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ORIGINAL PAPER

On existence of weak solutions for a p-Laplacian system at resonance

Bui Quoc Hung · Hoang Quoc Toan

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Abstract This article shows the existence of weak solutions of a resonance problem for uniformly p-Laplacian system in a bounded domain in R^N. Our arguments are based on the Saddle Point Theorem (P.H.Rabinowitz) and rely on a generalization of the Landesman–Lazer type condition.

Keywords Semilinear elliptic equation · Saddle point theorem · Landesman–Lazer condition

Mathematics Subject Classification 35J20 · 35J60 · 58E05

1 Introduction and preliminaries

Let Ω be a bounded domain in R^N, $(N \geq 3)$, with smooth boundary $\partial\Omega$. In the present paper we consider the existence of weak solutions of the following Dirichlet problem at resonance for p-Laplacian system:

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{\alpha-1}|v|^{\beta-1}v + f(x,u,v) - k_1(x) \\ -\Delta_p v = \lambda_1 |u|^{\alpha-1}|v|^{\beta-1}u + g(x,u,v) - k_2(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

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B. Q. Hung (✉)
Faculty of Information Technology, Le Quy Don Technical University,
236 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam
e-mail: quochung2806@yahoo.com

H. Q. Toan
Department of Mathematics, Hanoi University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
e-mail: hq_toan@yahoo.com

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where

$$p \geq 2, \alpha \geq 1, \beta \geq 1, \alpha + \beta = p \tag{1.2}$$

and $f, g : \Omega \times R^2 \to R$ are Carathéodory functions which will be specified later.

$$k_i(x) \in L^{p'}(\Omega),\ p' = \frac{p}{p-1}, k_i(x) > 0,\ \text{for a.e } x \in \bar{\Omega}, i = 1, 2.$$

λ_1 denotes the first eigenvalue of the problem:

$$\begin{cases} -\Delta_p u = & \lambda|u|^{\alpha-1}|v|^{\beta-1}v \\ -\Delta_p v = & \lambda|u|^{\alpha-1}|v|^{\beta-1}u, \end{cases} \tag{1.3}$$

where $(u, v) \in E = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $p \geq 2, \alpha \geq 1, \beta \geq 1, \alpha + \beta = p$.

It's well-known that the principle eigenvalue $\lambda_1 = \lambda_1(p)$ of (1.3) is obtained using the Ljusternick–Schnirelmann theory by minimizing the functional

$$J(u, v) = \frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx$$

on the set:

$$S = \left\{(u, v) \in E = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : A(u, v) = 1\right\},$$

where

$$A(u, v) = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uvdx$$

that is $\lambda_1 = \lambda_1(p)$ can be variational characterized as

$$\lambda_1 = \lambda_1(p) = \inf_{A(u,v)>0} \frac{J(u, v)}{A(u, v)}. \tag{1.4}$$

Moreover the eigenpair (φ_1, φ_2) associated with λ_1 is componentwise positive and unique (up to multiplication by nonzero scalar) (see Theorem 2.2 in [3] and Remark 5.4 in [5]). As usual $W_0^{1,p}(\Omega)$ denotes Sobolev space which can be defined as the completion of $C_0^\infty(\Omega)$ under the norm:

$$||u||_{W_0^{1,p}} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$$

and

$$\text{for } w = (u, v) \in E : ||w||_E = \left(||u||^p_{W_0^{1,p}} + ||v||^p_{W_0^{1,p}}\right)^{\frac{1}{p}}.$$

Observe that the existence of weak solutions of (p, q)-Laplacian systems at resonance in bounded domains with Dirichlet boundary condition, was first considered by Zographopoulos in [9]. Later in [4] Kandilakis and Magiropoulos have studied a quasilinear elliptic system with resonance part and nonlinear boundary condition in an unbounded domain by assuming the nonlinearities f and g depending only one variable u or v. In [8] Zeng-Qi Ou and Chen Lei Tang have considered the same system as in [4] with Dirichlet condition in a bounded domain. In these the existence of weak solutions is obtained by critical point theory (the Minimum Principle or the Saddle Point Theorem) under a Landesman–Lazer type condition.

In this paper by introducing a generalization of Landesman–Lazer type condition we shall prove an existence result for a p-Laplacian system on resonance in bounded domain with the nonlinearities f and g to be functions depending on both variables u and v.

Our arguments are based on the Saddle Point Theorem (P.H.Rabinowitz) and generalization of the Landesman–Lazer type condition.

We have the following definition.

Definition 1.1 Function $w = (u, v) \in E$ is called a weak solution of the problem (1.1) if and only if, for all $\bar{w} = (\bar{u}, \bar{v}) \in E$

$$\alpha \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \bar{u} dx + \beta \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla \bar{v} dx$$
$$-\lambda_1 \int_\Omega (\alpha |u|^{\alpha-1}|v|^{\beta-1} v\bar{u} + \beta |u|^{\alpha-1}|v|^{\beta-1} u\bar{v})dx$$
$$-\int_\Omega (\alpha f(x,u,v)\bar{u} + \beta g(x,u,v)\bar{v})dx + \int_\Omega (\alpha k_1(x)\bar{u} + \beta k_2(x)\bar{v})dx = 0.$$

We will use the following conditions

(H_1)

(i) For a.e $x \in \Omega : f(x, .), g(x, .) \in C^1(R^2)$ and $f(x, 0, 0) = 0, g(x, 0, 0) = 0$.

