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PULLBACK ATTRACTORS FOR STRONG SOLUTIONS OF 2D NON-AUTONOMOUS *g*-NAVIER-STOKES EQUATIONS

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Abstract Considered here is the first initial boundary value problem for the 2D non-autonomous g-Navier-Stokes equations in bounded domains. We prove the existence of a pullback attractor in V_g for the continuous process generated by strong solutions to the problem. We also prove the exponential growth in V_g and in $H^2(\Omega, g)$ for the pullback attractor, when time goes to $-\infty$.

Keywords g-Navier-Stokes equations · Strong solution · Pullback attractor · Tempered behavior

Mathematics Subject Classification (2010) 35B41 · 35Q30 · 37L30 · 35D35

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. In this paper, we study the long-time behavior of strong solutions to the following 2D nonautonomous g-Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \text{ in } (\tau, +\infty) \times \Omega, \\ \nabla \cdot (gu) = 0 \text{ in } (\tau, +\infty) \times \Omega, \\ u = 0 \text{ on } (\tau, +\infty) \times \Gamma, \\ u(\tau, x) = u_\tau(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where $u = u(t, x) = (u_1, u_2)$ is the unknown velocity vector, $p = p(t, x)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, and u_τ is the initial velocity.

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The g-Navier-Stokes equation is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv$ const, we get the usual Navier-Stokes equations. The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [17] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [17], good properties of the 2D g-Navier-Stokes equations can lead to an initial study of the Navier-Stokes equations on the thin 3D domain $\Omega_g = \Omega \times (0, g)$. In the last few years, the existence and long-time behavior of weak solutions to 2D g-Navier-Stokes equations have been studied extensively in both autonomous and nonautonomous cases (see, e.g., [3, 7–9, 13, 14, 16, 18]). In a recent work [4], we proved the existence and numerical approximation of strong solutions to the 2D g-Navier-Stokes equations. The long-time behavior of the strong solutions was studied more recently in [5] in the autonomous case in terms of existence of a global attractor and stability of a unique stationary solution.

In this paper, we continue in studying the long-time behavior of strong solutions to 2D g-Navier-Stokes equations in the nonautonomous case, i.e., when the external force f may depend on time t, in terms of existence and exponential growth of a pullback attractor. To do this, we make the following assumptions:

(G) $g \in W^{1,\infty}(\Omega)$ such that:

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A defined in Section 2 below);

(F) $f \in L^2_{\text{loc}}(\mathbb{R}; H_g)$ and satisfies

$$\int_{-\infty}^0 e^{\mu s} |f(s)|^2 ds < +\infty \quad \text{for some } \mu \in (0, 2\nu\gamma_0\lambda_1), \tag{1.2}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} > 0$.

When the external force f is time-dependent, to study the long-time behavior of strong solutions to problem (1.1), we will use the theory of pullback attractors. This theory is a natural generalization of the theory of global attractors for autonomous dynamical systems and allows considering a number of different problems of nonautonomous dynamical systems under a large class of nonautonomous forcing terms. The existence of pullback attractors has been proved for many dissipative partial differential equations. Recently, the regularity and tempered behavior in various function spaces of pullback attractors have been proved for the reaction-diffusion equations in [1, 2] and for 2D Navier-Stokes equations in [11, 12]. In this paper, using some ideas in those papers, we study the existence and exponential growth of a pullback attractor for the continuous process generated by strong solutions to problem (1.1).

The paper is organized as follows. In Section 2, we prove the existence of a pullback attractor in V_g by using an energy method which relies on the continuity of strong solutions. The exponential growth of the pullback attractor in V_g and in $H^2(\Omega, g)$ is studied in the last section under some suitable additional assumptions of the external force.

In the rest of this section, we recall some notations frequently used in the paper (see [4] for more details). Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \quad u, v \in L^2(\Omega, g),$$

and

$$((u,v))_g=\int_\Omega \sum_{j=1}^{2}\nabla u_j\cdot\nabla v_j g dx,\quad u=(u_1,u_2), v=(v_1,v_2)\in H_0^1(\Omega,g),$$

and norms $|u|^2=(u,u)_g,\ ||u||^2=((u,u))_g$. By H_g, V_g, we denote the completions of

$$\mathcal{V}=\left\{u\in (C_0^\infty(\Omega))^2:\nabla\cdot(gu)=0\right\}$$

in $L^2(\Omega,g)$ and $H_0^1(\Omega,g)$, respectively. We use $||.||_*$ for the norm in V_g', the dual space of V_g, and $\langle .,.\rangle$ for duality pairing between V_g and V_g'.

