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POLYNOMIAL VECTOR VARIATIONAL INEQUALITIES UNDER POLYNOMIAL CONSTRAINTS AND APPLICATIONS*

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Abstract. By using a scalarization method and some properties of semi-algebraic sets, we prove that both the proper Pareto solution set and the weak Pareto solution set of a vector variational inequality, where the convex constraint set is given by polynomial functions and all the components of the basic operators are polynomial functions, have finitely many connected components, provided that the Mangasarian–Fromovitz constraint qualification is satisfied at every point of the constraint set. In addition, if the proper Pareto solution set is dense in the Pareto solution set, then the latter also has finitely many connected components. Consequences of the results for vector optimization problems are discussed in detail.

Key words. polynomial vector variational inequality, solution set, connectedness structure, scalarization, semi-algebraic set

AMS subject classifications. 90C31, 90C29, 49J40, 49J53, 49K40

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1. Introduction.

This paper is our new attempt to study the connectedness structure of the Pareto solution set and the weak Pareto solution set of a vector variational inequality. Recall that, introduced by Giannessi in [5], the concept of vector variational inequality (VVI) has received a lot of attention from researchers; see, e.g., [6], [7], [9], [13], and the references therein. One reason is that many questions concerning vector optimization problems can be solved via a unified theory of VVIs.

Since the Pareto solution set (resp., the weak Pareto solution set) of a VVI can be disconnected, the problem of finding a sharp upper bound for the number of its connected components is important. But, before speaking about any upper bound of this type, the primary question one certainly wants to solve is the following: *Does the Pareto solution set (resp., the weak Pareto solution set) have a finite number of connected components, or not?*

Recently, using a scalarization method and properties of semi-algebraic sets, we have shown [7, Theorem 3.1] that *both the Pareto solution set and the weak Pareto solution set of a VVI with polynomial criteria under linear constraints have finitely many connected components*. One of the key arguments in the proof of the theorem is that the normal cone to the polyhedral constraint set at a point from a given pseudoface doesn't depend on the choice of the point. Moreover, the normal cone is

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polyhedral convex.

It is of interest to know whether the techniques of proving the main results in [7] can also be applied to the case where the convex constraint set is given by finitely many polyhedral functions. This paper is aimed at solving the problem.

By modifying the proof scheme of [7, Theorem 3.1] in a suitable way, we will prove that both the proper Pareto solution set and the weak Pareto solution set of a VVI under polynomial constraints, where all the components of the basic operators are polynomial functions, have finitely many connected components, provided that the Mangasarian–Fromovitz constraint qualification [11] is satisfied at every point of the convex constraint set. In addition, if the proper Pareto solution set is dense in the Pareto solution set, then the latter also has finitely many connected components.

Applying the above result to vector optimization problems under polynomial constraints, where all the components of the basic operators are polynomial functions, we obtain some topological properties of the stationary point set, as well as the weak Pareto solution set, of the problem in question.

Our results extend the preceding results in [7] from the case of polyhedral convex constraint sets to the case of convex constraint sets given by (possibly nonconvex) polynomial functions. However, the new results do not cover the preceding ones. The reason is that formally the Mangasarian–Fromovitz constraint qualification need not be satisfied at every point of a constraint set given by finitely many affine inequalities. (Nevertheless, if one uses the relative interior of the constraint set and considers the restrictions of the related functions on the affine hull of the latter, then it is possible to derive the preceding results in [7] from the results in this paper.)

The paper's organization is as follows. Section 2 recalls some definitions, notation, and auxiliary results. The main results are established in section 3. In section 4, we apply the obtained results on the connectedness structure of the solution sets of VVIs to polynomial vector optimization problems.

2. Auxiliary results. We denote the scalar product of two vectors x, y in a Euclidean space by $\langle x, y\rangle$. For a matrix M, the symbol M^T indicates the transpose of M.

2.1. Vector variational inequalities. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex subset. Given m vector-valued functions $F_i : K \to \mathbb{R}^n$, $i = 1, \dots, m$, we let $F = (F_1, \dots, F_m)$ and

$$F(x)(u) = (\langle F_1(x), u\rangle, \dots, \langle F_m(x), u\rangle)^T \qquad \forall x \in K,\ \forall u \in \mathbb{R}^n.$$

Denoting the nonnegative orthant of $\mathbb{R}^m$ by $\mathbb{R}^m_+$, we consider the set

$$\Sigma = \left\{ \xi = (\xi_1, \dots, \xi_m)^T \in \mathbb{R}^m_+ \,:\, \sum_{l=1}^{m} \xi_l = 1 \right\},$$

whose relative interior is given by $\mathrm{ri}\Sigma = \{\xi \in \Sigma \,:\, \xi_l > 0 \ \text{for all}\ l = 1, \dots, m\}$.

