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PULLBACK ATTRACTORS FOR 2D g-NAVIER-STOKES EQUATIONS WITH INFINITE DELAYS

Dao Trong Quyet

Abstract. We consider the first initial boundary value problem for the 2D non-autonomous g-Navier-Stokes equations with infinite delays. We prove the existence of a pullback $\mathcal{D}$-attractor for the continuous process associated to the problem with respect to a large class of non-autonomous forcing terms.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary Γ. In this paper we study the existence of a pullback attractor for the 2D non-autonomous g-Navier-Stokes equations with infinite delays:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p & = f(t) + F(t,u_t) \text{ in } (\tau,T)\times\Omega,\\ \nabla\cdot(gu) & = 0 \ \text{ in } (\tau,T)\times\Omega,\\ u & = 0 \ \text{ on } (\tau,T)\times\Gamma,\\ u(\tau+s,x) & = \phi(s,x),\ s\in(-\infty,0],\ x\in\Omega, \end{cases}$$

where $u = u(x,t) = (u_1,u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient.

The 2D g-Navier-Stokes equations is a generalization of the 2D Navier-Stokes equations, and arise in a natural way when we study the 3D Navier-Stokes equations in thin domains (see [17]). As mentioned in [12, 17], good properties of the 2D g-Navier-Stokes equations may be useful for the study of the Navier-Stokes equations on the thin three dimensional domain $\Omega_g = \Omega\times(0,g)$. In the last few years, the existence and long-time behavior of solutions (in terms of the existence and properties of attractors) to the 2D g-Navier-Stokes equations have been studied extensively (see e.g. [1, 2, 3, 8, 10, 11, 12, 14, 15, 17, 19]).

However, there are situations in which the model is better described if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system (in a certain sense) by applying

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a force which takes into account not only the present state, but the complete history of the solutions. It is noticed that the 2D Navier-Stokes equations with delays have been studied recently in [4, 5, 13].

In the previous paper [2], we considered the 2D g-Navier-Stokes equations with infinite delays and proved the existence of weak solutions and the exponential stability of the stationary solution under some restriction of the term containing delays. In this paper, we continue studying the long-time behavior of solutions to problem (1.1) in terms of existence of a pullback attractor for the process associated to the problem under a large class of non-autonomous forcing terms. To do this, we also use the theory of pullback attractors that has been developed recently and has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems (see e.g. the monograph [7]). The results obtained, in particular, recover and extend some existing ones for the 2D Navier-Stokes equations with infinite delays in [13].

It is known that there are some technical difficulties in dealing with partial differential equations with infinite delays due to the unboundedness of the delay involved. This introduces a major obstacle for proving the existence of pullback attractors. To overcome these difficulties, in this paper we try to exploit the techniques used in [13] in dealing with the infinite delays.

Let X be a Banach space. Given a function $u : (-\infty, T) \to X$, for each $t < T$ we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t+s)$, $s \in (-\infty, 0]$.

One possibility to deal with infinite delays, and which we will use here, is to consider, for any $\gamma > 0$, the space

$$C_\gamma(H_g) = \{\varphi \in C((-\infty, 0]; H_g) : \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in H_g\},$$

which is a Banach space with the norm

$$||\varphi||_\gamma := \sup_{s\in(-\infty,0]} e^{\gamma s}|\varphi(s)|.$$

Here the space H_g is defined in Section 2 below and $|\cdot|$ denotes the norm in H_g.

In order to study problem (1.1), we make the following assumptions:

$(H1)$ $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0\lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A is defined in Section 2 below).

$(H2)$ $F(t, u_t) : (\tau, T) \times C_\gamma(H_g) \to L^2(\Omega, g)$ such that:
(i) $\forall \xi \in C_\gamma(H_g)$, the mapping $(\tau, T)\ni t \mapsto F(t, \xi)$ is measurable;
(ii) $F(t, 0) = 0$ for all $t \in (\tau, T)$;
(iii) there exists a positive constant $L_F < \nu\lambda_1\gamma_0/2$, where $\gamma_0 = 1 -$

$\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$, such that for all $t \in (\tau, T)$ and $\xi, \eta \in C_\gamma(H_g)$,

$$|F(t,\xi) - F(t,\eta)| \le L_F||\xi - \eta||_\gamma.$$

$(H3)$ $f \in L^2(\tau, T; V_g')$, where the space V_g' is defined in Section 2, satisfies

$$\int_{-\infty}^{0} e^{(\nu\lambda_1\gamma_0 - 2L_F)s}||f(s)||_*^2 ds < +\infty.$$

We now give an example of the delay term $F(t, u_t)$. Let $F : (\tau, T^*) \times C_\gamma(H_g) \to L^2(\Omega, g)$ be defined as follows

$$F(t,\xi) = \int_{-\infty}^{0} G(t, s, \xi(s))ds, \ \forall t \in (\tau, T^*), \ \xi \in C_\gamma(H_g),$$

where the function $G : (\tau, T^*) \times (-\infty, 0) \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following assumptions:

(1) $G(t, s, 0) = 0$ for all $(t, s) \in (\tau, T^*) \times (-\infty, 0)$;

(2) There exists a function $\kappa : (-\infty, 0) \to (0, \infty)$ such that

$$\|G(t,s,u) - G(t,s,v)\|_{\mathbb{R}^2} \le \kappa(s)\|u - v\|_{\mathbb{R}^2} \quad \forall u, v \in \mathbb{R}^2, \forall (t,s) \in (\tau, T^*) \times (-\infty, 0),$$

and the function κ satisfies that $\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot} \in L^2(-\infty, 0)$ for some ε.

