - x^k\|^2. \tag{3.4}$$

Since $\bar{w} \in \partial_2 f(x^k, x^k)$, we have

$$f(x^k, y^k) \geq f(x^k, x^k) + \langle \bar{w}, y^k - x^k \rangle = \langle \bar{w}, y^k - x^k \rangle.$$

Combining with (3.4) yields

$$f(x^k, y^k) + \frac{\rho}{2}\|y^k - x^k\|^2 \geq 0,$$

which contradicts to the fact that

$$f(x^k, y^k) + \frac{\rho}{2}\|y^k - x^k\|^2 < 0.$$

Thus, the linesearch is well defined.
Now we show that C_k is nonempty. Indeed, by the assumption $S_M \neq \emptyset$, then for each $x^* \in S_M$, we get $f(y, x^*) \leq 0,\ \forall y \in C$, so $f(z^k, x^*) \leq 0,\ \forall k$.
From the convexity of $f(z^k, .)$ we have

$$f(z^k, y) \geq f(z^k, z^k) + \langle w^k, y - x^k \rangle,\ \forall y \in C.$$

Therefore,

$$0 \geq f(z^k, x^*) \geq \langle w^k, x^* - x^k \rangle,$$

i.e., $x^* \in C_k,\ \forall k$.
Since

$$\langle w^k, x^k - z^k \rangle = \eta_k \langle w^k, x^k - y^k \rangle.$$

Combining with the linesearch rule (3.1), we get

$$\langle w^k, x^k - z^k \rangle \geq \frac{\eta_k \rho}{2}\|y^2 - x^k\|^2,$$

as desired.

□

Now we are in the position to prove the convergence of the proposed algorithm.

Theorem 3.1 *Suppose that the solution set S_M of MEP(C, f) is nonempty. Then under assumptions (A1), (A2), the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to a solution x^* of EP(C, f).*

Proof. Firstly, we show that any weak accumulation point $\bar{x}$ of the sequence $\{x^k\}$ belongs to C_k for all k. Indeed, assume that $\{x^{k_j}\} \subset \{x^k\}$, $x^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$, and there exists k_0 such that $\bar{x} \notin C_{k_0}$. Then by the closedness and convexity of C_{k_0}, this implies C_{k_0} is also weakly closed, so there exists $k_{j_0} > k_0$ such that $x^{k_j} \notin C_{k_0}$ for all $k_j \geq k_{j_0}$, especially $x^{k_{j_0}} \notin C_{k_0}$. This contradicts to the fact that $x^{k_{j_0}} \in C_{k_{j_0}-1} \subset ... \subset C_{k_0+1} \subset C_{k_0}$. Therefore $\bar{x} \in C_k$, $\forall k$, or $\bar{x} \in \cap_{k=0}^{\infty} C_k$. Since $C_k \subset H_k$, $\forall k$, we also get $\bar{x} \in \cap_{k=0}^{\infty} H_k$. By Lemma 3.1, $S_M \subset \cap_{k=0}^{\infty} C_k$, it implies that $\cap_{k=0}^{\infty} C_k$ is nonempty.

Now, take any $\bar{x} \in \cap_{k=0}^{\infty} C_k$, from Step 4, $x^{k+1} = P_{C_{k+1}}(x^k)$ and Lemma 2.1, we have

$$\begin{aligned} \|x^{k+1} - \bar{x}\|^2 &\leq \|x^k - \bar{x}\|^2 - \|x^{k+1} - x^k\|^2 \\ &\leq \|x^k - \bar{x}\|^2 - d^2(x^k, H_k), \ \forall k. \end{aligned} \tag{3.5}$$

Hence, $\{\|x^{k+1} - \bar{x}\|^2\}$ is a nonincreasingly convergent sequence. Thus, $\{x^k\}$ is bounded and from (3.5) it implies that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{3.6}$$