Vector variational inequality [5, p. 167] defined by F, K, and the cone $C := \mathbb{R}^m_+$ is the following problem:

$$\text{(VVI)} \qquad \text{Find } x \in K \text{ such that } F(x)(y - x) \nleq_{C\setminus\{0\}} 0 \quad \forall y \in K,$$

where the inequality $v \nleq_{C\setminus\{0\}} 0$ for $v \in \mathbb{R}^m$ means that $-v \notin C\setminus\{0\}$. To this problem one associates [3] the following one:

$$\text{(VVI)}^w \qquad \text{Find } x \in K \text{ such that } F(x)(y - x) \nleq_{\mathrm{int}C} 0 \quad \forall y \in K,$$

where int$C$ denotes the interior of C and the inequality $v \nleq_{\mathrm{int}C} 0$ indicates that $-v \notin \mathrm{int}C$. The solution sets of (VVI) and (VVI)^w are abbreviated, respectively, as Sol(VVI) and Sol^w(VVI). The elements of the first set (resp., of the second) are said to be the *Pareto solutions* (resp., the *weak Pareto solutions*) of (VVI).

For $m = 1$, one has $F = F_1 : K \to \mathbb{R}^n$ and $C = \mathbb{R}_+$; hence both problems (VVI) and (VVI)^w coincide with the classical *variational inequality* problem [8, p. 13]

$$\text{(VI)} \qquad \text{Find } x \in K \text{ such that } \langle F(x), y - x\rangle \geq 0 \quad \forall y \in K,$$

whose solution set is denoted by Sol(VI).

For each $\xi \in \Sigma$, consider the variational inequality

$$\text{(VI)}_\xi \qquad \text{Find } x \in K \text{ such that } \left\langle \sum_{l=1}^{m} \xi_l F_l(x), y - x \right\rangle \geq 0 \quad \forall y \in K,$$

and denote its solution set by $\mathrm{Sol(VI)}_\xi$. The following definition seems to be new.

Definition 2.1. *If $x \in K$ and there exists $\xi \in \mathrm{ri}\Sigma$ such that $x \in \mathrm{Sol(VI)}_\xi$, then x is said to be a proper Pareto solution of* (VVI).

The proper Pareto solution set of (VVI) is abbreviated as Sol^{pr}(VVI). Clearly, by taking the union of $\mathrm{Sol(VI)}_\xi$ on $\xi \in \mathrm{ri}\Sigma$, we find the whole set Sol^{pr}(VVI). Combining this with some known results (see, e.g., [9, Theorem 2.1]), we have

$$\bigcup_{\xi \in \mathrm{ri}\Sigma} \mathrm{Sol(VI)}_\xi = \mathrm{Sol}^{pr}(\mathrm{VVI}) \subset \mathrm{Sol(VVI)} \subset \mathrm{Sol}^{w}(\mathrm{VVI}) = \bigcup_{\xi \in \Sigma} \mathrm{Sol(VI)}_\xi. \tag{2.1}$$

A result from [10] says that the first inclusion in (2.1) holds as an equality if K is a polyhedral convex set. The next example is designed to show that the inclusion can be strict even if K is given by a unique strongly convex polynomial inequality. In this example, both components of the objective function are constant vector functions that are monotone but not strongly monotone. (The definitions of monotonicity, strong monotonicity, and a classification of VVIs can be found in [9].)

Example 2.2. Consider problem (VVI) with $m = n = 2$, $F_1(x) = (1, 0)^T$, and $F_2(x) = (0, 1)^T$ for every $x = (x_1, x_2)^T \in \mathbb{R}^2$ and

$$K = \left\{x = (x_1, x_2)^T \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 - 1 \leq 0\right\}.$$

By (2.1), $x \in \mathrm{Sol}^w(\mathrm{VVI})$ if and only if there exists $\xi \in \Sigma$ such that $x \in \mathrm{Sol(VI)}_\xi$ or

$$\langle \xi_1 F_1(x) + \xi_2 F_1(x), y - x\rangle \geq 0 \quad \forall y \in K.$$

This condition can be written equivalently as

$$0 \in \xi_1 F_1(x) + \xi_2 F_1(x) + N_K(x), \tag{2.2}$$

where

$$N_K(x) := \{x^* \in \mathbb{R}^2 \ : \ \langle x^*, y - x\rangle \leq 0 \ \forall y \in K\} \tag{2.3}$$

for $x \in K$ and $N_K(x) := \emptyset$ for $x \notin K$ denotes the normal cone to K at x. Since $\xi_1 F_1(x) + \xi_2 F_2(x) = \binom{\xi_1}{\xi_2}$ and $N_K(x) = \{0\}$ for every $x \in \mathrm{int}K$, $\mathrm{Sol(VI)}_\xi \cap \mathrm{int}K = \emptyset$.