Set $A:V_g\to V_g'$ by $\langle Au,v\rangle=((u,v))_g$, $B:V_g\times V_g\to V_g'$ by $\langle B(u,v),w\rangle=b(u,v,w)$, where

$$b(u,v,w)=\sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j g dx.$$

Denote $D(A)=\left\{u\in V_g:Au\in H_g\right\}$, then $D(A)=H^2(\Omega,g)\cap V_g$ and $Au=-P_g\Delta u$ for every $u\in D(A)$, where P_g is the ortho-projector from $L^2(\Omega,g)$ onto H_g.

2 Existence of a pullback attractor in V_g

Definition 2.1 Given $f\in L^2(\tau,T;H_g)$ and $u_\tau\in V_g$, a strong solution on (τ,T) of problem (1.1) is a function $u\in L^2(\tau,T;D(A))\cap L^\infty(\tau,T;V_g)$ with $u(\tau)=u_\tau$, such that for all $v\in V_g$,

$$\frac{d}{dt}(u(t),v)_g+\nu((u(t),v))_g+\nu(Cu(t),v)_g+b(u(t),u(t),v)=(f(t),v)_g \tag{2.1}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,T)$.

Remark 2.1 From the above definition, we see that if u is a strong solution of (1.1) on (τ,T), then $u'\in L^2(\tau,T;H_g)$ and $u\in C([\tau,T];V_g)$. Moreover, u satisfies the energy equality

$$\begin{aligned}|u(t)|^2\ +\ 2\nu\int_s^t\|u(r)\|^2dr+2\nu\int_s^t b\left(\frac{\nabla g}{g},u(r),u(r)\right)dr\\ =|u(s)|^2+2\int_s^t(f(r),u(r))_g\,dr.\end{aligned} \tag{2.2}$$

The following existence theorem was proved in [4].

Theorem 2.1 *Suppose that $f\in L^2_{loc}(\mathbb{R};H_g)$ and $u_\tau\in V_g$ are given. Then for any $T>\tau$, there exists a unique strong solution u of problem (1.1) on (τ,T). Moreover, the map $u_\tau\mapsto u(t)$ is continuous on V_g for all $t\in[\tau,T]$, that is, the strong solution depends continuously on the initial data.*

Under assumptions of Theorem 2.1, we can define a continuous process $U(t,\tau):V_g\to V_g$ as follows

$$U(t,\tau)u_\tau=u(t;\tau,u_\tau)\ \ \forall\,u_\tau\in V_g\ \forall\,\tau\le t.$$

Denote by $\mathcal{D}_\mu^{H_g}$, the class of all families of nonempty subsets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H_g)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\mu\tau}\sup_{v\in D(\tau)}|v|^2\right) = 0.$$

We now prove the following result.

Lemma 2.1 *For any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_\mu^{H_g}$, there exists $\tau_1(\hat{D}, t) < t-3$, such that for any $\tau \le \tau_1(\hat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds*

$$\begin{cases} |u(r)|^2 & \le \rho_1(t) \ \ \textit{for all} \ \ r \in [t-3, t], \\ ||u(r)||^2 & \le \rho_2(t) \ \ \textit{for all} \ \ r \in [t-2, t], \\ \int_{r-1}^{r} |Au(\theta)|^2 d\theta & \le \rho_3(t) \ \ \textit{for all} \ \ r \in [t-1, t], \\ \int_{r-1}^{r} |u'(\theta)|^2 d\theta & \le \rho_4(t) \ \ \textit{for all} \ \ r \in [t-1, t], \end{cases} \tag{2.3}$$

where

$$\rho_1(t) = 1 + \frac{e^{\mu(3-t)}}{2\nu\gamma_0\lambda_1 - \mu}\int_{-\infty}^{t} e^{\mu\theta}|f(\theta)|^2 d\theta, \tag{2.4}$$