Then one can check that (see [2] for details) the function F satisfies $(H2)$.

The rest of the paper is organized as follows. In the next section, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the g-Navier-Stokes equations and abstract results on the existence of pullback attractors. In Section 3, we prove the continuity of the process $U(t, \tau)$ associated to the problem, and then the existence of a pullback attractor in $C_\gamma(H_g)$ for the process $U(t, \tau)$ by showing this process has a pullback absorbing set and is pullback asymptotically compact.

2. Preliminary results

2.1. Function spaces and inequalities for the nonlinear terms

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \ u, v \in L^2(\Omega, g),$$

and

$$((u, v))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g dx, \ u = (u_1, u_2), \ v = (v_1, v_2) \in H_0^1(\Omega, g),$$

and norms $|u|^2 = (u, u)_g$, $||u||^2 = ((u, u))_g$. Thanks to assumption $(H1)$, the norms $|\cdot|$ and $||\cdot||$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega, g)$. They are Hilbert spaces. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $||\cdot||_*$ for the norm in V_g', and $\langle \cdot, \cdot \rangle$ for duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u, v, w) = -b(u, w, v).$$

Hence

$$b(u, v, v) = 0, \ \forall u, v \in V_g.$$

Set $A : V_g \to V_g'$ by $\langle Au, v \rangle = ((u, v))_g$, $B : V_g \times V_g \to V_g'$ by $\langle B(u, v), w \rangle = b(u, v, w)$. Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $Au = -P_g \Delta u, \forall u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g. For $u \in D(A)$, we have

$$\langle A^{1/2}u, A^{1/2}u \rangle = \langle Au, u \rangle = \langle P_g[-\frac{1}{g}(\nabla \cdot g\nabla)u], u \rangle = \int_\Omega (\nabla u \cdot \nabla u) g dx,$$

which implies that

$$|A^{1/2}u|^2 = |\nabla u|^2 = ||u||^2 \quad \text{for all } \ u \in V_g.$$

We have the following results.

Lemma 2.1 ([1]). *If $n = 2$, then*

$$|b(u, v, w)| \le \begin{cases} c_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, & \forall u, v, w \in V_g, \\ c_2|u|^{1/2}\|u\|^{1/2}\|v\||Aw|^{1/2}|w|^{1/2}, & \forall u \in V_g, v \in D(A), w \in H_g, \\ c_3|u|^{1/2}|Au|^{1/2}\|v\||w|, & \forall u \in D(A), v \in V_g, w \in H_g, \\ c_4|u|\|v\||w|^{1/2}|w|^{1/2}|Aw|^{1/2}, & \forall u \in H_g, v \in V_g, w \in D(A), \end{cases}$$

where $c_i, i = 1, \ldots, 4$, are appropriate constants.

Lemma 2.2 ([3]). *Let $u \in L^2(\tau, T; V_g)$. Then the function Bu defined by*

$$(Bu(t), v)_g = b(u(t), u(t), v), \ \forall v \in V_g, \ \text{a.e. } t \in [\tau, T],$$

belongs to $L^2(\tau, T; V_g')$.

Lemma 2.3 ([3]). *Let $u \in L^2(\tau, T; V_g)$. Then the function Cu defined by*

$$(Cu(t), v)_g = ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = b(\frac{\nabla g}{g}, u, v), \ \forall v \in V_g,$$

belongs to $L^2(\tau, T; H_g)$, *and hence also belongs to* $L^2(\tau, T; V_g')$. *Moreover,*

$$|Cu(t)| \leq \frac{|\nabla g|_\infty}{m_0}.\|u(t)\| \quad \textit{for a.e. } t \in (\tau, T),$$

and

$$||Cu(t)||_* \leq \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}.||u(t)|| \quad \textit{for a.e. } t \in (\tau, T).$$

Since

$$-\frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - (\frac{\nabla g}{g} \cdot \nabla)u,$$

we have

$$(-\Delta u, v)_g = ((u, v))_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = (Au, v)_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g, \forall u, v \in V_g.$$

2.2. Pullback attractors

Let (X, d) be a metric space. For $A, B \subset X$, we define the Hausdorff semi-distance between A and B by

$$dist(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

A process on X is a family of two-parameter mappings $\{U(t, \tau)\}$ in X having the following properties:

$$U(t, r)U(r, \tau) = U(t, \tau) \quad \text{for all } t \geq r \geq \tau,$$
$$U(\tau, \tau) = Id \quad \text{for all } \tau \in \mathbb{R}.$$