If x is taken from the boundary of K, then $N_K(x) = \{\lambda x \ : \ \lambda \geq 0\}$; hence (2.2) is satisfied if and only if

$$x_1 = -\frac{\xi_1}{\sqrt{(\xi_1)^2 + (\xi_2)^2}}, \qquad x_2 = -\frac{\xi_2}{\sqrt{(\xi_1)^2 + (\xi_2)^2}}.$$

It follows that $\mathrm{Sol}^w(\mathrm{VVI}) = \Gamma$, where

$$\Gamma := \left\{x \in \mathbb{R}^2 \ : \ -1 \leq x_1 \leq 0, \ \ x_2 = -\sqrt{1 - x_1^2}\right\}$$

is a closed circle-arc. As a by-product of the above calculations, we have

$$\mathrm{Sol}^{pr}(\mathrm{VVI}) = \left\{x \in \mathbb{R}^2 \ : \ -1 < x_1 < 0, \ \ x_2 = -\sqrt{1 - x_1^2}\right\}.$$

It is not difficult to show by definition that both end-points $\bar{x} := (-1, 0)^T$ and $\hat{x} := (0, -1)^T$ of Γ belong to Sol(VVI). So we have $\mathrm{Sol(VVI)} = \mathrm{Sol}^w(\mathrm{VVI}) = \Gamma$, while $\mathrm{Sol}^{pr}(\mathrm{VVI}) = \Gamma \setminus \{\bar{x}, \hat{x}\}$.

The interested reader is referred to [9, section 6] for another interesting example of (VVI) with K being a closed ball in $\mathbb{R}^2$ and F_1 and F_2 strongly monotone affine operators, where the weak Pareto solution set coincides with the Pareto solution set, which is a closed circle-arc, and the proper Pareto solution set $\mathrm{Sol}^{pr}(\mathrm{VVI})$ is obtained by eliminating the end-points from that circle-arc.

2.2. Sets having finitely many connected components. Let X be a topological space. The space X is said to be *connected* if one cannot find any disjoint nonempty open sets U, V of X such that $X = U \cup V$. A nonempty subset $A \subset X$ is said to be a *connected component* of X if A—equipped with the induced topology—is connected, and if it is not a proper subset of any connected subset of X.

The following lemma will be useful for our further investigations.

LEMMA 2.3 (see [7, Lemma 2.1]). *Let Ω be a subset of a topological space X with the closure denoted by $\overline{\Omega}$. If Ω has k connected components, then any subset $M \subset X$ with the property $\Omega \subset M \subset \overline{\Omega}$ can have at most k connected components.*

2.3. Semi-algebraic sets. The proofs of the main results of this paper rely on some theorems on semi-algebraic sets, which are recalled below.

The ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[x_1, \ldots, x_n]$.

DEFINITION 2.4 (see [2, Definition 2.1.4]). *A* semi-algebraic subset *of $\mathbb{R}^n$ is a subset of the form*

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \left\{x \in \mathbb{R}^n \ : \ f_{i,j}(x) *_{i,j} 0\right\},$$

where $f_{i,j} \in \mathbb{R}[x_1, \ldots, x_n]$ and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$, with s and r_i being arbitrary natural numbers.

Thus, every semi-algebraic subset of $\mathbb{R}^n$ can be represented as a finite union of sets of the form

$$\left\{x \in \mathbb{R}^n \ : \ f_1(x) = \cdots = f_\ell(x) = 0, \ g_1(x) < 0, \ldots, g_m(x) < 0\right\}, \qquad (2.4)$$

where ℓ and m are natural numbers and $f_1, \ldots, f_\ell,\ g_1, \ldots, g_m$ are in $\mathbb{R}[x_1, \ldots, x_n]$. Since

$$\{x : g_i(x) \leq 0\} = \{x : g_i(x) = 0\} \cup \{x : g_i(x) < 0\},$$

it doesn't matter if one replaces some of the strict inequalities in (2.4) by nonstrict inequalities.

Open balls, closed balls, spheres, and unions of finitely many of those sets are some typical examples of semi-algebraic subsets in $\mathbb{R}^n$. Semi-algebraic subsets of $\mathbb{R}$ are exactly the finite unions of points and open intervals (bounded or unbounded).

By induction, from [2, Theorem 2.2.1] we can easily derive the following useful result.

THEOREM 2.5. *Let S be a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the natural projection onto the space of the first n coordinates; i.e.,*

$$\Phi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (x_1, \ldots, x_n)^T$$

for every $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^T \in \mathbb{R}^n \times \mathbb{R}^m$. Then $\Phi(S)$ is a semi-algebraic subset of $\mathbb{R}^n$.

We proceed with the concept of semi-algebraically connected subset.