(ii) There exists function $\tau \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$ such that:

$$|f(x,s,t)| \le \tau(x), |g(x,s,t)| \le \tau(x), \text{ for a.e } x \in \Omega, \forall (s,t) \in R^2.$$

(iii) For $(s, t) \in R^2$:

$$\alpha \frac{\partial f(x,s,t)}{\partial t} = \beta \frac{\partial g(x,s,t)}{\partial s} \qquad \text{for a.e } x \in \Omega. \tag{1.5}$$

For $(u, v) \in R^2$, a.e $x \in \Omega$, define

$$H(x,u,v) = \frac{\alpha}{2}\int_0^u (f(x,s,v) + f(x,s,0))]ds + \frac{\beta}{2}\int_0^v (g(x,u,t) + g(x,0,t))dt. \tag{1.6}$$

By hypotheses (1.5), from (1.6) with some simple computations we deduce that:

$$\frac{\partial H(x,s,t)}{\partial s} = \alpha f(x,s,t), \; \frac{\partial H(x,s,t)}{\partial t} = \beta g(x,s,t), \text{ for a.e } x \in \Omega, \forall (s,t) \in R^2. \tag{1.7}$$

Now, for $i, j = 1, 2$ we define

$$F_i(x) = \lim_{\tau \to +\infty} \frac{1}{\tau}\int_0^\tau \left\{ f\left(x, (-1)^{1+i} y\varphi_1, (-1)^{1+i}\tau\varphi_2\right) + f\left(x, (-1)^{1+i} y\varphi_1, 0\right)\right\} dy$$
$$G_j(x) = \lim_{\tau \to +\infty} \frac{1}{\tau}\int_0^\tau \left\{ g\left(x, (-1)^{1+j} \tau\varphi_1, (-1)^{1+j} y\varphi_2\right) + g\left(x, 0, (-1)^{1+j} y\varphi_2\right)\right\} dy \tag{1.8}$$

and

$$\lim_{\substack{s\to+\infty\\t\to+\infty}} f(x,s,t) = f^{+\infty}(x), \qquad \lim_{\substack{s\to+\infty\\t\to+\infty}} g(x,s,t) = g^{+\infty}(x)$$
$$\lim_{\substack{s\to-\infty\\t\to-\infty}} f(x,s,t) = f^{-\infty}(x), \qquad \lim_{\substack{s\to-\infty\\t\to-\infty}} g(x,s,t) = g^{-\infty}(x).$$

Assume that
(H_2)

(i)

$$f^{+\infty}(x) < k_1(x) < f^{-\infty}(x)$$
$$g^{+\infty}(x) < k_2(x) < g^{-\infty}(x) \qquad \text{for a.e } x \in \Omega \tag{1.9}$$

(ii)

$$\begin{aligned} &\int_\Omega \left\{ \frac{1}{2}(\alpha F_2(x)\varphi_1(x) + \beta G_2(x)\varphi_2(x)) - \frac{\alpha}{p} f^{-\infty}(x)\varphi_1(x) - \frac{\beta}{p} g^{-\infty}(x)\varphi_2(x) \right\} dx \\ &< \left(1 - \frac{1}{p}\right) \int_\Omega (\alpha k_1(x)\varphi_1(x) + \beta k_2(x)\varphi_2(x))]dx \\ &< \int_\Omega \left\{ \frac{1}{2}(\alpha F_1(x)\varphi_1(x) + \beta G_1(x)\varphi_2(x)) - \frac{\alpha}{p} f^{+\infty}(x)\varphi_1(x) - \frac{\beta}{p} g^{+\infty}(x)\varphi_2(x) \right\} dx. \end{aligned} \tag{1.10}$$

The main result of this paper can be described in the following theorem:

Theorem 1.1 *Assuming conditions* (H_1), (H_2) *are fulfilled. Then the problem* (1.1) *has at least a nontrivial weak solution in* E.

Proof of Theorem 1.1 is based on variational techniques and the Saddle Point Theorem (P.H.Rabinowitz).

Theorem 1.2 (Saddle Point Theorem, P.H.Rabinowitz in [6]) *Let* $E = X \oplus Y$ *be a Banach space with* Y *closed in* E *and* $dim X < \infty$. *For* $\varrho > 0$ *define*

$$M := \{u \in X : ||u|| \leq \varrho\} \qquad M_0 := \{u \in X : ||u|| = \varrho\}$$

Let $F \in C^1(E, R)$ *be such that*

$$b := \inf_{u \in Y} F(u) > a := \max_{u \in M_0} F(u)$$

If F *satisfies the* $(PS)_c$ *condition with*

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} F(\gamma(u)) \qquad \text{where } \Gamma := \{\gamma \in C(M, E) : \gamma|_{M_0} = I\},$$

then c *is a critical value of* F.

2 Proof of the main result

We define the Euler–Lagrange functional associated to the problem (1.1) by

$$\begin{aligned} I(w) &= \frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx - \lambda_1 \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u.v dx \\ &\quad - \int_\Omega H(x, u, v)dx + \int_\Omega (\alpha k_1(x)u + \beta k_2(x)v)dx \\ &= J(w) + T(w), \quad \text{for } w = (u, v) \in E, \end{aligned} \tag{2.1}$$

where

$$J(w) = \frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx. \tag{2.2}$$

$$T(w) = -\lambda_1 \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u.vdx - \int_\Omega H(x,u,v)dx + \int_\Omega (\alpha k_1(x)u + \beta k_2(x)v)dx. \tag{2.3}$$

We deduce that $I \in C^1(E)$.

Remark 2.1 By similar arguments as those in the proof of Lemma 2.3 in [10] and Lemma 5 in [4], we infer that the functional $A : E \to R$ and the operator $B : E \to E^*$ given by, for any $(u, v), (\bar{u}, \bar{v}) \in E$

$$A(u, v) = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u.vdx$$

and

$$< B(u, v), (\bar{u}, \bar{v}) >= \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}\bar{u}vdx + \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u\bar{v}dx,$$

are compact.

Remark 2.2 Applying Theorem 1.6 in [6, p9] we deduce that the functional $J : E \to R$ given by (2.2) is weakly lower semicontinuous on E. Hence the functional $I = T + J$ is also weakly lower semicontinuous on E.

Proposition 2.1 *Assuming the hypotheses* (H_1) *and* (H_2) *are fulfilled. The functional* I : $E \to R$ *given by* (2.1) *satisfies the* (PS) *condition on* E.

Proof Let $\{w_m = (u_m, v_m)\}$ be a Palais–Smale sequence in E, i.e:

$$|I(w_m)| \leq M, M \text{ is positive constant} \tag{2.4}$$

$$I'(w_m) \to 0 \text{ in } E^* \text{ as } m \to +\infty \tag{2.5}$$

First, we shall prove that $\{w_m\}$ is bounded in E. We suppose by contradiction that $\{w_m\}$ is not bounded in E. Without loss of generality we assume that

$$||w_m||_E \to +\infty \text{ as } m \to +\infty.$$

Let $\widehat{w}_m = \frac{w_m}{||w_m||_E} = (\widehat{u}_m, \widehat{v}_m)$ that is $\widehat{u}_m = \frac{u_m}{||w_m||_E}$ and $\widehat{v}_m = \frac{v_m}{||w_m||_E}$.