Suppose that $\mathcal{B}(X)$ is the family of all nonempty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 2.1. The process $\{U(t, \tau)\}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

Definition 2.2. The family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process $U(t, \tau)$ if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \leq t$ such that

$$\bigcup_{\tau \leq \tau_0} U(t, \tau)D(\tau) \subset B(t).$$

Definition 2.3. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $\{U(t, \tau)\}$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;

(2) $\hat{\mathcal{A}}$ is invariant, i.e.,

$$U(t, \tau)A(\tau) = A(t) \quad \text{for all } t \geq \tau;$$

(3) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty} \text{dist}(U(t,\tau)D(\tau), A(t)) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D}, \text{ and all } t \in \mathbb{R};$$

(4) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.1 ([6]). *Let $\{U(t,\tau)\}$ be a continuous process such that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)B(\tau)}.$$

3. Existence of a pullback $\mathcal{D}$-attractor

Definition 3.1. A weak solution of problem (1.1) is a function $u \in C((-\infty, T]; H_g) \cap L^2(\tau, T; V_g)$ with $u_\tau = \phi$, and such that for all $v \in V_g$,

$$\begin{aligned} &\frac{d}{dt}(u(t),v)_g + \nu((u(t),v))_g + b(u(t),u(t),v) + \nu(Cu(t),v)_g \\ &= \langle f(t),v\rangle + (F(t,u_t),v)_g, \end{aligned} \tag{3.1}$$

in the sense of $\mathcal{D}'(\tau, T)$.

It is noticed that if u is a weak solution of (1.1), then u satisfies the following energy equality

$$\begin{aligned} &|u(t)|^2 + 2\nu\int_s^t \|u(r)\|^2 dr + 2\nu\int_s^t b(\frac{\nabla g}{g}, u(r), u(r))dr \\ = \ & |u(s)|^2 + 2\int_s^t \Big[\langle f(r),u(r)\rangle + (F(r,u_r),u(r))_g\Big]dr. \end{aligned}$$

The following existence theorem was proved in [2].

Theorem 3.1. *Assume that hypotheses $(H1)$-$(H3)$ hold. Suppose that $\phi \in C_\gamma(H_g)$ are given, and that $2\gamma > \nu\lambda_1\gamma_0$, where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$. Then there exists a unique weak solution u of problem* (1.1)*.*

We now prove the continuity of the weak solutions with respect to the initial data.

Proposition 3.2. *Under the assumptions of Theorem 3.1, the solutions of* (1.1) *are continuous with respect to the initial condition. Namely, denoting u^i, for $i = 1, 2$, the corresponding solution to initial datum $\phi^i \in C_\gamma(H_g)$, the following estimates hold*:

$$\begin{aligned} &\max_{r\in[\tau,t]} |u^1(r) - u^2(r)|^2 \\ &\le \big(|\phi^1(0)-\phi^2(0)|^2 + \frac{L_F}{2\gamma}||\phi^1 - \phi^2||_\gamma^2\big)\exp\{\int_\tau^t (3L_F + \frac{c^2}{\nu\gamma_0}||u^1(s)||^2)ds\}, \end{aligned} \tag{3.2}$$

$$||u_t^1 - u_t^2||_\gamma^2 \leq \big(1 + \frac{L_F}{2\gamma}\big)||\phi^1 - \phi^2||_\gamma^2 \exp\{\int_\tau^t (3L_F + \frac{c^2}{\nu\gamma_0}||u^1(s)||^2)ds\}, \tag{3.3}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} > 0$.

Proof. Consider the equations satisfied by u^i for $i = 1, 2$, acting on the element $u^1 - u^2$, and taking the difference. This gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u^1(t) - u^2(t)|^2 + \nu||u^1(t) - u^2(t)||^2 \\
&+ \nu b\big(\frac{\nabla g}{g}, u^1(t), u^1(t) - u^2(t)\big) - \nu b\big(\frac{\nabla g}{g}, u^2(t), u^1(t) - u^2(t)\big) \\
&+ b\big(u^1(t), u^1(t), u^1(t) - u^2(t)\big) - b\big(u^2(t), u^1(t), u^1(t) - u^2(t)\big) \\
= &\big(F(t, u_t^1 - F(t, u_t^2), u^1(t) - u^2(t)\big)_g.
\end{aligned}$$

Using Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u^1(t) - u^2(t)|^2 + \nu||u^1(t) - u^2(t)||^2 \\
\leq\ &c|u^1(t) - u^2(t)|||u^1(t) - u^2(t)||||u^1(t)|| \\
&+ \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u^1(t) - u^2(t)||^2 + L_F||u_t^1 - u_t^2||_\gamma |u^1(t) - u^2(t)|.
\end{aligned}$$

Using the Cauchy inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u^1(t) - u^2(t)|^2 + \nu\gamma_0||u^1(t) - u^2(t)||^2 \\
\leq\ &\frac{\nu\gamma_0}{2}||u^1(t) - u^2(t)||^2 + \frac{c^2}{2\nu\gamma_0}|u^1(t) - u^2(t)|^2||u^1(t)||^2 \\
&+ L_F||u_t^1 - u_t^2||_\gamma |u^1(t) - u^2(t)|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u^1(t) - u^2(t)|^2 \\
\leq\ &\frac{c^2}{2\nu\gamma_0}|u^1(t) - u^2(t)|^2||u^1(t)||^2 + L_F||u_t^1 - u_t^2||_\gamma |u^1(t) - u^2(t)|.
\end{aligned}$$