DEFINITION 2.6 (see [2, Definition 2.4.2]). *A semi-algebraic subset S of $\mathbb{R}^n$ is* semi-algebraically connected *if for every pair of disjoint semi-algebraic sets F_1 and F_2, which are closed in S and satisfy $F_1 \cup F_2 = S$, one has $F_1 = S$ or $F_2 = S$.*

The connectedness structure of semi-algebraic sets is clearly described by the next fundamental theorem of real algebraic geometry.

THEOREM 2.7 (see [2, Theorem 2.4.5]). *A semi-algebraic subset S of $\mathbb{R}^n$ is semi-algebraically connected if and only if it is connected. Every semi-algebraic set has a finite number of connected components, which are semi-algebraic.*

3. Properties of the solution sets. Throughout this section, *we let*

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, p,\ h_j(x) = 0,\ j = 1, \ldots, s\}, \tag{3.1}$$

where $g_i, h_j \in \mathbb{R}[x_1, \ldots, x_n]$ for all $i = 1, \ldots, p$, $j = 1, \ldots, s$, and assume that K is convex.

Remark 3.1. If the functions g_i are convex and the functions h_j are affine, i.e., $h_j(x) = \langle a_j, x\rangle + b_j,\ a_j \in \mathbb{R}^n,\ b_j \in \mathbb{R}$ for all $j = 1, \ldots, s$, then the set K given by (3.1) is convex.

In order to have an explicit formula for computing the normal cone to K at every point $x \in K$, we have to impose a regularity condition on the functions appearing in (3.1). This condition was proposed by Mangasarian and Fromovitz in 1967.

DEFINITION 3.2 (see [11, p. 44]). *One says that the* Mangasarian–Fromovitz constraint qualification *is satisfied at a point $x \in K$ if*

(i) *the column gradient vectors $\{\nabla h_j(x) : j = 1, \ldots, s\}$ are linearly independent;*
(ii) *there exists $v \in \mathbb{R}^n$ with $\langle \nabla h_j(x), v\rangle = 0$ for all $j = 1, \ldots, s$, and $\langle \nabla g_i(x), v\rangle < 0$ for all $i \in I(x)$, where $I(x) := \{i : g_i(x) = 0\}$.*

We will study the connectedness structure of the solution sets of VVIs of the form (VVI) under two assumptions:

(a1) *All the components of $F_l,\ l = 1, \ldots, m$, are polynomial functions in the variables $x_1, \ldots, x_n$; i.e., for each l one has $F_l = (F_{l1}, \ldots, F_{ln})$ with $F_{lj} \in \mathbb{R}[x_1, \ldots, x_n]$ for all $j = 1, \ldots, n$.*

(a2) *The Mangasarian–Fromovitz constraint qualification is satisfied at every point of K.*

Our main result can be formulated as follows.

THEOREM 3.3. *If* (a1) *and* (a2) *are fulfilled, then the following hold:*

(i) *The weak Pareto solution set* $\mathrm{Sol}^w(\mathrm{VVI})$ *is a semi-algebraic subset of* $\mathbb{R}^n$ *(so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$*).*

(ii) *The proper Pareto solution set* $\mathrm{Sol}^{pr}(\mathrm{VVI})$ *is a semi-algebraic subset of* $\mathbb{R}^n$ *(so it has finitely many connected components and each of them is a semi-algebraic subset of* $\mathbb{R}^n$*).*

(iii) *If the set* $\mathrm{Sol}^{pr}(\mathrm{VVI})$ *is dense in* $\mathrm{Sol}(\mathrm{VVI})$*, then* $\mathrm{Sol}(\mathrm{VVI})$ *has a finite number of connected components.*

Proof. (i) Let $I = \{1, \ldots, p\}$ and $J = \{1, \ldots, s\}$. To every subset $\alpha \subset I$ (the case $\alpha = \emptyset$ is not excluded) we associate the (possibly *curved*) *pseudoface* of K:

$$\mathcal{F}_\alpha = \{x \in \mathbb{R}^n \,:\, g_i(x) = 0 \ \ \forall i \in \alpha, \ \ g_i(x) < 0 \ \ \forall i \notin \alpha, \ \ h_j(x) = 0 \ \ \forall j \in J\}.$$

Using the notation $I(x)$ given in Definition 3.2, we have $I(x) = \alpha$ for all $x \in \mathcal{F}_\alpha$.