Combining this fact with Lemma 2.6, we deduce that $\{y^k\}$, $\{z^k\}$, $\{w^k\}$ are also bounded.
Next, we show that $\{x^k\}$ converges weakly to some point $x^* \in \cap_{k=0}^{\infty} C_k$. Indeed, if $\bar{x}$ and x^* are two weak accumulation points of $\{x^k\}$, then there exist $\{x^{k_i}\} \subset \{x^k\}$, $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_i} \rightharpoonup \bar{x}$; $x^{k_j} \rightharpoonup x^*$, by the above argument, we have $\bar{x},\ x^* \in \cap_{k=0}^{\infty} C_k$. From (3.5) it yields $\lim_{k\to\infty} \|x^k - \bar{x}\|^2 = \alpha$ and $\lim_{k\to\infty} \|x^k - x^*\|^2 = \beta$. We have

$$\begin{aligned} \alpha = \lim_{k\to\infty} \|x^k - \bar{x}\|^2 &= \lim_{j\to\infty} \|x^{k_j} - \bar{x}\|^2 \\ &= \lim_{j\to\infty} (\|x^{k_j} - x^*\|^2 + 2\langle x^{k_j} - x^*, x^* - \bar{x}\rangle + \|x^* - \bar{x}\|^2) \\ &= \lim_{j\to\infty} (\|x^{k_j} - x^*\|^2 + \|x^* - \bar{x}\|^2) \\ &= \lim_{k\to\infty} (\|x^k - x^*\|^2 + \|x^* - \bar{x}\|^2) \\ &= \lim_{k\to\infty} (\|x^k - \bar{x}\|^2 + 2\|x^* - \bar{x}\|^2) \\ &= \alpha + 2\|x^* - \bar{x}\|^2. \end{aligned}$$

Hence, $\|x^* - \bar{x}\| = 0$, or $\{x^k\}$ converges weakly to x^*.
Now, we have only to show that x^* solves EP(C, f).
Indeed, because $\{w^k\}$ is bounded, there exists $L > 0$ such that $\|w^k\| \leq L$, $\forall k$.

Combining with (3.3) we get

$$\begin{aligned}\|x^{k+1} - x^k\| = d(x^k, C_k) \geq d(x^k, H_k) &= \frac{|\langle w^k, x^k - z^k \rangle|}{\|w^k\|} \\ &\geq L^{-1} \frac{\eta_k \rho}{2} \|x^k - y^k\|^2.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\lim_{k\to\infty} \eta_k \|x^k - y^k\|^2 = 0. \tag{3.8}$$

We now consider two distinct cases:

Case 1. $\limsup_{k\to\infty} \eta_k > 0$. Then there exists $\bar{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}$, $\forall i$, and by (3.8), one has

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{3.9}$$

Since $x^k \rightharpoonup x^*$ and (3.9), it implies that $y^{k_i} \rightharpoonup x^*$ as $i \to \infty$.
By definition of y^{k_i}:

$$y^{k_i} = \arg\min\{f(x^{k_i}, y) + \frac{\rho}{2}\|y - x^{k_i}\|^2 : \ y \in C\},$$

we have

$$0 \in \partial_2 f(x^{k_i}, y^{k_i}) + \rho(y^{k_i} - x^{k_i}) + N_C(y^{k_i}),$$

so there exists $v^{k_i} \in \partial_2 f(x^{k_i}, y^{k_i})$ such that

$$\langle v^{k_i}, y - y^{k_i}\rangle + \rho\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \geq 0, \ \forall y \in C.$$

Combining with

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) \geq \langle v^{k_i}, y - y^{k_i}\rangle, \ \forall y \in C,$$

yields

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) + \rho\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \geq 0, \ \forall y \in C. \tag{3.10}$$

In addition,

$$\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \leq \|y^{k_i} - x^{k_i}\|\|y - y^{k_i}\|,$$

we receive from (3.10) that

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) + \rho\|y^{k_i} - x^{k_i}\|\|y - y^{k_i}\| \geq 0. \tag{3.11}$$

Letting $i \to \infty$, by jointly weak continuity of f and (3.9), we obtain in the limit that

$$f(x^*, y) - f(x^*, x^*) \geq 0.$$

Hence

$$f(x^*, y) \geq 0, \ \forall y \in C,$$

which means that x^* is a solution of EP(C, f).