Since the Lipschitz assumption on F, and the fact that, for $s \in [\tau, t]$,

$$\begin{aligned}
||u_s^1 - u_s^2||_\gamma &= \sup_{\theta\leq 0} e^{\gamma\theta}|u^1(s+\theta) - u^2(s+\theta)| \\
&= \max\Big\{\sup_{\theta\in(-\infty,\tau-s)} e^{\gamma\theta}|\phi^1(s-\tau+\theta) - \phi^2(s-\tau+\theta)|; \\
&\qquad \sup_{\theta\in[\tau-s,0]} e^{\gamma\theta}|u^1(s+\theta) - u^2(s+\theta)|\Big\} \\
&\leq \max\{e^{\gamma(\tau-s)}||\phi^1 - \phi^2||_\gamma; \max_{\theta\in[\tau,s]}|u^1(\theta) - u^2(\theta)|\},
\end{aligned} \tag{3.4}$$

so we conclude that, for all $t \in [\tau, T]$,

$$\begin{aligned}
&\frac{1}{2}|u^1(t) - u^2(t)|^2 \\
\leq\ &\frac{1}{2}|\phi^1(0) - \phi^2(0)|^2 + L_F||\phi^1 - \phi^2||_\gamma \int_\tau^t e^{\gamma(\tau - s)}|u^1(s) - u^2(s)|ds \\
&+ L_F \int_\tau^t |u^1(s) - u^2(s)| \max_{\theta \in [\tau, s]} |u^1(\theta) - u^2(\theta)|ds \\
&+ \frac{c^2}{2\nu\gamma_0} \int_\tau^t |u^1(s) - u^2(s)|^2||u^1(s)||^2 ds.
\end{aligned}$$

If we now substitute t by $r \in [\tau, t]$ and consider the maximum when varying this r, from the above we can conclude that

$$\begin{aligned}
\max_{r \in [\tau, t]} |u^1(t) - u^2(t)|^2 \leq |\phi^1(0) - \phi^2(0)|^2 + \frac{L_F}{2\gamma}||\phi^1 - \phi^2||_\gamma^2 \\
+ \int_\tau^t (3L_F + \frac{c^2}{\nu\gamma_0}||u^1(s)||^2) \max_{r \in [\tau, s]} |u^1(s) - u^2(s)|^2 ds.
\end{aligned}$$

Hence, by the Gronwall lemma, we obtain (3.2). Finally, (3.3) follows from (3.2) and (3.4). □

By Theorem 3.1 and Proposition 3.2, one can define a continuous process $U(t, \tau) : C_\gamma(H_g) \to C_\gamma(H_g)$, with $\tau \leq t$, given by

$$U(t, \tau)\phi = u_t,$$

where u is the unique weak solution of (1.1) with the initial datum ϕ.

To prove the existence of a pullback attractor for the process $U(t, \tau)$, we first show that $U(t, \tau)$ has a family of $\widehat{D}$-pullback absorbing sets.

Proposition 3.3. *Under the assumptions* (*H*1)-(*H*3), *the family* $\widehat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\}$, *with* $D(t) = B_{C_\gamma(H_g)}(0, \rho(t))$, *where*

$$\rho^2(t) = 1 + \frac{2}{\nu\gamma_0} \int_{-\infty}^t e^{-(\nu\lambda_1\gamma_0 - 2L_F)(t-s)}||f(s)||_*^2 ds,$$

is a pullback $\mathcal{D}$*-absorbing set for the process* $U(t, \tau)$.

Proof. From (3.1) substituting v by $u(t)$, we obtain

$$\begin{aligned}
(3.5) \quad &\frac{d}{dt}(u(t), u(t))_g + \nu(Au(t), u(t))_g + \nu(Cu(t), u(t))_g + b(u(t), u(t), u(t)) \\
&= \langle f(t), u(t)\rangle + (F(t, u_t), u(t))_g.
\end{aligned}$$

Because $b(u(t), u(t), u(t)) = 0$ and $(Cu(t), u(t))_g = b(\frac{\nabla g}{g}, u(t), u(t))$, from (3.5) we have

$$\begin{aligned}
&\frac{d}{dt}(u(t), u(t))_g + \nu(Au(t), u(t))_g + \nu b(\frac{\nabla g}{g}, u(t), u(t)) \\
&= \langle f(t), u(t)\rangle + (F(t, u_t), u(t))_g,
\end{aligned}$$

and therefore,

$$\frac{d}{dt}|u(t)|^2 + 2\nu||u(t)||^2 = 2\langle f(t), u(t)\rangle + 2(F(t,u_t),u(t))_g - 2\nu b(\frac{\nabla g}{g}, u(t), u(t)). \tag{3.6}$$