By (2.1), we have

$$\mathrm{Sol}^w(\mathrm{VVI}) = \bigcup_{\xi \in \Sigma} \mathrm{Sol}(\mathrm{VI})_\xi \tag{3.2}$$

with $(\mathrm{VI})_\xi$ denoting the following variational inequality:

$$\text{Find } \ x \in K \ \text{ such that } \ \langle F(x, \xi), y - x \rangle \geq 0 \quad \forall y \in K,$$

where $F(x, \xi) := \sum_{l=1}^m \xi_l F_l(x)$ for every $\xi = (\xi_1, \ldots, \xi_m)^T \in \Sigma$. Using the notation $F(x, \xi)$ and the definition of normal cone in (2.3), we can rewrite the inclusion $x \in \mathrm{Sol}(\mathrm{VI})_\xi$ equivalently as

$$-F(x, \xi) \in N_K(x). \tag{3.3}$$

We have $K = \bigcup_{\alpha \subset I} \mathcal{F}_\alpha$, $\mathcal{F}_\alpha \cap \mathcal{F}_{\widetilde{\alpha}} = \emptyset$ if $\alpha \neq \widetilde{\alpha}$, and, therefore,

$$\mathrm{Sol}^w(\mathrm{VVI}) = \bigcup_{\alpha \subset I} [\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha]. \tag{3.4}$$

Since a finite union of semi-algebraic subsets of $\mathbb{R}^n$ is again a semi-algebraic subset of $\mathbb{R}^n$, by (3.4) and by Theorem 2.7 we see that the proof of our assertion will be completed if we can establish the following claim.

CLAIM 1. *For every index set* $\alpha \subset I$*, the intersection* $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ *is a semi-algebraic subset of* $\mathbb{R}^n$*.*

By virtue of the assumption (a2), the result formulated in [1, remark on p. 151] gives us an explicit formula for computing the *Clarke tangent cone* [4, p. 51] to K at every point $x \in K$:

$$T_K(x) = \{v \in \mathbb{R}^n \,:\, \langle \nabla g_i(x), v \rangle \leq 0 \ \ \forall i \in I(x), \ \ \langle \nabla h_j(x), v \rangle = 0 \ \ \forall j \in J\}.$$

Since K is convex, the normal cone $N_K(x)$ defined in (2.3) coincides [4, Proposition 2.4.4] with the *Clarke normal cone* [4, p. 51] to K at x, which is the negative dual cone of the Clarke tangent cone $T_K(x)$. Thus,

$$N_K(x) = (T_K(x))^* = \{x^* \in \mathbb{R}^n \ : \ \langle x^*, v\rangle \le 0 \ \ \forall v \in T_K(x)\}.$$

This means that a vector $x^* \in \mathbb{R}^n$ belongs to $N_K(x)$ if and only if the inequality $\langle x^*, v\rangle \le 0$ is a consequence of the system

$$\langle \nabla g_i(x), v\rangle \le 0 \quad \forall i \in I(x), \qquad \langle \nabla h_j(x), v\rangle = 0 \quad \forall j \in J.$$

Hence, by Farkas' lemma [12, Corollary 22.3.1], $x^* \in N_K(x)$ if and only if there exist $\lambda_i \ge 0$, $i \in I(x)$, $\mu_j \in \mathbb{R}$, $j \in J$, such that

$$x^* = \sum_{i\in I(x)} \lambda_i \nabla g_i(x) + \sum_{j\in J} \mu_j \nabla h_j(x).$$

It follows that

$$N_K(x) = \left\{ \sum_{i\in\alpha} \lambda_i \nabla g_i(x) + \sum_{j\in J} \mu_j \nabla h_j(x) \ : \ \lambda_i \ge 0 \ \ \forall i \in \alpha, \quad \mu_j \in \mathbb{R} \ \ \forall j \in J \right\} \tag{3.5}$$

for every $\alpha \subset I$ and for every $x \in \mathcal{F}_\alpha$.

According to the formulas (3.2), (3.3), and (3.5),

$$\begin{aligned} \mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \bigcup_{\xi\in\Sigma} \Bigg\{ x \in \mathcal{F}_\alpha \ : \quad -\sum_{l=1}^m \xi_l F_l(x) = \sum_{i\in\alpha} \lambda_i \nabla g_i(x) + \sum_{j\in J} \mu_j \nabla h_j(x), \\ \lambda_i \ge 0 \ \forall i \in \alpha, \ \ \mu_j \in \mathbb{R} \ \forall j \in J \Bigg\}. \end{aligned} \tag{3.6}$$

The equality in (3.6) is equivalent to the following system of polynomial equalities in the variables $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_m$, λ_i, $i \in \alpha$, μ_j, $j \in J$:

$$-\sum_{l=1}^m \xi_l F_{lk}(x) = \sum_{i\in\alpha} \lambda_i \frac{\partial g_i(x)}{\partial x_k} + \sum_{j\in J} \mu_j \frac{\partial h_j(x)}{\partial x_k}, \quad k = 1, \ldots, n, \tag{3.7}$$

with $\frac{\partial g_i(x)}{\partial x_k}$ denoting the partial derivative of g_i at x with respect to x_k. Denote the power of α by $|\alpha|$, and put

$$\begin{aligned} \Omega_\alpha = \Bigg\{ (x, \xi, \lambda, \mu) \in \mathcal{F}_\alpha \times \Sigma \times \mathbb{R}_+^{|\alpha|} \times \mathbb{R}^s \ : \quad -\sum_{l=1}^m \xi_l F_l(x) = \sum_{i\in\alpha} \lambda_i \nabla g_i(x) \\ + \sum_{j\in J} \mu_j \nabla h_j(x) \Bigg\}. \end{aligned} \tag{3.8}$$