Thus $\widehat{w}_m$ is bounded in E. Then there exists a subsequence $\{\widehat{w}_{m_k} = (\widehat{u}_{m_k}, \widehat{v}_{m_k})\}_k$ which converges weakly to $\widehat{w} = (\widehat{u}, \widehat{v})$ in E. Since the embedding $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, the sequences $\{\widehat{u}_{m_k}\}$ and $\{\widehat{v}_{m_k}\}$ converge strongly to $\widehat{u}$ and $\widehat{v}$ in $L^p(\Omega)$ respectively.

From (2.4) we have

$$\lim_{k\to+\infty}\sup\left\{\frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}_{m_k}|^p dx - \lambda_1\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx \right.$$
$$\left. - \int_\Omega \frac{H(x, w_{m_k})}{||w_{m_k}||_E^p}dx + \int_\Omega \frac{\alpha k_1\widehat{u}_{m_k} + \beta k_2\widehat{v}_{m_k}}{||w_{m_k}||_E^{p-1}}dx\right\} \leq 0. \tag{2.6}$$

By hypotheses (H_1), we deduce that

$$H(x,w_{mk}) = \frac{\alpha}{2}\int_0^{u_{m_k}} (f(x,s,v_{mk}) + f(x,s,0))ds + \frac{\beta}{2}\int_0^{v_{m_k}} (g(x,u_{mk},t) + g(x,0,t))dt.$$

This implies that $|H(x,w_{mk})| \le c.\tau(x)(|u_{mk}| + |v_{mk}|)$, c is positive constant.

Hence,

$$\left|\int_\Omega \frac{H(x,w_{mk})}{||w_{mk}||^p}\right| \le \frac{c}{||w_{mk}||_E^{p-1}}||\tau||_{L^{p'}(\Omega)}\big(||\widehat{u}_{mk}||_{L^p(\Omega)} + ||\widehat{v}_{mk}||_{L^p(\Omega)}\big).$$

Since $\widehat{u}_{m_k}, \widehat{v}_{m_k}$ converge strongly in $L^p(\Omega)$ then bounded in $L^p(\Omega)$, hence

$$\lim_{k\to+\infty}\sup\int_\Omega \frac{H(x,w_{mk})}{||w_{mk}||_E^p} = 0 \tag{2.7}$$

and

$$\lim_{k\to+\infty}\int_\Omega \frac{\alpha k_1\widehat{u}_{m_k} + \beta k_2\widehat{v}_{m_k}}{||w_{m_k}||_E^{p-1}}dx = 0.$$

From the compactness of operator A it follows that

$$\lim_{k\to+\infty}\lambda_1\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx = \lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}.\widehat{v}dx. \tag{2.8}$$

Using the weak lower semicontinuity of the functional J and the variational characterization of λ_1 from (2.6) we get

$$\begin{aligned}
\lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}.\widehat{v}dx &\le \frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}|^p dx \\
&\le \lim_{k\to+\infty}\inf\left\{\frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}_{m_k}|^p dx\right\} \\
&\le \lim_{k\to+\infty}\sup\left\{\frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}_{m_k}|^p dx\right\} \le \lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}.\widehat{v}dx.
\end{aligned} \tag{2.9}$$

Thus, theses inequalities are indeed equalities and we have

$$\begin{aligned}
\lim_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}_{m_k}|^p dx\right\} &= \frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}|^p dx \\
&= \lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}.\widehat{v}dx.
\end{aligned} \tag{2.10}$$

We shall prove that $\widehat{u} \neq 0$ and $\widehat{v} \neq 0$.

By contradiction suppose that $\widehat{u} = 0$, thus $\widehat{u}_{m_k} \to 0$ in $L^p(\Omega)$ as $k \to +\infty$. We have

$$\begin{aligned}
|A(\widehat{u}_{m_k}, \widehat{v}_{m_k})| &= \left|\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx\right| \\
&\le ||\widehat{u}_{m_k}||^\alpha_{L^p(\Omega)}.||\widehat{v}_{m_k}||^\beta_{L^p(\Omega)}.
\end{aligned}$$

Since $||\widehat{u}_{m_k}||_{L^p(\Omega)} \to 0$, letting $k \to +\infty$ shows that

$$\lim_{k\to+\infty} A(\widehat{u}_{m_k}, \widehat{v}_{m_k}) = 0. \tag{2.11}$$

From (2.6) taking $\lim_{k\to+\infty}\sup$ with (2.7) and (2.10) we arrive at

$$\lim_{k\to+\infty}\sup\left\{\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^p dx\right\}=0. \tag{2.12}$$

On the other hand, since $||\widehat{w}_{m_k}||_E=1$ and

$$\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^p dx\geq\min\left(\frac{\alpha}{p},\frac{\beta}{p}\right).||\widehat{w}_{m_k}||_E=\min\left(\frac{\alpha}{p},\frac{\beta}{p}\right)>0$$

which contradicts (2.11). Thus $\widehat{u}\neq 0$. Similary we have $\widehat{v}\neq 0$.

By again the definition of λ_1 from (2.10) we deduce that

$$\widehat{w}=(\widehat{u},\widehat{v})=(\varphi_1,\varphi_2)\text{ or }\widehat{w}=(\widehat{u},\widehat{v})=(-\varphi_1,-\varphi_2),$$

where (φ_1,φ_2) is eigenpair associated with λ_1 of the problem (1.3).

Next, we shall consider following two cases:

Firstly, assume that $\widehat{u}_{m_k}\to\varphi_1$, $\widehat{v}_{m_k}\to\varphi_2$ in $L^p(\Omega)$ as $k\to+\infty$.