Using Lemma 2.3, we get

$$\frac{d}{dt}|u(t)|^2 + 2\nu||u(t)||^2 \le 2\langle f(t), u(t)\rangle + 2(F(t,u_t),u(t))_g + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u(t)||^2,$$

and using the Cauchy inequality, we have

$$\frac{d}{dt}|u(t)|^2 + 2\nu\gamma_0||u(t)||^2 \le \frac{\nu\gamma_0}{2}||u(t)||^2 + \frac{2\|f(t)\|_*^2}{\nu\gamma_0} + 2L_F||u_t||_\gamma^2,$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$. Then, we have

$$\frac{d}{dt}|u(t)|^2 + \frac{3\nu\gamma_0}{2}||u(t)||^2 \le 2\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big). \tag{3.7}$$

Noting that $||u(t)||^2 \ge \lambda_1|u(t)|^2$, we also have

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1\gamma_0|u(t)|^2 + \frac{\nu\gamma_0}{2}||u(t)||^2 \le 2\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big).$$

Hence

$$\begin{aligned}|u(t)|^2 &+ \frac{\nu\gamma_0}{2}\int_\tau^t e^{-\nu\lambda_1\gamma_0(t-s)}||u(s)||^2 ds \\ &\le e^{-\nu\lambda_1\gamma_0(t-s)}|u(\tau)|^2 + 2\int_\tau^t e^{-\nu\lambda_1\gamma_0(t-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds.\end{aligned} \tag{3.8}$$

Furthermore,

$$\begin{aligned}||u_t||_\gamma^2 \le \max\Big\{&\sup_{\theta\in(-\infty,\tau-t]} e^{2\gamma\theta}|\phi(\theta+t-\tau)|^2; \sup_{\theta\in[\tau-t,0]}\big[e^{2\gamma\theta-\nu\lambda_1\gamma_0(t-\tau+\theta)}|u(\tau)|^2 \\ &+ 2e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1\gamma_0(t+\theta-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds\big]\Big\}.\end{aligned}$$

On one hand,

$$\sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}|\phi(\theta+t-\tau)| = \sup_{\theta\le 0} e^{\gamma(\theta-(t-\tau))}|\phi(\theta)| = e^{-\gamma(t-\tau)}||\phi||_\gamma.$$

On the other hand, as we are assuming that $2\gamma > \nu\lambda_1\gamma_0$,

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta-\nu\lambda_1\gamma_0(t-\tau+\theta)}|u(\tau)|^2 \le e^{-\nu\lambda_1\gamma_0(t-\tau)}|u(\tau)|^2,$$

and

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1\gamma_0(t+\theta-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds$$

$$\leq \int_{\tau}^{t+\theta} e^{-\nu\lambda_1\gamma_0(t-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds.$$

Collecting these inequalities we deduce that

$$||u_t||_\gamma^2 \leq e^{-\nu\lambda_1\gamma_0(t-\tau)}||\phi||_\gamma^2 + 2\int_{\tau}^{t} e^{-\nu\lambda_1\gamma_0(t-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds.$$

By the Gronwall lemma we have

$$(3.9)\quad ||u_t||_\gamma^2 \leq e^{-(\nu\lambda_1\gamma_0-2L_F)(t-\tau)}||\phi||_\gamma^2 + \frac{2}{\nu\gamma_0}\int_{\tau}^{t} e^{-(\nu\lambda_1\gamma_0-2L_F)(t-s)}\|f(s)\|_*^2 ds.$$

This completes the proof. □

Proposition 3.4. *Under the assumptions of Proposition 3.3, the process $U(t,\tau)$ is $\mathcal{D}$-asymptotically compact.*

Proof. Let $t_0 \in \mathbb{R}$, $u^n(\cdot)$ be a sequence of solutions in their respective intervals $[\tau_n, t_0]$, with initial data $\phi^n \in B_0(\tau_n) = B_{C_\gamma(H_g)}(0, \rho(\tau_n))$, where $\tau_n \to -\infty$ as $n \to +\infty$. Then we will prove that sequence $\xi^n = u_{t_0}^n$ is relatively compact in $C_\gamma(H_g)$.

Step 1. Denote $\sigma = \nu\lambda_1\gamma_0 - 2L_F$. Consider two arbitrary values $0 < \overline{T} < T$, we will prove that $\xi^n|_{[-\overline{T},0]}$ is relatively compact in $C([-\overline{T},0]; H_g)$.