By (3.7), we have

$$\Omega_\alpha = \Big\{(x,\xi,\lambda,\mu) \in \mathbb{R}^{n+m+|\alpha|+s} \ : \quad g_i(x)=0,\ i\in\alpha,\ \ g_i(x)<0,\ i\notin\alpha,$$
$$-\sum_{l=1}^m \xi_l F_{lk}(x) = \sum_{i\in\alpha}\lambda_i \frac{\partial g_i(x)}{\partial x_k} + \sum_{j=1}^s \mu_j \frac{\partial h_j(x)}{\partial x_k},\ \ k=1,\dots,n,$$
$$\sum_{l=1}^m \xi_l = 1,\ \xi_l \geq 0,\ \ l=1,\dots,m,$$
$$\lambda_i \geq 0,\ i\in\alpha\Big\}.$$

Since Ω_α is determined by $|\alpha|+n+1$ polynomial equations, $m+|\alpha|$ polynomial inequalities, and $p-|\alpha|$ strict polynomial inequalities of the variables $(x,\xi,\mu) = (x_1,\dots,x_n,\xi_1,\dots,\xi_m,\mu_1,\dots,\mu_s) \in \mathbb{R}^{n+m+s}$ and $\lambda_i,\ i\in\alpha$, it is a semi-algebraic set.

From (3.6) we get $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \Phi(\Omega_\alpha)$, where $\Phi : \mathbb{R}^{n+m+|\alpha|+s} \to \mathbb{R}^n$ is the natural projection onto the space of the first n coordinates. According to Theorem 2.5, $\mathrm{Sol}^w(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic set. This proves Claim 1.

We have thus shown that the weak Pareto solution set $\mathrm{Sol}^w(\mathrm{VVI})$ is a semi-algebraic subset of $\mathbb{R}^n$. Then, according to Theorem 2.7, $\mathrm{Sol}^w(\mathrm{VVI})$ has finitely many connected components and each of them is a semi-algebraic subset of $\mathbb{R}^n$.

(ii) Combining (2.1) with the formula $K = \bigcup_{\alpha\subset I}\mathcal{F}_\alpha$ yields

$$\mathrm{Sol}^{pr}(\mathrm{VVI}) = \bigcup_{\alpha\subset I} [\mathrm{Sol}^{pr}(\mathrm{VVI}) \cap \mathcal{F}_\alpha]. \tag{3.9}$$

Due to (3.9), our assertion will be proved if we can obtain the following.

Claim 2. *For every index set $\alpha \subset I$, the intersection $\mathrm{Sol}^{pr}(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic subset of $\mathbb{R}^n$.*

Let Ω_α and Φ be defined as above. Instead of (3.6), now we have

(3.10)
$$\mathrm{Sol}^{pr}(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \bigcup_{\xi\in \mathrm{ri}\Sigma} \Big\{ x\in\mathcal{F}_\alpha : -\sum_{l=1}^m \xi_l F_l(x) = \sum_{i\in\alpha}\lambda_i \nabla g_i(x) + \sum_{j\in J}\mu_j \nabla h_j(x),$$
$$\lambda_i \geq 0\ \ \forall i\in\alpha,\quad \mu_j\in\mathbb{R}\ \ \forall j\in J\Big\}.$$

Put

(3.11)
$$\widetilde{\Omega}_\alpha = \Big\{(x,\xi,\lambda,\mu) \in \mathcal{F}_\alpha \times \mathrm{ri}\Sigma \times \mathbb{R}_+^{|\alpha|} \times \mathbb{R}^s \ :\ -\sum_{l=1}^m \xi_l F_l(x) = \sum_{i\in\alpha}\lambda_i\nabla g_i(x) + \sum_{j\in J}\mu_j \nabla h_j(x)\Big\}.$$

The formula

$$\widetilde{\Omega}_\alpha = \Bigg\{(x,\xi,\lambda,\mu) \in \mathbb{R}^{n+m+|\alpha|+s} \ : \quad g_i(x) = 0,\ i \in \alpha, \quad g_i(x) < 0,\ i \notin \alpha, \\ -\sum_{l=1}^{m} \xi_l F_{lk}(x) = \sum_{i\in\alpha} \lambda_i \frac{\partial g_i(x)}{\partial x_k} \\ + \sum_{j=1}^{s} \mu_j \frac{\partial h_j(x)}{\partial x_k},\ \ k = 1,\dots,n, \\ \sum_{l=1}^{m} \xi_l = 1,\ \xi_l > 0,\ l = 1,\dots,m, \\ \lambda_i \geq 0,\ \ i \in \alpha \Bigg\}$$

shows that $\widetilde{\Omega}_\alpha$ is the solution set of a system of $|\alpha| + n + 1$ polynomial equations, $|\alpha|$ polynomial inequalities, and $p - |\alpha| + m$ strict polynomial inequalities of the variables $(x, \xi, \mu) = (x_1, \dots, x_n, \xi_1, \dots, \xi_m, \mu_1, \dots, \mu_s) \in \mathbb{R}^{n+m+s}$ and $\lambda_i,\ i \in \alpha$. Hence $\widetilde{\Omega}_\alpha$ is a semi-algebraic set.