$$\begin{aligned}\rho_2(t) = \max_{r\in[t-2,t]}\Bigg\{&\left[\frac{\rho_1(r)}{\nu\gamma_0} + \left(\frac{1}{\nu^2\gamma_0^2\lambda_1} + \frac{2}{\nu}\right)\int_{r-1}^{r}|f(\theta)|^2 d\theta\right] \\ &\times \exp\left[\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0} + 2c_2'\rho_1(r)\left(\frac{\rho_1(r)}{\nu\gamma_0} + \frac{1}{\nu^2\gamma_0^2\lambda_1}\right)\int_{r-1}^{r}|f(\theta)|^2 d\theta\right]\Bigg\},\end{aligned} \tag{2.5}$$

$$\rho_3(t) = \frac{1}{\nu}[\rho_2(t) + 2c_2'\rho_1(t)\rho_2^2(t) + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\rho_2(t) + \frac{2}{\nu}\int_{t-2}^{t}|f(\theta)|^2 d\theta], \tag{2.6}$$

$$\rho_4(t) = \nu\rho_2(t) + 3c_1^2\rho_2(t)\rho_3(t) + \frac{3\nu^2|\nabla g|_\infty^2}{m_0^2}\rho_2(t) + 3\int_{t-2}^{t}|f(\theta)|^2 d\theta. \tag{2.7}$$

Proof For each integer $n \ge 1$, we denote by $u^n(s) = u^n(s; \tau, u_\tau)$ the Galerkin approximation of the solution $u(s; \tau, u_\tau)$ of (1.1), which is given by

$$u^n(s) = \sum_{j=1}^{n}\gamma_{nj}(s)v_j,$$

and is the solution of

$$\frac{d}{dt}\left(u^n(s), v_j\right)_g + \nu\left(Au^n(s), v_j\right)_g + \nu\left(Cu^n(s), v_j\right)_g + b\left(u^n(s), u^n(s), v_j\right) = (f(s), v_j)_g, \tag{2.8}$$

for any $j = 1, \dots, n$, and $(u^n(\tau), v_j)_g = (u_\tau, v_j)_g$.

Multiplying by $\gamma_{nj}(s)$ in (2.8), and summing from $j = 1$ to n, we obtain

$$\frac{d}{d\theta}|u^n(\theta)|^2 + 2\nu||u^n(\theta)||^2 = 2\left(f(\theta), u^n(\theta)\right)_g - 2\nu\left(Cu^n(\theta), u^n(\theta)\right)_g, \quad \text{a.e. } \theta > \tau, \tag{2.9}$$

and therefore

$$\begin{aligned}&\frac{d}{d\theta}(e^{\mu\theta}|u^n(\theta)|^2)+2\nu e^{\mu\theta}||u^n(\theta)||^2\\&\quad=\mu e^{\mu\theta}|u^n(\theta)|^2+2e^{\mu\theta}(f(\theta),u^n(\theta))_g-2\nu e^{\mu\theta}(Cu^n(\theta),u^n(\theta))_g\\&\quad\le\frac{1}{\lambda_1}\mu e^{\mu\theta}||u^n(\theta)||^2+2\nu e^{\mu\theta}\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u^n(\theta)||^2+2e^{\mu\theta}(f(\theta),u^n(\theta))_g.\end{aligned}$$

Then, we have

$$\begin{aligned}&\frac{d}{d\theta}(e^{\mu\theta}|u^n(\theta)|^2)+\frac{1}{\lambda_1}e^{\mu\theta}(2\nu\gamma_0\lambda_1-\mu)||u^n(\theta)||^2\\&\quad\le 2e^{\mu\theta}(f(\theta),u^n(\theta))_g\le\frac{e^{\mu\theta}}{2\nu\gamma_0\lambda_1-\mu}|f(\theta)|^2+\frac{e^{\mu\theta}}{\lambda_1}(2\nu\gamma_0\lambda_1-\mu)||u^n(\theta)||^2,\end{aligned}$$

where $\gamma_0=1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}>0$. Hence, we deduce that

$$\frac{d}{d\theta}\left(e^{\mu\theta}|u^n(\theta)|^2\right)\le\frac{e^{\mu\theta}}{2\nu\gamma_0\lambda_1-\mu}|f(\theta)|^2,$$

and therefore

$$e^{\mu r}|u^n(r)|^2\le e^{\mu\tau}|u_\tau|^2+\frac{1}{2\nu\gamma_0\lambda_1-\mu}\int_{-\infty}^r e^{\mu\theta}|f(\theta)|^2d\theta\quad\forall r\ge\tau. \tag{2.10}$$