From $(H2)$ and $(H3)$, there exits $n_0(t_0,T)$ such that $\tau_n \leq t_0 - T$ for $n \geq n_0(t_o,T)$, and with

$$R(t_0,T) := 1 + \frac{2}{\nu\gamma_0}e^{-\sigma(t_0-T)}\int_{-\infty}^{t_0} e^{\sigma s}||f(s)||_*^2 ds,$$

so we have

$$(3.10)\qquad ||u_t^n||_\gamma^2 \leq R(t_0,T)\ \ \forall\, t \in [t_0-T, t_0],\ \forall n \geq n_0(t_0,T).$$

Thus

$$(3.11)\qquad \begin{gathered} |u^n(t)|^2 \leq R(t_0,T)\ \ \forall\, t \in [t_0-T,t_0],\ \forall n \geq n_0(t_0,T), \\ ||u_{t_0-T}^n||_\gamma^2 \leq R(t_0,T)\ \forall n \geq n_0(t_0,T). \end{gathered}$$

Let $y^n(\cdot) = u^n(\cdot + t_0 - T)$, then for each $n \geq 1$ such that $\tau_n < t_0 - T$, the function $y^n(\cdot)$ is a solution on $[0,T]$ of a similar problem to (1.1), namely with f replaced by $\widetilde{f}(s) = f(t_0 - T + s)$ and F replaced by $\widetilde{F}(s,\cdot) = F(t_0 - T + s, \cdot)$, and with $y_0^n = u_{t_0-T}^n$, $y_T^n = u_{t_0}^n = \xi^n$. Then $||y_0^n||_\gamma$ satisfies the estimates in (3.11), for all $n \geq n_0(t_0,T)$.

On the other hand, from (3.8), we have:

$$\begin{aligned} &e^{-\nu\lambda_1\gamma_0(t-\tau)}\frac{2}{\nu\gamma_0}\int_{\tau}^{t}||u(s)||^2 ds \\ &\leq |u(\tau)|^2 + 2\int_{\tau}^{t} e^{-\nu\lambda_1\gamma_0(t-s)}\Big(\frac{\|f(t)\|_*^2}{\nu\gamma_0} + L_F||u_t||_\gamma^2\Big)ds. \end{aligned}$$

Combining this with (3.9) and applying the Fubini theorem, we have

$$\begin{aligned}\frac{2}{\nu\gamma_0}\int_\tau^t ||u(s)||^2 ds \leq e^{\nu\lambda_1\gamma_0(t-\tau)}|u(\tau)|^2 + e^{2L_F(t-\tau)}||\phi||_\gamma^2 \\ + \frac{2}{\nu\gamma_0}e^{-\nu\lambda_1\gamma_0\tau}\int_\tau^t e^{\nu\lambda_1\gamma_0 s}||f(s)||_*^2 ds \\ + \frac{2}{\nu\gamma_0}e^{2L_F t-\nu\lambda_1\gamma_0\tau}\int_\tau^t e^{(\nu\lambda_1\gamma_0-2L_F)s}||f(s)||_*^2 ds.\end{aligned} \tag{3.12}$$

Then, we have

$$||y^n||_{L^2(0,T;V_g)}^2 \leq K(t_0,T).$$

Hence, $\{y^n\}$ is bounded in $L^\infty(0,T;H_g)$ and $L^2(0,T;V_g)$, and $\{(y^n)'\}$ is bounded in $L^2(0,T;V_g')$. Thus, exists a subsequence (relabelled the same) such that

- $\{y^n\}$ converges weakly-star to y in $L^\infty(0,T;H_g)$,
- $\{y^n\}$ converges weakly to y in $L^2(0,T;V_g)$,
- $\{(y^n)'\}$ converges weakly to y' in $L^2(0,T;V_g')$,
- $\{y^n\}$ converges strongly to y in $L^2(0,T;H_g)$,
- $\{y^n(t)\}$ converges to $y(t)$ a.e. $t\in(0,T)$.

Moreover, the same argument in proof Theorem 3.1, we obtain $y^n(t_n) \rightharpoonup y(t_0)$ weakly in H_g if $t_n \to t_0 \in [0,T]$.

From $(H2)$(iii) and (3.10), we have

$$\int_0^t |\widetilde{F}(s,y_s^n)|^2 ds \leq Ct,$$

where $C>0$ independ on n and t, and also

$$\begin{aligned}&\widetilde{F}(\cdot,y_\cdot^n) \rightharpoonup \xi \ \text{ weakly in } \ L^2(0,T;L^2(\Omega,g)),\\ &\int_s^t |\widetilde{F}(r,y_r^n)|^2 dr \leq C(t-s),\\ &\int_s^t |\xi(s)|^2 ds \leq \liminf_{n\to+\infty}\int_s^t |\widetilde{F}(r,y_r^n)|^2 dr \leq C(t-s), \ \ \forall\, 0\leq s\leq t\leq T.\end{aligned}$$

Then, we can prove that $y(\cdot)$ is the unique weak solution to the problem

$$\begin{cases}\dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= \widetilde{f}(t) + \widetilde{\xi}(t)\\ \nabla\cdot(gu) &= 0\\ u &= 0\\ u(0,x) &= y(0,x), \ x\in\Omega.\end{cases} \tag{3.13}$$

By the energy inequality

$$\frac{1}{2}|z(t)|^2 + \nu(1-\frac{|\nabla g|_\infty}{m_0\lambda_1})\int_s^t ||z(r)||^2 dr$$

$$\leq \frac{1}{2}|z(s)|^2 + \int_s^t \langle \widetilde{f}(r), z(r)\rangle dr + C(t-s), \ \forall\, 0 \leq s \leq t \leq T,$$

where $z = y^n$ or $z = y$, the maps $J_n, J : [0,T] \to \mathbb{R}$ defined by

$$J(t) = \frac{1}{2}|y(t)|^2 - \int_0^t \langle \widetilde{f}(r), y(r)\rangle dr - Ct,$$

$$J_n(t) = \frac{1}{2}|y^n(t)|^2 - \int_0^t \langle \widetilde{f}(r), y^n(r)\rangle dr - Ct,$$

are non-increasing and continuous.