Case 2. $\lim_{k\to\infty} \eta_k = 0$.
In this case, from the boundedness of $\{y^k\}$, it deduces that there exists $\{y^{k_i}\} \subset \{y^k\}$ such that $y^{k_i} \rightharpoonup \bar{y}$ as $i \to \infty$.
Replacing y by x^{k_i} in (3.10) we get

$$f(x^{k_i}, y^{k_i}) + \rho\|y^{k_i} - x^{k_i}\|^2 \leq 0. \tag{3.12}$$

In the other hand, by the Armijo linesearch rule (3.1), for $m_{k_i} - 1$, there exists $w^{k_i,m_{k_i}-1} \in \partial_2 f(z^{k_i,m_{k_i}-1}, z^{k_i,m_{k_i}-1})$ such that

$$\langle w^{k_i,m_{k_i}-1}, x^{k_i} - y^{k_i}\rangle < \frac{\rho}{2}\|y^{k_i} - x^{k_i}\|^2. \tag{3.13}$$

By the convexity of $f(z^{k_i,m_{k_i}-1}, .\ )$ we get

$$\begin{aligned} f(z^{k_i,m_{k_i}-1}, y^{k_i}) &\geq f(z^{k_i,m_{k_i}-1}, z^{k_i,m_{k_i}-1}) + \langle w^{k_i,m_{k_i}-1}, y^{k_i} - z^{k_i,m_{k_i}-1}\rangle \\ &= (1 - \eta^{k_i,m_{k_i}-1})\langle w^{k_i,m_{k_i}-1}, y^{k_i} - x^{k_i}\rangle \\ &\geq -(1 - \eta^{k_i,m_{k_i}-1})\frac{\rho}{2}\|y^{k_i} - x^{k_i}\|^2, \end{aligned}$$

where the last inequality deduces from (3.13). Combining with (3.12) we get

$$f(z^{k_i,m_{k_i}-1}, y^{k_i}) \geq -(1-\eta^{k_i,m_{k_i}-1})\frac{\rho}{2}\|y^{k_i}-x^{k_i}\|^2 \geq \frac{1}{2}(1-\eta^{k_i,m_{k_i}-1})f(x^{k_i}, y^{k_i}). \tag{3.14}$$

According to the algorithm, $z^{k_i,m_{k_i}-1} = (1-\eta^{m_{k_i}-1})x^{k_i}+\eta^{m_{k_i}-1}y^{k_i}$, $\eta^{k_i,m_{k_i}-1} \to 0$ and x^{k_i} converges weakly to x^*, y^{k_i} converges weakly to $\bar{y}$, it implies that $z^{k_i,m_{k_i}-1} \rightharpoonup x^*$ as $i \to \infty$. Beside that $\{\|y^{k_i} - x^{k_i}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{i\to+\infty} \|y^{k_i} - x^{k_i}\|^2$ exists. Hence, we get in the limit (3.14) that

$$f(x^*, \bar{y}) \geq -\frac{\rho}{2}\lim_{i\to+\infty} \|y^{k_i} - x^{k_i}\|^2 \geq \frac{1}{2}f(x^*, \bar{y}). \tag{3.15}$$

Therefore, $f(x^*, \bar{y}) = 0$ and $\lim_{i\to+\infty} \|y^{k_i} - x^{k_i}\|^2 = 0$. By the Case 1, it is immediate that x^* is a solution of EP(C, f). □

4 Strong convergence algorithm for EP(C, f)

Let $T : C \to C$ be a nonexpansive mapping i.e., $\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C$ such that $Fix(T) = \{u \in C : Tu = u\} \neq \emptyset$. For obtaining a fixed point of mapping T, Takahashi *et al.* [27] introduced the following iterative method, known as the shrinking projection method:

Algorithm 3

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\alpha \in [0, 1)$, $\{\alpha_k\} \subset [0, \alpha]$ and set $C_0 = C$.

Iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Compute

$$y^k = \alpha_k x^k + (1 - \alpha_k)Tx^k,$$
$$C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}.$$

Step 2. Compute $x^{k+1} = P_{C_{k+1}}(x^g)$, and go to Step 1 with k is replaced by $k+1$.

They proved that $\{x^k\}$ generated by Algorithm 3 converges strongly to $x^* = P_{Fix(T)}(x^g)$. In spired by Algorithm 3 and recent works [7, 10], we now present the following algorithm for solving EP(C, f).

Algorithm 4

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\eta \in (0, 1)$, $\rho > 0$ and set $B_0 = C$.