From (2.4) we have

$$\begin{aligned}-M\leq&-\frac{\alpha}{p}\int_\Omega|\nabla u_{m_k}|^p dx-\frac{\beta}{p}\int_\Omega|\nabla v_{m_k}|^p dx+\lambda_1\int_\Omega|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}v_{m_k}dx\\&+\int_\Omega H(x,w_{m_k})dx-\int_\Omega(\alpha k_1u_{m_k}+\beta k_2v_{m_k})dx\leq M.\end{aligned} \tag{2.13}$$

Moreover, from (2.5) there exists the sequence ϵ_k, $\epsilon_k\to 0^+$, $k\to+\infty$ such that

$$|<I'(w_{m_k}),\left(\frac{u_{m_k}}{p},\frac{v_{m_k}}{p}\right)>|\leq\epsilon_k.\frac{1}{p}||w_m||_E.$$

This implies

$$\begin{aligned}-\epsilon_k.\frac{1}{p}||w_{m_k}||_E\leq&\,\alpha\int_\Omega|\nabla u_{m_k}|^{p-2}\nabla u_{m_k}\nabla\left(\frac{u_{m_k}}{p}\right)dx+\beta\int_\Omega|\nabla v_{m_k}|^{p-2}\nabla v_{m_k}\nabla\left(\frac{v_{m_k}}{p}\right)dx\\&-\lambda_1\int_\Omega\left(\alpha|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}v_{m_k}\left(\frac{u_{m_k}}{p}\right)+\beta|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}\left(\frac{v_{m_k}}{p}\right)\right)dx\\&-\int_\Omega\left(\alpha f(x,w_{m_k})\frac{u_{m_k}}{p}+\beta g(x,w_{m_k})\frac{v_{m_k}}{p}\right)dx+\int_\Omega\left(\alpha k_1\frac{u_{m_k}}{p}+\beta k_2\frac{v_{m_k}}{p}\right)dx\\&\leq\epsilon_k.\frac{1}{p}||w_{m_k}||_E.\end{aligned}$$

Remark that $\alpha+\beta=p$, we get

$$\begin{aligned}-\epsilon_k.\frac{1}{p}||w_{m_k}||_E\leq&\,\frac{\alpha}{p}\int_\Omega|\nabla u_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega|\nabla v_{m_k}|^p dx\\&-\lambda_1\int_\Omega(\alpha|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}v_{m_k})dx-\int_\Omega\left(\alpha f(x,w_{m_k})\frac{u_{m_k}}{p}+\beta g(x,w_{m_k})\frac{v_{m_k}}{p}\right)dx\\&+\int_\Omega\left(\frac{\alpha}{p}k_1u_{m_k}+\frac{\beta}{p}k_2v_{m_k}\right)dx\leq\epsilon_k.\frac{1}{p}||w_{m_k}||_E.\end{aligned} \tag{2.14}$$

Hence, summing (2.13), (2.14) we obtain

$$-M - \frac{\epsilon_k}{p}||w_{m_k}||_E \le \int_\Omega \left(H(x, w_{m_k}) - \left(\frac{\alpha}{p} f(x, w_{m_k}) u_{m_k} + \frac{\beta}{p} g(x, w_{m_k}) v_{m_k} \right) \right) dx$$
$$- \int_\Omega \left(\alpha \left(1 - \frac{1}{p} \right) k_1 u_{m_k} + \beta \left(1 - \frac{1}{p} \right) k_2 v_{m_k} \right) dx \le M + \frac{\epsilon_k}{p}||w_{m_k}||_E. \tag{2.15}$$

After dividing (2.15) by $||w_{m_k}||_E$, letting $\lim\limits_{k\to+\infty}\sup$ we deduce that

$$\lim_{k\to+\infty}\sup \int_\Omega \left\{ \frac{H(x, w_{m_k})}{||w_{m_k}||_E} - \frac{\alpha}{p} f(x, w_{m_k}) \widehat{u}_{m_k} - \frac{\beta}{p} g(x, w_{m_k}) \widehat{v}_{m_k} \right\} dx$$
$$= \left(1 - \frac{1}{p} \right) \int_\Omega (\alpha k_1 \varphi_1 + \beta k_2 \varphi_2) dx. \tag{2.16}$$

We remark that, from (1.6) by some standard computations we get

$$\lim_{k\to+\infty}\sup \int_\Omega \frac{H(x, w_{m_k})}{||w_{m_k}||_E} dx = \frac{1}{2} \int_\Omega (\alpha F_1 \varphi_1 + \beta G_1 \varphi_2) dx,$$

where $F_1(x)$, $G_1(x)$ are given by (1.8).

Letting $\lim\limits_{k\to+\infty}\sup$ (2.16) we obtain

$$\int_\Omega \left\{ \frac{1}{2}(\alpha F_1 \varphi_1 + \beta G_1 \varphi_2) - \frac{\alpha}{p} f^{+\infty} \varphi_1 - \frac{\beta}{p} g^{+\infty} \varphi_2 \right\} dx$$
$$= \left(1 - \frac{1}{p} \right) \int_\Omega (\alpha k_1 \varphi_1 + \beta k_2 \varphi_2) dx,$$

which contradicts $(H_2(ii))$.

Similarly, in the case when $\widehat{u}_{m_k} \to -\varphi_1$, $\widehat{v}_{m_k} \to -\varphi_2$, in $L^p(\Omega)$ as $k \to +\infty$, by similar computations, we also have

$$\int_\Omega \left\{ \frac{1}{2}(\alpha F_2 \varphi_1 + \beta G_2 \varphi_2) - \frac{\alpha}{p} f^{-\infty} \varphi_1 - \frac{\beta}{p} g^{-\infty} \varphi_2 \right\} dx$$
$$= \left(1 - \frac{1}{p} \right) \int_\Omega (\alpha k_1 \varphi_1 + \beta k_2 \varphi_2) dx,$$

where $F_2(x)$, $G_2(x)$ are given by (1.8), which contradicts $(H_2(ii))$.

This implies that the (PS) sequence $\{w_m\}$ is bounded in E. Then there exists a subsequence w_{m_k} which converges weakly to $w_0 = (u_0, v_0) \in E$.

We shall prove that w_{m_k} converges strongly to $w_0 = (u_0, v_0) \in E$.