The same argument in proof Theorem 3.1, for a fixed $t_0 > 0$, using a sequence $\widetilde{t}_k$ with $\widetilde{t}_k \nearrow t_0$, we have the convergence of the norms. and with the weak convergence already proved, deduce that $y^n \to y$ in $C([\delta, T]; H_g)$, for any $\delta > 0$.

Now, with $\overline{T} < T$, we obtain that $\xi^n \to \psi$ in $C([-\overline{T}, 0]; H_g)$, where $\psi(s) = y(s+T)$, for $s \in [-\overline{T}, 0]$. Repeating the same procedure for $2\overline{T}, 3\overline{T}, \ldots,$, for a diagonal subsequence (relabelled the same) we can obtain a continuous function $\psi : (-\infty, 0] \to H_g$ and a subsequence such that $\xi^n \to \psi$ in $C([-\overline{T}, 0]; H_g)$ on every interval $[-\overline{T}, 0]$. Moreover, for a fixed $T > 0, u^n(s+t_0)$, with $s \in [-T, 0]$, satisfies the estimate (3.11) for any $n \geq n_0(t_0, T)$, it is clear that we also have

$$|\psi(s)|^2 \leq 1 + Me^{\sigma T}, \ \ \forall s \in [-T, 0], \ \ \forall\, T > 0, \tag{3.14}$$

where

$$M = \frac{2}{\nu\gamma_0} e^{-\sigma t_0} \int_{-\infty}^{t_0} e^{\sigma s} ||f(s)||_*^2 ds.$$

Step 2. We will prove that ξ^n converges to ψ in $C_\gamma(H_g)$. Indeed, we have to show that for every $\epsilon > 0$, there exists n_ϵ such that

$$\sup_{s \in (-\infty, 0]} |\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \ \ \forall\, n \geq n_\epsilon. \tag{3.15}$$

Fix $T_\epsilon > 0$ such that $\max\{e^{-2\gamma T_\epsilon}, Me^\sigma e^{(\sigma-2\gamma)T_\epsilon}\} \leq \epsilon/8$, and take $n_\epsilon \geq n_0(t_0, T_\epsilon)$ such that

$$|\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \ \ \forall\, s \in [-T_\epsilon, 0], \ \ \tau_n \leq t_0 - T_\epsilon, \ \ \forall\, n \geq n_\epsilon.$$

(This is possible since the convergence of ξ^n to ψ holds in compact intervals of time.) So, in order to prove (3.15) we only have to check that

$$\sup_{s \in (-\infty, -T_\epsilon]} |\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \epsilon \ \ \forall\, n \geq n_\epsilon.$$

By (3.14) and the choice of T_ϵ, it is not difficult to check that, for all $k \in \mathbb{N}$, and for all $s \in [-(T_\epsilon + k + 1), -(T_\epsilon + k)]$, it holds that

$$\begin{aligned} e^{2\gamma s}|\psi|^2 &\leq e^{-2\gamma(T_\epsilon + k)}(1 + Me^{\sigma(T_\epsilon + k + 1)}) \\ &= e^{-2\gamma T_\epsilon} e^{-2\gamma k} + Me^\sigma e^{(\sigma - 2\gamma)T_\epsilon} e^{k(\sigma - 2\gamma)} \\ &\leq \epsilon/8 + \epsilon/8 \\ &= \epsilon/4. \end{aligned}$$

Thus, we have

$$\sup_{s\in(-\infty,-T_\epsilon]} |\xi^n(s)|^2 e^{2\gamma s} \le \epsilon/4 \ \ \forall\, n \ge n_\epsilon.$$

We remember that ξ^n has two parts:

$$\xi^n(s) = \begin{cases} \phi^n(s+t_0-\tau_n) & \text{if } s \in (-\infty, \tau_n - t_0), \\ u^n(s+t_0), & \text{if } s \in [\tau_n - t_0, 0]. \end{cases}$$

Thus, the proof is finished if we prove that

$$\max\{\sup_{s\in(-\infty,\tau_n-t_0)} e^{2\gamma s}|\phi^n(s+t_0-\tau_n)|^2; \sup_{s\in[\tau_n-t_0,-T_\epsilon]} e^{2\gamma s}|u^n(s+t_0)|^2\} \le \epsilon/4.$$