At each iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min\left\{f(x^k, y) + \frac{\rho}{2}\|y - x^k\|^2 : \ y \in C\right\} \qquad CP(x^k)$$

to obtain its unique solution y^k.

If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m satisfying

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k, \\ w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}), \\ \langle w^{k,m}, x^k - y^k\rangle \geq \frac{\rho}{2}\|y^k - x^k\|^2. \end{cases} \tag{4.1}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$, $w^k = w^{k,m_k}$.

Step 3. Define

$$H_k = \{x \ \in \mathbb{H} : \langle w^k, x - z^k\rangle \leq 0\}, \ C_k = C \cap H_k, \tag{4.2}$$

and take $u^k = P_{C_k}(x^k)$, and go to Step 4.

Step 4. Compute

$$x^{k+1} = P_{B_{k+1}}(x^g),$$

where $B_{k+1} = \{x \in B_k : \|x - u^k\| \leq \|x - x^k\|\}$, and go to Step 1 with k is replaced by $k + 1$.

The following theorem establishes the strong convergence of the proposed algorithm.

Theorem 4.1 *Suppose that the set S_M is nonempty. Then under assumptions (A1), (A2), the sequence $\{x^k\}$, $\{u^k\}$ generated by Algorithm 2 converge strongly to a solution x^* of EP(C, f).*

Proof. Firstly, we observe that $S_M \subset \cap_{k=0}^{\infty} C_k \subset \cap_{k=0}^{\infty} B_k$.

Indeed, take $\bar{x} \in S_M$, i.e., $f(y, \bar{x}) \leq 0,\ \forall y \in C$, especially $f(z^k, \bar{x}) \leq 0,\ \forall k$. Since $f(z^k, .)$ is convex on C, we have

$$f(z^k, \bar{x}) \geq f(z^k, z^k) + \langle w^k, \bar{x} - z^k \rangle,$$

so

$$\langle w^k, \bar{x} - z^k \rangle \leq 0,\ \forall k.$$

Thus, $\bar{x} \in H_k,\ \forall k$, therefore $\bar{x} \in C_k,\ \forall k$. Hence, $S_M \subset \cap_{k=0}^{\infty} C_k$.
From Step 3, $u^k = P_{C_k}(x^k)$, it implies that

$$\|x - u^k\|^2 \leq \|x - x^k\|^2 - \|x^k - u^k\|^2, \quad \forall x \in C_k.$$

Consequently,

$$\|x - u^k\| \leq \|x - x^k\|,\ \forall x \in \cap_{k=0}^{\infty} C_k, \quad \forall k. \tag{4.3}$$

Thus, $\cap_{k=0}^{\infty} C_k \subset \cap_{k=0}^{\infty} B_k$.
By definition of x^k: $x^k = P_{B_k}(x^g)$, we have

$$\|x^k - x^g\| \leq \|x - x^g\|,\ \forall x \in B_k, \tag{4.4}$$

so,

$$\|x^k - x^g\| \leq \|\bar{x} - x^g\|,\ \forall k,\ \forall \bar{x} \in \cap_{k=0}^{\infty} B_k. \tag{4.5}$$

Therefore, $\{x^k\}$ is bounded. Take into account with Lemma 2.2 yields $\{w^k\}$ is bounded. Combining with (4.3) we get $\{u^k\}$ is also bounded.
In addition, $x^{k+1} \in B_{k+1}$ and $B_{k+1} \subset B_k$, we get from (4.4) that

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\|,\ \forall k, \tag{4.6}$$

combining this fact with the boundedness of $\{x^k\}$, we receive

$$\lim_{k \to \infty} \|x^k - x^g\| = \alpha \geq 0. \tag{4.7}$$

Next, we show that $\{x^k\}$ is asymptotically regular, that is $\|x^{k+1} - x^k\| \to 0$ as $k \to \infty$. Indeed, we have

$$\begin{aligned}\|x^{k+1} - x^k\|^2 &= \|x^{k+1} - x^g + x^g - x^k\|^2 \\ &= \|x^{k+1} - x^k\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^g, x^g - x^k\rangle \\ &= \|x^{k+1} - x^k\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^k, x^g - x^k\rangle - 2\|x^g - x^k\|^2 \\ &\le \|x^{k+1} - x^g\|^2 - \|x^k - x^g\|^2,\end{aligned}$$

where the last inequality follows from the fact that $x^k = P_{B_k}(x^g)$ and $x^{k+1} \in B_k$, then $\langle x^{k+1} - x^k, x^g - x^k\rangle \le 0$.
By (4.7), we obtain