From (2.10), we see that for each $t\in\mathbb{R}$ and $\hat{D}\in\mathcal{D}_\mu^{H_g}$, there exists $\tau_1(\hat{D},t)<t-3$ such that for any $n\ge 1$,

$$|u^n(r;\tau,u_\tau)|^2\le\rho_1(t)\ \ \forall r\in[t-3,t],\ \ \tau\le\tau_1(\hat{D},t),\ \ u_\tau\in D(\tau), \tag{2.11}$$

where $\rho_1(t)$ is given by (2.4).

Now, multiplying in (2.8) by $\lambda_j\gamma_{nj}(s)$, where λ_j is the eigenvalue associated to the eigenfunction v_j, and summing from $j=1$ to n, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\theta}||u^n(\theta)||^2+\nu|Au^n(\theta)|^2+\nu\left(Cu^n(\theta),Au^n(\theta)\right)_g+b\left(u^n(\theta),u^n(\theta),Au^n(\theta)\right)\\&\quad=(f(\theta),Au^n(\theta)),\quad\text{a.e. }\theta>\tau.\end{aligned}\tag{2.12}$$

By Lemmas 2.1 and 2.3 in [4], (2.12) implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{d\theta}||u^n(\theta)||^2+\nu|Au^n(\theta)|^2&\le\frac{\nu}{4}|Au^n(\theta)|^2+\frac{1}{\nu}|f(\theta)|^2\\&\quad+c_2|u^n(\theta)|^{1/2}|Au^n(\theta)|^{3/2}||u^n(\theta)||+\frac{\nu|\nabla g|_\infty}{m_0}||u^n(\theta)|||Au^n(\theta)|.\end{aligned}$$

Using Young's inequality and Cauchy's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\theta}||u^n(\theta)||^2+\nu|Au^n(\theta)|^2\\&\quad\le\frac{\nu}{4}|Au^n(\theta)|^2+\frac{1}{\nu}|f(\theta)|^2+\frac{\nu}{4}|Au^n(\theta)|^2+c_2'|u^n(\theta)|^2||u^n(\theta)||^4\\&\quad+\frac{\nu|\nabla g|_\infty}{2m_0\lambda_1^{1/2}}|Au^n(\theta)|^2+\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{2m_0}|u^n(\theta)|^2.\end{aligned}$$

Then, we have

$$\frac{d}{d\theta}||u^n(\theta)||^2 + \nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})|Au^n(\theta)|^2$$
$$\leq \frac{2}{\nu}|f(\theta)|^2 + \left(2c_2'|u^n(\theta)|^2||u^n(\theta)||^2 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\right)||u^n(\theta)||^2, \text{ a.e. } \theta > \tau. \quad (2.13)$$

From this inequality, in particular, we get

$$||u^n(r)||^2 \leq ||u^n(s)||^2 + \frac{2}{\nu}\int_{r-1}^{r}|f(\theta)|^2 d\theta$$
$$+ \int_{r-1}^{r}\left(2c_2'|u^n(\theta)|^2||u^n(\theta)||^2 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\right)||u^n(\theta)||^2 d\theta$$

for all $\tau \leq r-1 \leq s \leq r$, and therefore, by Gronwall's inequality,

$$||u^n(r)||^2 \leq \left(||u^n(s)||^2 + \frac{2}{\nu}\int_{r-1}^{r}|f(\theta)|^2 d\theta\right)$$
$$\times \exp\left(\int_{r-1}^{r}\left(2c_2'|u^n(\theta)|^2||u^n(\theta)||^2 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\right)d\theta\right)$$

for all $\tau \leq r-1 \leq s \leq r$. Integrating this inequality for s between $r-1$ and r, we obtain

$$||u^n(r)||^2 \leq \left(\int_{r-1}^{r}||u^n(s)||^2 ds + \frac{2}{\nu}\int_{r-1}^{r}|f(\theta)|^2 d\theta\right)$$
$$\times \exp\left(\int_{r-1}^{r}\left(2c_2'|u^n(\theta)|^2||u^n(\theta)||^2 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\right)d\theta\right). \quad (2.14)$$