By (3.10) we have $\mathrm{Sol}^{pr}(\mathrm{VVI}) \cap \mathcal{F}_\alpha = \Phi(\widetilde{\Omega}_\alpha)$ with Φ being defined as above. Thus, Theorem 2.5 assures that $\mathrm{Sol}^{pr}(\mathrm{VVI}) \cap \mathcal{F}_\alpha$ is a semi-algebraic set. This establishes Claim 2.

We have shown that $\mathrm{Sol}^{pr}(\mathrm{VVI})$ is a semi-algebraic subset. So, by Theorem 2.7, $\mathrm{Sol}^{pr}(\mathrm{VVI})$ has finitely many connected components, and each of them is a semi-algebraic subset of $\mathbb{R}^n$.

(iii) Since the set $\mathrm{Sol}^{pr}(\mathrm{VVI})$ is assumed to be dense in Sol(VVI), we have

$$\mathrm{Sol}^{pr}(\mathrm{VVI}) \subset \mathrm{Sol}(\mathrm{VVI}) \subset \overline{\mathrm{Sol}^{pr}(\mathrm{VVI})}, \tag{3.12}$$

where $\overline{\mathrm{Sol}^{pr}(\mathrm{VVI})}$ is the closure of $\mathrm{Sol}^{pr}(\mathrm{VVI})$ in the Euclidean topology of $\mathbb{R}^n$. Since $\mathrm{Sol}^{pr}(\mathrm{VVI})$ has finitely many connected components, from (3.12) and Lemma 2.3 it follows that Sol(VVI) has a finite number of connected components.

The proof is complete. ∎

Note that, in the problem discussed in Example 2.2, the assumption *"the set* $\mathrm{Sol}^{pr}(\mathrm{VVI})$ *is dense in* Sol(VVI)*"* in Theorem 3.3 is satisfied.

4. Applications to vector optimization. By using Theorem 3.3 we will obtain some facts about the connectedness structure of the stationary point sets and of the weak Pareto solution sets of polynomial vector optimization problems under polynomial constraints.

First, let us recall several solution concepts for a general differentiable vector optimization problem.

4.1. Some solution concepts in vector optimization. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex subset, and $f = (f_1, \dots, f_m) : \Omega \to \mathbb{R}^m$ a continuously differentiable function defined on an open set $\Omega \subset \mathbb{R}^n$ containing K. The vector optimization problem with the constraint set K and the vector objective function f is written formally as follows:

$$\text{(VP)} \qquad \text{Minimize } \ f(x) \quad \text{subject to} \quad x \in K.$$

Definition 4.1. *A point $x \in K$ is said to be an* efficient solution *(or a* Pareto solution*) of* (VP) *if* $(f(K) - f(x)) \cap (-\mathbb{R}^m_+ \setminus \{0\}) = \emptyset$.

Definition 4.2. *One says that $x \in K$ is a* weakly efficient solution *(or a* weak Pareto solution*) of* (VP) *if* $(f(K) - f(x)) \cap (-\operatorname{int} \mathbb{R}^m_+) = \emptyset$.

For any $\xi \in \Sigma$, where Σ is defined as in section 2, we consider the parametric variational inequality $(\mathrm{VI})_\xi$, which now takes the following form:

$$\text{Find } x \in K \text{ such that } \left\langle \sum_{l=1}^{m} \xi_l \nabla f_l(x), y - x \right\rangle \geq 0 \quad \forall y \in K. \tag{4.1}$$

According to [9, Theorem 3.1(i)], if $x \in \mathrm{Sol}^w(\mathrm{VP})$, then there exists $\xi \in \Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$. The converse is true if the functions f_l, $l = 1, \ldots, m$, are convex (see [9, Theorem 3.1(ii)]).

Definition 4.3. *If $x \in K$ and there is $\xi \in \Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then we call x a stationary point of* (VP).

It is well known [9, Theorem 3.1(iii)] that if the functions f_l, $l = 1, \ldots, m$, are convex and if there is $\xi \in \mathrm{ri}\Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then x is a Pareto solution of (VP). This sufficient optimality condition is a motivation for introducing the next concept.