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{4.8}$$

Because $x^{k+1} \in B_{k+1}$, one has

$$\begin{aligned}\|x^k - u^k\| &\le \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\| \\ &\le 2\|x^k - x^{k+1}\|\end{aligned}$$

Take into account with (4.8) we get

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{4.9}$$

Next, we show that $\{x^k\}$, $\{u^k\}$ converge strongly to $x^* = P_{\cap_{k=0}^{\infty} B_k}(x^g)$.
Indeed, we observe that any weak accumulation point $\bar{x}$ of the sequence $\{x^k\}$ belongs to B_k for all k. By contradiction, if $\{x^{k_j}\} \subset \{x^k\}$, $x^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$, and there exists k_0 such that $\bar{x} \notin B_{k_0}$. Then, by the closedness and convexity of B_{k_0}, it implies that B_{k_0} is also weakly closed, so $x^{k_j} \notin B_{k_0}$ for all $k_j \ge k_{j_0}$, for some $k_{j_0} > k_0$, especially $x^{k_{j_0}} \notin B_{k_0}$. This contradicts to the fact that $x^{k_{j_0}} \in B_{k_{j_0}-1} \subset ... \subset B_{k_0+1} \subset B_{k_0}$. Therefore, $\bar{x} \in B_k$, $\forall k$, or $\bar{x} \in \cap_{k=0}^{\infty} B_k$. Now, we set $x^* = P_{\cap_{k=0}^{\infty} B_k}(x^g)$.
From (4.5) one has,

$$\|x^k - x^g\| \le \|x^* - x^g\|, \ \forall k. \tag{4.10}$$

It is immediate from Lemma 2.4 that x^k converges strongly to x^*. Combining with (4.9) we have u^k also converges strongly to x^*.
To finish the proof, we need showing that x^* solves EP(C, f).
Since $\{w^k\}$ is bounded, then there exists $L > 0$ such that $\|w^k\| \le L$, $\forall k$. Combining with $u^k = P_{C_k}(x^k)$ we get

$$\begin{aligned}\|u^k - x^k\| = d(x^k, C_k) \ge d(x^k, H_k) &= \frac{|\langle w^k, x^k - z^k\rangle|}{\|w^k\|} \\ &\ge L^{-1}\frac{\eta_k\rho}{2}\|x^k - y^k\|^2.\end{aligned} \tag{4.11}$$

From (4.9) and (4.11), we obtain

$$\lim_{k\to\infty} \eta_k \|x^k - y^k\|^2 = 0. \tag{4.12}$$

We now consider two distinct cases:

Case 1. $\limsup_{k\to\infty} \eta_k > 0$.
Then there exists $\bar{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}$, $\forall i$, and from (4.12), one has

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{4.13}$$

Remember that $x^k \to x^*$ and (4.13), it implies that $y^{k_i} \to x^*$ as $i \to \infty$.
By definition of y^{k_i} we have

$$f(x^{k_i}, y) + \frac{\rho}{2}\|y - x^{k_i}\|^2 \geq f(x^{k_i}, y^{k_i}) + \frac{\rho}{2}\|y^{k_i} - x^{k_i}\|^2, \ \forall y \in C. \tag{4.14}$$

Letting $i \to \infty$, by jointly weak continuity of f and $x^{k_i} \to x^*$, $y^{k_i} \to x^*$, we obtain in the limit that

$$f(x^*, y) + \frac{\rho}{2}\|y - x^*\|^2 \geq 0.$$

By Lemma 2.3, we conclude that

$$f(x^*, y) \geq 0, \ \forall y \in C.$$

Therefore, x^* is a solution of EP(C, f).