By (2.9), we have

$$\frac{d}{d\theta}|u^n(\theta)|^2 + 2\nu||u^n(\theta)||^2 \leq \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u^n(\theta)||^2 + \gamma_0\nu||u^n(\theta)||^2 + \frac{1}{\gamma_0\nu\lambda_1}|f(\theta)|^2,$$

or

$$\frac{d}{d\theta}|u^n(\theta)|^2 + \nu\gamma_0||u^n(\theta)||^2 \leq \frac{1}{\gamma_0\nu\lambda_1}|f(\theta)|^2,$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$. Hence,

$$\nu\gamma_0\int_{r-1}^{r}||u^n(\theta)||^2 d\theta \leq |u^n(r-1)|^2 + \frac{1}{\gamma_0\nu\lambda_1}\int_{r-1}^{r}|f(\theta)|^2 d\theta.$$

Therefore, from (2.11) and (2.14), we deduce that for any $n \geq 1$,

$$||u^n(r;\tau,u_\tau)||^2 \leq \rho_2(t) \text{ for all } r \in [t-2,t],\ \tau \leq \tau_1(\hat{D},t), u_\tau \in D(\tau), \quad (2.15)$$

where $\rho_2(t)$ is given by (2.5).

Now, by (2.13),

$$\frac{d}{d\theta}||u^n(\theta)||^2 + \nu\gamma_0|Au^n(\theta)|^2$$
$$\leq \frac{2}{\nu}|f(\theta)|^2 + 2c_2'|u^n(\theta)|^2||u^n(\theta)||^4 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}||u^n(\theta)||^2, \text{ a.e. } \theta > \tau.$$

Hence

$$\nu\gamma_0 \int_{r-1}^{r} |Au^n(\theta)|^2 d\theta \le ||u^n(r-1)||^2 + \frac{2}{\nu}\int_{r-1}^{r} |f(\theta)|^2 d\theta + 2c_2' \times \int_{r-1}^{r} |u^n(\theta)|^2 ||u^n(\theta)||^4 d\theta + \frac{\nu|\nabla g|_\infty \lambda_1^{1/2}}{m_0}\int_{r-1}^{r} ||u^n(\theta)||^2 d\theta$$

for all $\tau \le r-1$ and therefore, by (2.11) and (2.15), for every $n \ge 1$,

$$\int_{r-1}^{r} |Au^n(\theta;\tau,u_\tau)|^2 d\theta \le \rho_3(t) \tag{2.16}$$

for all $r \in [t-1,t]$, $\tau \le \tau_1(\hat{D},t)$, $u_\tau \in D(\tau)$, where $\rho_3(t)$ is given by (2.6).

On the other hand, multiplying by the derivative $\gamma_{nj}'(s)$ in (2.8), and summing from $j=1$ until n, we obtain

$$\begin{aligned} &|(u^n(\theta))'|^2 + \frac{\nu}{2}\frac{d}{d\theta}||u^n(\theta)||^2 + \nu(Cu^n(\theta),(u^n(\theta))')_g \\ &\quad + b(u^n(\theta),u^n(\theta),(u^n(\theta))') = (f(\theta),(u^n(\theta))')_g, \text{ a.e. } \theta > \tau. \end{aligned} \tag{2.17}$$

By Lemmas 2.1 and 2.3 in [4], and Cauchy's inequality, (2.17) implies that

$$\begin{aligned} 2|(u^n(\theta))'|^2 + \nu\frac{d}{d\theta}||u^n(\theta)||^2 \le\ & \frac{1}{3}|(u^n(\theta))'|^2 + 3|f(\theta)|^2 \\ & + \frac{1}{3}|(u^n(\theta))'|^2 + 3c_1^2|Au^n(\theta)|^2||u^n(\theta)||^2 \\ & + \frac{1}{3}|(u^n(\theta))'|^2 + \frac{3\nu^2|\nabla g|_\infty^2}{m_0^2}||u^n(\theta)||^2 \text{ a.e. } \theta > \tau, \end{aligned}$$

hence

$$|(u^n(\theta))'|^2 + \nu\frac{d}{d\theta}||u^n(\theta)||^2 \le 3|f(\theta)|^2 + 3c_1^2|Au^n(\theta)|^2||u^n(\theta)||^2 + \frac{3\nu^2|\nabla g|_\infty^2}{m_0^2}||u^n(\theta)||^2.$$