Definition 4.4. *If $x \in K$ and there is $\xi \in \mathrm{ri}\Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then we call x a proper stationary point of* (VP).

The Pareto solution set, the weak Pareto solution set, the stationary point set, and the proper stationary point set of (VP) are abbreviated as Sol(VP), $\mathrm{Sol}^w(\mathrm{VP})$, Stat(VP), and Pr(VP), respectively.

By the above definitions and remarks we have

$$\bigcup_{\xi \in \mathrm{ri}\Sigma} \mathrm{Sol}(\mathrm{VI})_\xi = \mathrm{Pr}(\mathrm{VP}) \subset \mathrm{Sol}(\mathrm{VP}) \subset \mathrm{Sol}^w(\mathrm{VP}) \subset \mathrm{Stat}(\mathrm{VP}) = \bigcup_{\xi \in \Sigma} \mathrm{Sol}(\mathrm{VI})_\xi, \tag{4.2}$$

where the first inclusion is valid if all the functions f_i are convex. In addition, under the latter condition, the third inclusion in (4.2) becomes an equality.

4.2. Polynomial vector optimization. Let $f_l : \mathbb{R}^n \to \mathbb{R}$, $l = 1, \ldots, m$, be polynomial functions, and let $K \subset \mathbb{R}^n$ be given as in (3.1), where $g_i, h_j \in \mathbb{R}[x_1, \ldots, x_n]$ for all $i = 1, \ldots, p$, $j = 1, \ldots, s$. In this setting, the problem (VP) is called a *polynomial vector optimization problem under polynomial constraints* and is denoted by (pVP). The Pareto solution set, the weakly Pareto solution set, the stationary point set, and the proper stationary point set of (pVP) are abbreviated as Sol(pVP), $\mathrm{Sol}^w(\mathrm{pVP})$, Stat(pVP), and Pr(pVP), respectively.

Theorem 4.5. *If K is convex and assumption* (a2) *is fulfilled, then the following assertions hold:*

(i) *The set* Stat(pVP) *(resp., the set* Pr(pVP)*) is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components, and each of them is a semi-algebraic subset of $\mathbb{R}^n$).*

(ii) *If all the functions f_l are convex, then* $\mathrm{Sol}^w(\mathrm{pVP})$ *is a semi-algebraic subset of $\mathbb{R}^n$ (so it has finitely many connected components, and each of them is a semi-algebraic subset of $\mathbb{R}^n$).*

(iii) *If all the functions f_l are convex and the set* Pr(pVP) *is dense in* Sol(pVP)*, then* Sol(pVP) *has a finite number of connected components.*

Proof. (i) Since f_l, $l = 1, \dots, m$, are polynomial functions, f is continuously differentiable. By (4.2), we have

$$\text{Stat(pVP)} = \bigcup_{\xi \in \Sigma} \text{Sol(VI)}_\xi, \qquad \text{Pr(pVP)} = \bigcup_{\xi \in \text{ri}\Sigma} \text{Sol(VI)}_\xi. \tag{4.3}$$

Combining (4.3) with (2.1) and Theorem 3.3, we get the desired properties.

(ii) Since all the components of f are convex polynomials, we have

$$\text{Sol}^w\text{(pVP)} = \text{Stat(pVP)} = \bigcup_{\xi \in \Sigma} \text{Sol(VI)}_\xi.$$

Hence the assertion follows from (i).

(iii) Since all the functions f_l are convex and the set Pr(pVP) is dense in Sol(pVP), we have

$$\text{Pr(pVP)} \subset \text{Sol(pVP)} \subset \overline{\text{Pr(pVP)}}, \tag{4.4}$$

where $\overline{\text{Pr(pVP)}}$ is the closure of Pr(pVP) in the Euclidean topology of $\mathbb{R}^n$. By (i) we see that Pr(pVP) has finitely many connected components. Now, from (4.4) and Lemma 2.3 it follows that Sol(pVP) has a finite number of connected components.

The proof is complete. ∎

We conclude this section with an illustrative example, where the assumption *"the set* Pr(pVP) *is dense in* Sol(pVP)*"* in Theorem 4.5 is satisfied.

Example 4.6. Consider problem (pVP) with $m = n = 2$, $f_1(x) = x_1$ and $f_2(x) = x_2$ for every $x = (x_1, x_2)^T \in \mathbb{R}^2$, and

$$K = \left\{ x = (x_1, x_2)^T \in \mathbb{R}^2 \; : \; x_1^2 + x_2^2 - 1 \leq 0 \right\}.$$

Let Γ, $\bar{x}$, and $\hat{x}$ be defined as in Example 2.2. It is not difficult to verify that $\text{Sol(pVP)} = \text{Sol}^w\text{(pVP)} = \text{Stat(pVP)} = \Gamma$ and $\text{Pr(pVP)} = \Gamma \setminus \{\bar{x}, \hat{x}\}$.

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