Case 2. $\lim_{k\to\infty} \eta_k = 0$.
From the boundedness of $\{y^k\}$, it deduces that there exists $\{y^{k_i}\} \subset \{y^k\}$ such that $y^{k_i} \rightharpoonup \bar{y}$ as $i \to \infty$.
Replacing y by x^{k_i} in (3.10) we get

$$f(x^{k_i}, y^{k_i}) + \rho\|y^{k_i} - x^{k_i}\|^2 \leq 0. \tag{4.15}$$

In the other hand, by the Armijo linesearch rule (4.1), for $m_{k_i} - 1$, there exists $w^{k_i, m_{k_i}-1} \in \partial_2 f(z^{k_i, m_{k_i}-1}, z^{k_i, m_{k_i}-1})$ such that

$$\langle w^{k_i, m_{k_i}-1}, x^{k_i} - y^{k_i}\rangle < \frac{\rho}{2}\|y^{k_i} - x^{k_i}\|^2 \tag{4.16}$$

and by the convexity of $f(z^{k_i, m_{k_i}-1}, .)$ we get

$$\begin{aligned} f(z^{k_i, m_{k_i}-1}, y^{k_i}) &\geq f(z^{k_i, m_{k_i}-1}, z^{k_i, m_{k_i}-1}) + \langle w^{k_i, m_{k_i}-1}, y^{k_i} - z^{k_i, m_{k_i}-1}\rangle \\ &= (1 - \eta^{k_i, m_{k_i}-1})\langle w^{k_i, m_{k_i}-1}, y^{k_i} - x^{k_i}\rangle \\ &\geq -(1 - \eta^{k_i, m_{k_i}-1})\frac{\rho}{2}\|y^{k_i} - x^{k_i}\|^2, \end{aligned}$$

where the last inequality deduces from (4.16). Combining with (4.15) we get

$$f(z^{k_i,m_{k_i}-1},y^{k_i}) \geq -(1-\eta^{k_i,m_{k_i}-1})\frac{\rho}{2}\|y^{k_i}-x^{k_i}\|^2 \geq \frac{1}{2}(1-\eta^{k_i,m_{k_i}-1})f(x^{k_i},y^{k_i}). \quad (4.17)$$

According to the algorithm, we have $z^{k_i,m_{k_i}-1} = (1-\eta^{m_{k_i}-1})x^{k_i}+\eta^{m_{k_i}-1}y^{k_i}$, $\eta^{k_i,m_{k_i}-1} \to 0$ and x^{k_i} converges strongly to x^*, y^{k_i} converges weakly to $\bar{y}$, it implies that $z^{k_i,m_{k_i}-1} \to x^*$ as $i \to \infty$. Beside that $\{\|y^{k_i}-x^{k_i}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{i\to+\infty}\|y^{k_i}-x^{k_i}\|^2$ exists. Hence, we get in the limit (4.17) that

$$f(x^*,\bar{y}) \geq -\frac{\rho}{2}\lim_{i\to+\infty}\|y^{k_i}-x^{k_i}\|^2 \geq \frac{1}{2}f(x^*,\bar{y}).$$

Therefore, $f(x^*,\bar{y}) = 0$ and $\lim_{i\to+\infty}\|y^{k_i}-x^{k_i}\|^2 = 0$. By the Case 1, it is immediate that x^* is a solution of EP(C,f). □

5 Numerical examples

In this section, we applied the algorithm 1 for solving an equilibrium problem arising in Nash-Cournot oligopolistic electricity market equilibrium model. This model has been investigated in some research papers, for instance, [8, 23]. To test the proposed algorithm, we take the example in [23]. In this example, there are n^c companies, each company i may possess I_i generating units. Let n^g be number of all generating units and x be the vector whose entry x_i stands for the power generating by unit i. Following [8, 23] we suppose that the price p is a decreasing affine function of the σ with $\sigma = \sum_{i=1}^{n^g} x_i$, that is

$$p(x) = 378.4 - 2\sum_{i=1}^{n^g} x_i = p(\sigma).$$

Then the profit made by company i is given by

$$f_i(x) = p(\sigma)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

where $c_j(x_j)$ is the cost for generating x_j. As in [23] we suppose that the cost $c_j(x_j)$ is given by

$$c_j(x_j) := \max\{c_j^0(x_j), c_j^1(x_j)\}$$

with

$$c_j^0(x_j) := \frac{\alpha_j^0}{2}x_j^2 + \beta_j^0 x_j + \gamma_j^0,\ c_j^1(x_j) := \alpha_j^1 x_j + \frac{\beta_j^1}{\beta_j^1+1}\gamma_j^{-1/\beta_j^1}(x_j)^{(\beta_j^1+1)/\beta_j^1},$$

where $\alpha_j^k, \beta_j^k, \gamma_j^k$ $(k = 0, 1)$ are given parameters.