Integrating this last inequality, we deduce that

$$\begin{aligned} \int_{r-1}^{r} |(u^n(\theta))'|^2 d\theta \le\ & \nu||u^n(r-1)||^2 + 3\int_{r-1}^{r}|f(\theta)|^2 d\theta \\ & + 3c_1^2\int_{r-1}^{r}|Au^n(\theta)|^2||u^n(\theta)||^2 d\theta + \frac{3\nu^2|\nabla g|_\infty^2}{m_0^2}\int_{r-1}^{r}||u^n(\theta)||^2 d\theta, \end{aligned}$$

and therefore, by (2.11), (2.15), and (2.16), we obtain

$$\int_{r-1}^{r} |(u^n(\theta))'|^2 d\theta \le \rho_4(t) \tag{2.18}$$

for all $r \in [t-1,t]$, $\tau \le \tau_1(\hat{D},t)$, $u_\tau \in D(\tau)$, where $\rho_4(t)$ is given by (2.7).

From the facts that u^n converges to $u(.;\tau,u_\tau)$ weakly in $L^2(t-3,t;D(A))$, $(u^n)'$ converges to $u'(.;\tau,u_\tau)$ weakly in $L^2(t-3,t;H_g)$, and $u(.;\tau,u_\tau) \in C([t-3,t];V_g)$, using Lemma 11.2 in [15], we can pass to the limit when $n \to +\infty$ in (2.11), (2.15), (2.16), and (2.18), and it turns out that (2.3) holds. □

Remark 2.2 It is clear that under the assumptions of Lemma 2.1,

$$\lim_{t\to-\infty} e^{\mu t}\rho_1(t) = 0.$$

In other words, the family $\{\overline{B}_{H_g}(0, \rho_1^{1/2}(t)) : t \in \mathbb{R}\}$, where $\overline{B}_{H_g}(0, \rho_1^{1/2}(t))$ is the closed ball in H_g of center zero and radius $\rho_1^{1/2}(t)$, with $\rho_1(t)$ given by (2.4), belongs to $\mathcal{D}_\mu^{H_g}$.

We will denote by $\mathcal{D}_\mu^{H_g,V_g}$ the class of all families $\hat{D}_{V_g}$ of elements of $\mathcal{P}(V_g)$ of the form $\hat{D}_{V_g} = \{D(t) \cap V_g : t \in \mathbb{R}\}$, where $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{H_g}$. Now, the following result is immediate.

Lemma 2.2 *Under the assumptions of Lemma 2.1, the family*

$$\hat{D}_{0,V_g} = \left\{\overline{B}_{H_g}(0, \rho_1^{1/2}(t)) \cap V_g : t \in \mathbb{R}\right\}$$

belongs to $\mathcal{D}_\mu^{H_g,V_g}$ and satisfies that for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}_\mu^{H_g}$, there exists $\tau(\hat{D}, t) < t$ such that

$$U(t,\tau)D(\tau) \subset D_{0,V_g}(t) \quad for\ all\ \tau \le \tau(\hat{D}, t).$$

In particular, the family $\hat{D}_{0,V_g}$ is pullback $\mathcal{D}_\mu^{H_g,V_g}$-absorbing for the process U in V_g.

Now, we apply an energy method with continuous function in order to obtain the pullback asymptotic compactness in V_g for the universe $\mathcal{D}_\mu^{H_g,V_g}$.

Lemma 2.3 *Suppose that $f \in L^2_{loc}(\mathbb{R}; H_g)$ satisfies the condition (1.2). Then, the process $U(t,\tau)$ in V_g is pullback $\mathcal{D}_\mu^{H_g,V_g}$-asymptotically compact.*

Proof Let us fix $t \in \mathbb{R}$, a family $\hat{D}_{V_g} \in \mathcal{D}_\mu^{H_g,V_g}$, a sequence $\{\tau_n\}$ with $\tau_n \to -\infty$, and a sequence $\{u_{\tau_n}\} \subset V_g$ with $u_{\tau_n} \in D_{V_g}(\tau_n)$ for any n. We must prove that the sequence $\{u^n(t) = u^n(t; \tau_n, u_{\tau_n})\}$ is relatively compact in V_g.