Indeed, since $w_{m_k} \rightharpoonup w_0 = (u_0, v_0)$ in E and the embedding $W_0^{1,p} \times W_0^{1,p} \hookrightarrow L^p(\Omega) \times L^p(\Omega)$ is compact, the subsequences u_{m_k}, v_{m_k} converge strongly to u_0, v_0 in L^p respectively. We have

$$|T'(w_{m_k}, (w_{m_k} - w_0))| \le \lambda_1 \left\{ \int_\Omega \alpha |u_{m_k}|^{\alpha-1} |v_{m_k}|^\beta |u_{m_k} - u_0| dx \right.$$
$$\left. + \int_\Omega \beta |u_{m_k}|^\alpha |v_{m_k}|^{\beta-1} |v_{m_k} - v_0| dx \right\} + \int_\Omega \{ \alpha |f(x, w_{m_k})| |u_{m_k} - u_0|$$
$$+ \beta |g(x, w_{m_k})| |v_{m_k} - v_0| \} dx + \int_\Omega \{ \alpha k_1(x) |u_{m_k} - u_0| + \beta k_2(x) |v_{m_k} - v_0| \} dx$$

$$\leq \lambda_1 \left\{\alpha||u_{m_k}||_{L^p}^{\alpha-1}||v_{m_k}||_{L^p}^{\beta}||u_{m_k}-u_0||_{L^p}+\beta||u_{m_k}||_{L^p}^{\alpha}||v_{m_k}||_{L^p}^{\beta-1}||v_{m_k}-v_0||_{L^p}\right\}$$
$$+||\tau||_{L^{p'}}(\alpha||u_{m_k}-u_0||_{L^p}+\beta||v_{m_k}-v_0||_{L^p})$$
$$+\alpha||k_1||_{L^{p'}}||u_{m_k}-u_0||_{L^p}+\beta||k_2||_{L^{p'}}||u_{m_k}-u_0||_{L^p}. \tag{2.17}$$

Letting $k\to+\infty$ and remark that $||u_{m_k}-u_0||_{L^p}\to 0$, $||v_{m_k}-v_0||_{L^p}\to 0$. We obtain

$$\lim_{k\to+\infty} < T'(w_{m_k}),(w_{m_k}-w_0) >= 0.$$

Moreover,

$$\lim_{k\to+\infty}(J'(w_{m_k}),(w_{m_k}-w_0)) = \lim_{k\to+\infty}\left\{(I'(w_{m_k}),(w_{m_k}-w_0))-(T'(w_{m_k}),(w_{m_k}-w_0))\right\}.$$

We have

$$\lim_{k\to+\infty}(J'(w_{m_k}),(w_{m_k}-w_0))=0$$

i.e

$$(J'(w_{m_k}),(w_{m_k}-w_0))=\alpha\int_\Omega|\nabla u_{m_k}|^{p-2}|\nabla u_{m_k}|\nabla(u_{m_k}-u_0)dx$$
$$+\beta\int_\Omega|\nabla v_{m_k}|^{p-2}|\nabla v_{m_k}|\nabla(v_{m_k}-v_0)dx\to 0 \qquad \text{as } k\to+\infty. \tag{2.18}$$

Since $w_{m_k}\rightharpoonup w_0$ in E and $J'(w_0)\in E^*$,$(J'(w_0),(w_m-w_0))\to 0$ as $k\to+\infty$.

That is

$$(J'(w_0),(w_{m_k}-w_0))=\alpha\int_\Omega|\nabla u_0|^{p-2}|\nabla u_0|\nabla(u_{m_k}-u_0)dx$$
$$+\beta\int_\Omega|\nabla v_0|^{p-2}|\nabla v_0|\nabla(v_{m_k}-v_0)dx\to 0, \qquad \text{as } k\to+\infty. \tag{2.19}$$

Using the well-know inequality:

$$(|s|^{r-2}s-|\bar{s}|^{r-2})(s-\bar{s})\geq c_r|s-\bar{s}|^r,$$

for $s,\bar{s}\in R^N$, $r\geq 2$, we deduce that

$$< J'(w_{m_k})-J'(w_0),(w_{m_k}-w_0) >$$
$$=\alpha\int_\Omega(|\nabla u_{m_k}|^{p-2}\nabla u_{m_k}-|\nabla u_0|^{p-2}\nabla u_0)\nabla(u_{m_k}-u_0)dx$$
$$+\beta\int_\Omega(|\nabla v_{m_k}|^{p-2}\nabla v_{m_k}-|\nabla v_0|^{p-2}\nabla v_0)\nabla(v_{m_k}-v_0)dx$$
$$\geq c_1||u_{m_k}-u_0||_{W_0^{1,p}}+c_2||v_{m_k}-v_0||_{W_0^{1,p}}.$$

From (2.18), (2.19) it follows that the left-hand side of this inequality converges to zero as $k\to+\infty$. Then we arrive at $u_{m_k}\to u_0$, $v_{m_k}\to v_0$ as $k\to+\infty$ in $W_0^{1,p}(\Omega)$.

Hence, we deduce that $\{w_{m_k}\}$ converges strongly to w_0 in E.

Therefore, the functional I satisfies the Palais$-$Smale condition in E.

The proof of the Proposition 2.1 is complete.

Splitting E as the direct sum of X, Y: $E = X \oplus Y$ where

$$X = L(\varphi) = \{t\varphi = t(\varphi_1, \varphi_2), \quad t \in R\}$$
$$Y = \left\{ w = (u, v) \in E : \int_\Omega (u\varphi_1^{\alpha-1}\varphi_2^{\beta} + v\varphi_1^{\alpha}\varphi_2^{\beta-1})dx = 0 \right\},$$

where $\varphi = (\varphi_1, \varphi_2)$ is a nomarlized eigenpair associated with the eigenvalue λ_1 of the problem (1.3)

$$||(\varphi_1, \varphi_2)|| = \left(\int_\Omega |\nabla\varphi_1|^p dx + \int_\Omega |\nabla\varphi_2|^p dx \right)^{\frac{1}{p}} = 1.$$

Since $w = (u, v) \in E$, $w = t(\varphi_1, \varphi_2) + w_0$, $w_0 = (u_0, v_0) \in Y$.