The first term above can be estimated as follows:

$$\begin{aligned} &\sup_{s\le\tau_n-t_0} e^{2\gamma s}|\phi^n(s+t_0-\tau_n)|^2 \\ &= \sup_{s\le\tau_n-t_0} e^{2\gamma(s+t_0-\tau_n)} e^{2\gamma(\tau_n-t_0)}|\phi^n(s+t_0-\tau_n)|^2 \\ &= e^{2\gamma(\tau_n-t_0)}||\phi^n||_\gamma^2 \\ &\le e^{2\gamma(\tau_n-t_0)}\rho^2(\tau_n) \\ &\le e^{2\gamma(\tau_n-t_0)} + Me^{(2\gamma-\sigma)(\tau_n-t_0)} \\ &\le \epsilon/4. \end{aligned}$$

For the second term, we have

$$\begin{aligned} &\sup_{s\in[\tau_n-t_0,-T_\epsilon]} e^{2\gamma s}|u^n(s+t_0)|^2 \\ &= \sup_{s\in[\tau_n-t_0+T_\epsilon,0]} e^{2\gamma(\theta-T_\epsilon)}|u^n(t_0-T_\epsilon+\theta)|^2 \\ &\le e^{-2\gamma T_\epsilon}||u^n_{t_0-T_\epsilon}||_\gamma^2 \\ &\le e^{-2\gamma T_\epsilon} R(t_0,T_\epsilon) \\ &= e^{-2\gamma T_\epsilon} + Me^{(\sigma-2\gamma)T_\epsilon} \\ &\le \epsilon/4, \end{aligned}$$

where we have used (3.11) with $T = T_\epsilon$. □

Theorem 3.5. *Suppose* (*H*1)-(*H*3) *hold, and* $\nu\lambda_1\gamma_0 < 2\gamma$. *Then the process* $U(t,\tau)$ *defined in* $C_\gamma(H_g)$ *associated to problem* (1.1) *has a pullback* $\mathcal{D}$*-attractor* $\widehat{\mathcal{A}}_g = \{A_g(t) : t \in \mathbb{R}\}$.

Proof. The existence of the pullback attractor is a direct consequence of Propositions 3.3 and 3.4. □

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References

[1] C. T. Anh and D. T. Quyet, *Long-time behavior for 2D non-autonomous g-Navier-Stokes equations*, Ann. Poln. Math. **103** (2012), no. 3, 277–302.

[2] ______, *g-Navier-Stokes equations with infinite delays*, Vietnam J. Math. **40** (2012), no. 1, 57–78.

[3] H. Bae and J. Roh, *Existence of solutions of the g-Navier-Stokes equations*, Taiwanese J. Math. **8** (2004), no. 1, 85–102.

[4] T. Caraballo and J. Real, *Navier-Stokes equations with delays*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), no. 2014, 2441–2454.

[5] ______, *Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2040, 3181–3194.

[6] T. Caraballo, G. Łukaszewicz, and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal. **64** (2006), no. 3, 484–498.

[7] A. Carvalho, J. A. Langa, and J. C. Robinson, *Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems*, Springer, New York, 2013.

[8] J. Jiang and Y. Hou, *The global attractor of g-Navier-Stokes equations with linear dampness on* $\mathbb{R}^2$, Appl. Math. Comp. **215** (2009), no. 3, 1068–1076.

[9] ______, *Pullback attractor of 2D non-autonomous g-Navier-Stokes equations on some bounded domains*, Appl. Math. Mech. (English Ed.) **31** (2010), no. 6, 697–708.

[10] M. Kwak, H. Kwean, and J. Roh, *The dimension of attractor of the 2D g-Navier-Stokes equations*, J. Math. Anal. Appl. **315** (2006), no. 2, 436–461.

[11] H. Kwean, *The* H^1*-compact global attractor of two-dimensional g-Navier-Stokes equations*, Far East J. Dyn. Syst. **18** (2012), no. 1, 1–20.

[12] H. Kwean and J. Roh, *The global attractor of the 2D g-Navier-Stokes equations on some unbounded domains*, Commun. Korean Math. Soc. **20** (2005), no. 4, 731–749.

[13] P. Marín-Rubio, J. Real, and J. Valero, *Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case*, Nonlinear Anal. **74** (2011), no. 5, 2012–2030.

[14] D. T. Quyet, *Asymptotic behavior of strong solutions to 2D g-Navier-Stokes equations*, Comun. Korean Math. Soc. **29** (2014), no. 4, 505–518.

[15] ______, *Pullback attractor for strong solutions of 2D non-autonomous g-Navier-Stokes equations*, Acta Math. Vietnam. **40** (2015), no. 4, 637–651.

[16] J. Roh, *Convergence of the g-Navier-Stokes equations*, Taiwanese J. Math. **13** (2009), no. 1, 189–210.

[17] ______, *Dynamics of the g-Navier-Stokes equations*, J. Differential Equations **211** (2005), no. 2, 452–484.

[18] R. Temam, *Navier-Stokes Equations*: *Theory and Numerical Analysis*, 2nd edition, Amsterdam, North-Holland, 1979.

[19] D. Wu, *The finite-dimensional uniform attractors for the non-autonomous g-Navier-Stokes equations*, J. Appl. Math. **2009** (2009), Art. ID 150420, 17 pp.

Dao Trong Quyet
Faculty of Information Technology
Le Quy Don Technical University
100 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam
E-mail address: dtq100780@gmail.com