Let $x_j^{\min}$ and $x_j^{\max}$ be the lower and upper bounds for the power generating by the unit j. Then the strategy set of the model takes the form

$$C := \{x = (x_1, ..., x^{n^g})^T : \ x_j^{\min} \leq x_j \leq x_j^{\max}, \ \forall j\}.$$

By setting $q^i := (q_1^i, ..., q_{n^g}^i)^T$ with

$$q_j^i = \begin{cases} 1 & \text{if } j \in I_i \\ 0 & \text{if } j \notin I_i \end{cases},$$

and define

$$A := 2\sum_{i=1}^{n^c}(1 - q^i)(q^i)^T, \quad B := 2\sum_{i=1}^{n^c} q^i(q^i)^T, \tag{5.1}$$

$$a := -387.4\sum_{i=1}^{n^c} q^i, \text{and } c(x) := \sum_{j=1}^{n^g} c_j(x_j). \tag{5.2}$$

Then the oligopolistic equilibrium model under consideration can be formulated by the following equilibrium problem EP(C, f) (see [23, Page 155])

Find $x^* \in C : f(x^*, y) = [(A + B)x^* + By + a]^T(y - x^*) + c(y) - c(x^*) \geq 0, \ \forall y \in C.$

The authors in [23] pointed out that $f(x, y) + f(y, x) = -(y - x)^T A(y - x)$, and A is not positive semidefinite, the bifunction f is nonmonotone, nonsmooth, so they could not apply their algorithms proposed in [23] for the EP(C, f) directly.

We test Algorithm 1 for this problem with corresponds to the first model in [8] where $n^c = 3$, and the parameters are given in the following tables

Com.	Gen.	$x_{\min}^g$	$x_{\max}^g$	$x_{\min}^c$	$x_{\max}^c$
1	1	0	80	0	80
2	2	0	80	0	130
2	3	0	50	0	130
3	4	0	55	0	125
3	5	0	30	0	125
3	6	0	40	0	125

Table 1: The lower and upper bounds of the power generation of the generating units and companies.

We implement Algorithm 1 in Matlab R2013a running on a Desktop with Intel(R) Core(TM) 2Duo CPU E8400 3GHz with 3GB Ram. To terminate the Algorithm, we use the stopping criteria $\frac{\|x^{k+1} - x^k\|}{\max\{1, \|x^k\|\}} \leq \epsilon$ with a tolerance $\epsilon = 10^{-4}$.

Gen.	α_j^0	β_j^0	γ_j^0	α_j^1	β_j^1	γ_j^1
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 2: The parameters of the generating unit cost functions.

Iter(k)	ρ	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	Cpu(s)
0	0.1	0	0	0	0	0	0	
92		46.6488	32.1488	15.0054	21.8164	12.4822	12.4831	15.1789
0	0.5	0	0	0	0	0	0	
165		46.6528	32.1421	15.0053	21.6056	12.5865	12.5864	25.6466
0	0.9	0	0	0	0	0	0	
209		46.6574	32.1488	15.0039	21.7068	12.5350	12.5350	28.8134
0	0.1	30	20	10	15	10	10	
71		46.6484	32.1311	15.0017	21.7730	12.5104	12.5107	11.1385
0	0.5	30	20	10	15	10	10	
117		46.6526	32.1274	15.0169	21.6004	12.5889	12.5890	17.3161
0	0.9	30	20	10	15	10	10	
151		46.6572	32.1407	15.0114	21.6973	12.5395	12.5395	20.2489

Table 3: Results computed with some starting points and regularized parameters

The computation results are reported in Table 3 with some starting points and regularized parameters.

Conclusion. We have introduced two projection algorithms for finding a solution of a nonmonotone equilibrium problem in a real Hilbert space. The weak and strong convergence of the proposed algorithms are obtained. We then have applied a proposed algorithm for a Nash-Cournot oligopolistic equilibrium model of electricity market. Some computation results are reported.

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