$$u = t\varphi_1 + u_0 \tag{2.20}$$
$$v = t\varphi_2 + v_0 \tag{2.21}$$

Multiplying the equations in (2.20), (2.21) by $\varphi_1^{\alpha-1}\varphi_2^{\beta}\lambda_1$ and $\varphi_1^{\alpha}\varphi_2^{\beta-1}\lambda_1$ respectively, we have

$$\lambda_1 u\varphi_1^{\alpha-1}\varphi_2^{\beta} = \lambda_1 t\varphi_1^{\alpha}\varphi_2^{\beta} + \lambda_1 u_0\varphi_1^{\alpha-1}\varphi_2^{\beta}. \tag{2.22}$$
$$\lambda_1 v\varphi_1^{\alpha}\varphi_2^{\beta-1} = \lambda_1 t\varphi_1^{\alpha}\varphi_2^{\beta} + \lambda_1 v_0\varphi_1^{\alpha}\varphi_2^{\beta-1}. \tag{2.23}$$

We remark that

$$-\Delta_p\varphi_1 = -\text{div}(|\nabla\varphi_1|^{p-2}\nabla\varphi_1) = \lambda_1\varphi_1^{\alpha-1}\varphi_2^{\beta}.$$

From (2.22) we have $\lambda_1 u\varphi_1^{\alpha-1}\varphi_2^{\beta} = t(-\text{div}(|\nabla\varphi_1|^{p-2}\nabla\varphi_1))\varphi_1 + \lambda_1 u_0\varphi_1^{\alpha-1}\varphi_2^{\beta}$.

By integrating both sides of (2.22), we obtain that

$$\begin{aligned}\lambda_1 \int_\Omega u\varphi_1^{\alpha-1}\varphi_2^{\beta} dx &= t\int_\Omega \left(-\text{div}(|\nabla\varphi_1|^{p-2}\nabla\varphi_1)\right)\varphi_1 dx + \lambda_1 \int_\Omega u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} dx \\ &= t\int_\Omega |\nabla\varphi_1|^p dx + \lambda_1 \int_\Omega u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} dx.\end{aligned} \tag{2.24}$$

Similary, from (2.23) we also have

$$\lambda_1 \int_\Omega v\varphi_1^{\alpha}\varphi_2^{\beta-1} dx = t\int_\Omega |\nabla\varphi_2|^p dx + \lambda_1 \int_\Omega v_0\varphi_1^{\alpha}\varphi_2^{\beta-1} dx. \tag{2.25}$$

Hence combining (2.24) and (2.25) we obtain

$$\begin{aligned}\lambda_1 \int_\Omega \left(u\varphi_1^{\alpha-1}\varphi_2^{\beta} + v\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx &= t\int_\Omega |\nabla\varphi_1|^p dx + \lambda_1 \int_\Omega u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} dx \\ &\quad + t\int_\Omega |\nabla\varphi_2|^p dx + \lambda_1 \int_\Omega v_0\varphi_1^{\alpha}\varphi_2^{\beta-1} dx.\end{aligned}$$

Since $(u_0, v_0) \in Y$, we have

$$\int_\Omega \left(u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} + v_0\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx = 0.$$

Thus, for any $w \in E$ such that $w = t\varphi + w_0$, $w_0 \in Y$ we get

$$t = \frac{\lambda_1 \int_\Omega \left(u\varphi_1^{\alpha-1}\varphi_2^{\beta} + v\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx}{\int_\Omega |\nabla\varphi_1|^p dx + \int_\Omega |\nabla\varphi_2|^p dx} = \lambda_1 \int_\Omega \left(u\varphi_1^{\alpha-1}\varphi_2^{\beta} + v\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx. \quad (2.26)$$

Moreover, if $w = t\varphi + \tilde{w}$ where t is defined in (2.26) then $\tilde{w} \in Y$.

Therefore, $E = X \oplus Y$.

Lemma 2.1 *Exists $\bar{\lambda} > \lambda_1$ such that*

$$\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx \geq \bar{\lambda}\int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uv dx, \forall w = (u, v) \in Y.$$

Proof Let $\lambda = \inf\{\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx : (u, v) \in Y, \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uv dx = 1\}$.

We shall prove that this value is attained in Y.

Let $w_m = (u_m, v_m) \in Y$ be a minimizing sequence i.e

$$\int_\Omega |u_m|^{\alpha-1}|v_m|^{\beta-1}u_m v_m dx = 1, \quad \text{for } m = 1, 2, ...$$

and

$$\lim_{m\to+\infty} \frac{\alpha}{p}\int_\Omega |\nabla u_m|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v_m|^p dx = \lambda.$$

This implies that $\{w_m\}$ is bounded in E. Hence there exists a subsequence $\{w_{m_k}\}$ of $\{w_m\}$ which weakly converges to $w_0 = (u_0, v_0) \in E$ and the compactness of the embedding $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ implies that the subsequences $\{u_{m_k}\}$ and $\{v_{m_k}\}$ converge strongly to u_0 and v_0 respectively in $L^p(\Omega)$.

Observe further that with $\alpha + \beta = p$

$$\int_\Omega \left((u_{m_k} - u_0)\varphi_1^{\alpha-1}\varphi_2^{\beta} + (v_{m_k} - v_0)\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx$$
$$\leq ||u_{m_k} - u_0||_{L^p}||\varphi_1||_{L^p}^{\alpha-1}|\varphi_2||_{L^p}^{\beta} + ||v_{m_k} - v_0||_{L^p}||\varphi_1||_{L^p}^{\alpha}|\varphi_2||_{L^p}^{\beta-1}.$$

Since $||u_{m_k} - u_0||_{L^p(\Omega)} \to 0$, $||v_{m_k} - v_0||_{L^p(\Omega)} \to 0$ as $k \to +\infty$, we deduce that

$$\lim_{k\to+\infty}\int_\Omega \left(u_{m_k}\varphi_1^{\alpha-1}\varphi_2^{\beta} + v_{m_k}\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx = \int_\Omega \left(u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} + v_0\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx.$$

From this it follows that

$$\int_\Omega \left(u_0\varphi_1^{\alpha-1}\varphi_2^{\beta} + v_0\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx = 0,$$

hence $(u_0, v_0) \in Y$.

On the other hand, by the continuity of the operator A

$$\lim_{k\to+\infty}\int_\Omega |u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}v_{m_k} dx = \int_\Omega |u_0|^{\alpha-1}|v_0|^{\beta-1}u_0 v_0 dx.$$

This implies

$$\int_\Omega |u_0|^{\alpha-1}|v_0|^{\beta-1}u_0 v_0 dx = 1.$$

So $u_0 \neq 0$ and $v_0 \neq 0$.