By Lemma 2.1, we know that there exists a $\tau_1(\hat{D}_{V_g}, t) < t-3$, such that the subsequence $\{u^n : \tau_n \le \tau_1(\hat{D}_{V_g}, t)\} \subset \{u^n\}$ is uniformly bounded in $L^\infty(t-2, t; V_g) \cap L^2(t-2, t; D(A))$ with $\{(u^n)'\}$ also uniformly bounded in $L^2(t-2, t; H_g)$. Then, according to Aubin-Lions lemma [10, Chapter 1], there exists an element $u \in L^\infty(t-2, t; V_g) \cap L^2(t-2, t; D(A))$ with $u' \in L^2(t-2, t; H_g)$, such that for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{\star}{\rightharpoonup} u & \text{weak-star in } L^\infty(t-2, t; V_g), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2, t; H_g), \\ u^n \to u & \text{strongly in } L^2(t-2, t; V_g), \\ u^n(s) \to u(s) & \text{strongly in } V_g,\ \text{ a.e. } s \in (t-2, t). \end{cases} \tag{2.19}$$

Observe that $u \in C([t-2, t] : V_g)$. Due to (2.19), u satisfies (2.1) in the interval $(t-2, t)$.

From (2.19), we also deduce that $\{u^n\}$ is equicontinuous in H_g on $[t-2, t]$. Thus, taking into account that the sequence $\{u^n\}$ is uniformly bounded in $C([t-2, t]; V_g)$, by the compactness of the injection of V_g into H_g, and Ascoli-Arzela theorem, we obtain that

$$u^n \to u \quad \text{strongly in } C\left([t-2, t]; H_g\right). \tag{2.20}$$

Again, by the uniform boundedness of $\{u^n\}$ in $C([t-2, t]; V_g)$, we have that for all sequences $\{s_n\} \subset [t-2, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V_g, \tag{2.21}$$

where we have used (2.20) to identify the weak limit.

Actually, we claim that

$$u^n \to u \quad \text{strongly in } C([t-1, t]; V_g), \tag{2.22}$$

which in particular will imply the relative compactness. Indeed, if (2.22) does not hold, there exist $\epsilon > 0$, a sequence $\{t_n\} \subset [t-1, t]$ converging to some t_*, such that

$$||u^n(t_n) - u(t_*)|| \geq \epsilon \quad \forall\, n \geq 1. \tag{2.23}$$

From (2.21), we have

$$||u(t_*)|| \leq \liminf_{n\to\infty} ||u^n(t_n)||. \tag{2.24}$$

On the other hand, using the energy equality (2.2) for u and all u^n, and reasoning as for the derivation of (2.13), we have that for all $t-2 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} ||u^n(s_2)||^2 + \nu\gamma_0 \int_{s_1}^{s_2} |Au^n(r)|^2 dr &\leq ||u^n(s_1)||^2 + \frac{2}{\nu}\int_{s_1}^{s_2} |f(r)|^2 dr \\ &+ 2c_2' \int_{s_1}^{s_2} |u^n(r)|^2 ||u^n(r)||^4 dr + \frac{\nu|\nabla g|_\infty \lambda_1^{1/2}}{m_0} \int_{s_1}^{s_2} ||u^n(r)||^2 dr, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} ||u(s_2)||^2 + \nu\gamma_0 \int_{s_1}^{s_2} |Au(r)|^2 dr &\leq ||u(s_1)||^2 + \frac{2}{\nu}\int_{s_1}^{s_2} |f(r)|^2 dr \\ &+ 2c_2' \int_{s_1}^{s_2} |u(r)|^2 ||u(r)||^4 dr + \frac{\nu|\nabla g|_\infty \lambda_1^{1/2}}{m_0} \int_{s_1}^{s_2} ||u(r)||^2 dr. \end{aligned} \tag{2.26}$$