Moreover, since the functional J given by (2.2) is lower weakly semicontinuous, we obtain

$$\lambda \leq J(u_0, v_0) = \frac{\alpha}{p}\int_\Omega |\nabla u_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v_{m_k}|^p dx$$
$$\leq \lim_{m\to+\infty} inf \left\{\frac{\alpha}{p}\int_\Omega |\nabla u_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v_{m_k}|^p dx\right\} = \lambda,$$

hence

$$\lambda = J(u_0, v_0) = \frac{\alpha}{p}\int_\Omega |\nabla u_0|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v_0|^p dx.$$

It means that λ is attained at w_0.

Our goal is to show that $\lambda > \lambda_1$.

By the variational characterization of λ_1, it is clear that: $\lambda \geq \lambda_1$.

If $\lambda = \lambda_1$, by simplicity of λ_1 there exists $t \in R$ such that $w_0 = (u_0, v_0) = t(\varphi_1, \varphi_2)$. Since $w_0 = (u_0, v_0) \in Y$

$$0 = \int_\Omega \left(t\varphi_1\varphi_1^{\alpha-1}\varphi_2^{\beta} + t\varphi_2\varphi_1^{\alpha}\varphi_2^{\beta-1}\right) dx = t\int_\Omega \varphi_1^{\alpha}\varphi_2^{\beta} dx.$$

This contradicts the fact that

$$1 = \int_\Omega |u_0|^{\alpha-1}|v_0|^{\beta-1}u_0v_0 dx = t\int_\Omega \varphi_1^{\alpha}\varphi_2^{\beta} dx.$$

Thus, there exists $\bar{\lambda}$ such that: $\bar{\lambda} > \lambda_1$ and the proof of proposition is complete. □

Proposition 2.2 *The functional I given by* (2.1) *is coercive on Y provided hypotheses* (H_1) *and* (H_2) *hold.*

Proof Observe that by Holder inequality, Lemma 2.1, hypotheses (H_1), (H_2), we have

$$|I(w)| = |\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx - \lambda_1\int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uv dx$$
$$-\int_\Omega H(x, u, v)dx + \int_\Omega (\alpha k_1 u + \beta k_2 v)dx|$$
$$\geq |\min\left(\frac{\alpha}{p}; \frac{\beta}{p}\right)||w||_E^p - \frac{\lambda_1}{\bar{\lambda}}\left(\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx\right)$$
$$-\int_\Omega \tau(x)(|u| + |v|)dx - \alpha||k_1||_{L^{p'}}||u||_{L^p} - \beta||k_2||_{L^{p'}}||v||_{L^p}|$$
$$\geq |\left(1 - \frac{\lambda_1}{\bar{\lambda}}\right)\min\left(\frac{\alpha}{p}; \frac{\beta}{p}\right)||w||_E^p - (||\tau||_{L^{p'}}$$
$$+\alpha||k_1||_{L^{p'}})||u||_{L^p} - (||\tau||_{L^{p'}} + \beta||k_2||_{L^{p'}})||v||_{L^p}|$$
$$\geq |\left(1-\frac{\lambda_1}{\bar{\lambda}}\right)\min\left(\frac{\alpha}{p}; \frac{\beta}{p}\right)||w||_E^p - \max\left\{(||\tau||_{L^{p'}} + \alpha||k_1||_{L^{p'}}), (||\tau||_{L^{p'}} + \beta||k_2||_{L^{p'}})\right\}.$$
$$.c(||u||_{W_0^{1,p}} + ||v||_{W_0^{1,p}})|.$$

Since $||w_E|| \to +\infty$ and $\left(1 - \frac{\lambda_1}{\bar{\lambda}}\right) > 0$, $p \geq 2$, we obtain $I(w) \to +\infty$.

Thus the functional I given by (2.1) is coercive on Y and Proposition 2.2 is proved. □

From Proposition 2.1 the functional I is coercive on Y, so that

$$B_Y = \min_{w \in Y} I(w) > -\infty.$$

On the other hand, for every $t \in R$ we have

$$\frac{\alpha}{p}\int_\Omega |\nabla(t\varphi_1)|^p dx + \frac{\beta}{p}\int_\Omega |\nabla(t\varphi_2)|^p dx - \lambda_1 \int_\Omega |t\varphi_1|^{\alpha-1}|t\varphi_2|^{\beta-1}(t\varphi_1)(t\varphi_2)dx = 0$$

as follows from the definition of λ_1 and φ. Thus,

$$\begin{aligned} I(t\varphi) &= t\int_\Omega (\alpha k_1\varphi_1 + \beta k_2\varphi_2)dx - \int_\Omega H(x,t\varphi)dx \\ &= t\int_\Omega \left((\alpha k_1\varphi_1 + \beta k_2\varphi_2) - \frac{H(x,t\varphi)}{t}\right)dx. \end{aligned}$$

Remark that

$$\begin{aligned} \frac{H(x,t\varphi)}{t} &= \frac{1}{t}\left\{\frac{\alpha}{2}\int_0^{t\varphi_1}(f(x,s,t\varphi_2) + f(x,s,0))ds \right. \\ &\qquad \left. + \frac{\beta}{2}\int_0^{t\varphi_2}(g(x,t\varphi_1,\tau) + g(x,0,\tau))d\tau\right\} \\ &= \frac{1}{t}\left\{\frac{\alpha}{2}\int_0^{t}((f(x,y\varphi_1,t\varphi_2) + f(x,y\varphi_1,0))dy)\varphi_1 \right. \\ &\qquad \left. + \frac{\beta}{2}\int_0^{t}((g(x,t\varphi_1,y\varphi_2) + g(x,0,y\varphi_2))dy)\varphi_2\right\}. \end{aligned}$$

Hence,

$$\lim_{t\to+\infty}\frac{H(x,t\varphi)}{t} = \frac{1}{2}(\alpha F_1(x)\varphi_1 + \beta G_1(x)\varphi_2).$$

Therefore,

$$\begin{aligned} &\lim_{t\to+\infty} t\int_\Omega \left((\alpha k_1\varphi_1 + \beta k_2\varphi_2) - \frac{H(x,t\varphi)}{t}\right)dx \\ &= \lim_{t\to+\infty} t\int_\Omega \left\{(\alpha k_1\varphi_1 + \beta k_2\varphi_2) - \frac{1}{2}(\alpha F_1(x)\varphi_1 + \beta G_1(x)\varphi_2)\right\}dx. \end{aligned}$$

On the other hand, from $(H_2(i))$ we obtain

$$\frac{1}{p}\int_\Omega (\alpha f^{+\infty}\varphi_1 + \beta g^{+\infty}\varphi_2)dx < \frac{1}{p}\int_\Omega (\alpha k_1\varphi_1 + \beta k_2\varphi_2)\,dx.$$

It follows from $H_2(ii)$ that

$$\begin{aligned} &\int_\Omega \left\{\frac{1}{2}(\alpha F_1(x)\varphi_1 + \beta G_1(x)\varphi_2) - \frac{\alpha}{p}f^{+\infty}(x)\varphi_1 - \frac{\beta}{p}g^{+\infty}(x)\varphi_2\right\}dx \\ &> \left(1 - \frac{1}{p}\right)\int_\Omega (\alpha k_1\varphi_1 + \beta k_2\varphi_2)dx. \end{aligned}$$

Thus,

$$\int_\Omega \left\{\frac{1}{2}(\alpha F_1(x)\varphi_1 + \beta G_1(x)\varphi_2) - (\alpha k_1\varphi_1 + \beta k_2\varphi_2)\right\}dx > 0.$$

This shows that

$$\lim_{t\to+\infty} I(t\varphi) = -\infty.$$

Next, with $t < 0$ we also have

$$\begin{aligned}\frac{H(x,t\varphi)}{t} &= \frac{1}{t}\left\{\frac{\alpha}{2}\int_0^{t\varphi_1}(f(x,s,t\varphi_2)+f(x,s,0))ds\right.\\ &\quad\left.+\frac{\beta}{2}\int_0^{t\varphi_2}(g(x,t\varphi_1,\tau)+g(x,0,\tau))d\tau\right\}\\ &= -\frac{1}{|t|}\left\{\frac{\alpha}{2}\int_0^{-|t|\varphi_1}(f(x,s,-|t|\varphi_2)+f(x,s,0))ds\right.\\ &\quad\left.+\frac{\beta}{2}\int_0^{-|t|\varphi_2}(g(x,-|t|\varphi_1,\tau)+g(x,0,\tau))d\tau\right\}.\end{aligned}$$

Set $s = -y\varphi_1 \to ds = -\varphi_1 dy$ and $s = -|t|\varphi_1 = -y\varphi_1 \Rightarrow y = |t|$

$$\begin{aligned}\frac{H(x,t\varphi)}{t} = -\frac{1}{|t|}&\left\{\frac{\alpha}{2}\int_0^{-|t|}((f(x,-y\varphi_1,-|t|\varphi_2)+f(x,-y\varphi_1,0))dy)(-\varphi_1)\right.\\ &\left.+\frac{\beta}{2}\int_0^{-|t|}((g(x,-|t|\varphi_1,-y\varphi_2)+g(x,0,-y\varphi_2))dy)(-\varphi_2)\right\}.\end{aligned}$$

Now, letting $t \to -\infty$, we get

$$\lim_{t\to-\infty}\frac{H(x,t\varphi)}{t} = \frac{1}{2}\int_\Omega(\alpha F_2(x)\varphi_1 + \beta G_2(x)\varphi_2)dx.$$

We deduce that

$$\lim_{t\to-\infty} I(t\varphi) = \lim_{t\to-\infty} t\int_\Omega\left\{(\alpha k_1\varphi_1+\beta k_2\varphi_2) - \frac{1}{2}(\alpha F_2(x)\varphi_1+\beta G_2(x)\varphi_2)\right\}dx.$$

Similarly above from $(H_2(ii))$ we obtain

$$\frac{1}{2}\int_\Omega(\alpha F_2(x)\varphi_1+\beta G_2(x)\varphi_2)dx < \int_\Omega(\alpha k_1\varphi_1+\beta k_2\varphi_2)dx.$$

This implies that

$$\lim_{t\to-\infty} I(t\varphi) = -\infty.$$

Thus, there exists t_0 such that $|t_0|$ large enough, we have $I(t_0\varphi) < 0$.

Set $w_0(x) = (t_0\varphi_1, t_0\varphi_2)$ we get

$$I(w_0) = I(t_0\varphi) < B_Y \leq I(t\varphi).$$

Proof of theorem 1.1 By Propositions 2.1 and 2.2, applying the Saddle Point Theorem (P.H.Rabinowitz) (see Theorem 2.1), we deduce that the functional I attains its proper infimum at some $w_0 = (u_0, v_0) \in E$, so that the problem (1.1) has at least a weak solution $w_0 \in E$. Moreover w_0 is nontrivial weak solution of the Problem (1.1). The Theorem 1.1 is completely proved. □

Remark 2.3 We will get the same result as above if the hypotheses (H_2) is replaced by reverse inequalities as follows.

We assume that

$(H_2)^*$

$$\int_\Omega \left\{ \frac{1}{2}(\alpha F_2(x)\varphi_1(x) + \beta G_2(x)\varphi_2(x)) - \frac{\alpha}{p} f^{-\infty}(x)\varphi_1(x) - \frac{\beta}{p} g^{-\infty}(x)\varphi_2(x) \right\} dx$$
$$> \left(1 - \frac{1}{p}\right) \int_\Omega (\alpha k_1(x)\varphi_1(x) + \beta k_2(x)\varphi_2(x)) dx >$$
$$> \int_\Omega \left\{ \frac{1}{2}(\alpha F_1(x)\varphi_1(x) + \beta G_1(x)\varphi_2(x)) - \frac{\alpha}{p} f^{+\infty}(x)\varphi_1(x) - \frac{\beta}{p} g^{+\infty}(x)\varphi_2(x) \right\} dx. \tag{2.27}$$

This means that, if the conditions (H_1), $(H_2)^*$ holds, then the problem (1.1) has at least a nontrivial weak solution in E. This assertion is proved by using variational techniques, the Minimum Principle and generalization of the Landesman–Lazer type condition.

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