Then, we can define the functions

$$\begin{aligned} J_n(s) = {} & ||u^n(s)||^2 - \frac{2}{\nu}\int_{t-2}^{s} |f(r)|^2 dr - 2c_2' \int_{t-2}^{s} |u^n(r)|^2 ||u^n(r)||^4 dr \\ & - \frac{\nu|\nabla g|_\infty \lambda_1^{1/2}}{m_0} \int_{t-2}^{s} ||u^n(r)||^2 dr, \end{aligned}$$

$$\begin{aligned} J(s) = {} & ||u(s)||^2 - \frac{2}{\nu}\int_{t-2}^{s} |f(r)|^2 dr - 2c_2' \int_{t-2}^{s} |u(r)|^2 ||u(r)||^4 dr \\ & - \frac{\nu|\nabla g|_\infty \lambda_1^{1/2}}{m_0} \int_{t-2}^{s} ||u(r)||^2 dr. \end{aligned}$$

It is clear from the regularity of u and u^n that these functions are continuous on $[t-2,t]$. Moreover, from the definition of J_n and (2.25), we have

$$\begin{aligned} J_n(s_2)-J_n(s_1) &= ||u^n(s_2)||^2 - \frac{2}{\nu}\int_{t-2}^{s_2}|f(r)|^2dr - 2c_2'\int_{t-2}^{s_2}|u^n(r)|^2||u^n(r)||^4dr \\ &\quad -\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\int_{t-2}^{s_2}||u^n(r)||^2dr \\ &\quad -||u^n(s_1)||^2+\frac{2}{\nu}\int_{t-2}^{s_1}|f(r)|^2dr+2c_2'\int_{t-2}^{s_1}|u^n(r)|^2||u^n(r)||^4dr \\ &\quad +\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\int_{t-2}^{s_1}||u^n(r)||^2dr \\ &= ||u^n(s_2)||^2-||u^n(s_1)||^2-\frac{2}{\nu}\int_{s_1}^{s_2}|f(r)|^2dr \\ &\quad -2c_2'\int_{s_1}^{s_2}|u^n(r)|^2||u^n(r)||^4dr-\frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{m_0}\int_{s_1}^{s_2}||u^n(r)||^2dr \\ &\le -\nu\gamma_0\int_{s_1}^{s_2}|Au^n(r)|^2dr \\ &\le 0 \quad \text{for all } t-2\le s_1\le s_2\le t, \end{aligned}$$

and therefore all J_n are non-increasing functions in $[t-2,t]$. Arguing similarly as above, we deduce that J is also a non-increasing function in $[t-2,t]$. Observe now that by the last convergence in (2.19) and (2.20), $||u^n(s)||\to||u(s)||$ and $|u^n(s)|^2||u^n(s)||^4\to|u(s)|^2||u(s)||^4$, a.e. $s\in(t-2,t)$. Moreover, as the sequence $\{u^n\}$ is bounded in $L^\infty(t-2,t;V_g)\subset L^\infty(t-2,t;H_g)$, we have that the sequence $\{|u^n(s)|^2||u^n(s)||^4\}$ is bounded in $L^\infty(t-2,t)$. Therefore, from the Lebesgue dominated convergence theorem, we deduce that

$$\int_{t-2}^{s}|u^n(s)|^2||u^n(s)||^4\to\int_{t-2}^{s}|u(s)|^2||u(s)||^4 \ \text{ for all } s\in[t-2,t].$$

Thus

$$J_n(s)\to J(s) \quad \text{a.e. } s\in(t-2,t).$$

Hence, there exists a sequence $\{\widetilde{t_k}\}\subset(t-2,t_*)$, such that $\widetilde{t_k}\to t_*$ as $k\to+\infty$, and

$$\lim_{n\to+\infty}J_n(\widetilde{t_k})=J(\widetilde{t_k}) \quad \text{for all } k.$$

Fix an arbitrary value $\delta>0$. By the continuity of J, there exists k_δ, such that

$$|J(\widetilde{t_k})-J(t_*)|<\frac{\delta}{2} \quad \forall\, k\ge k_\delta.$$

Now consider $n(k_\delta)$ such that for all $n\ge n(k_\delta)$ it holds

$$t_n\ge\widetilde{t_{k_\delta}} \quad \text{and} \quad |J_n(\widetilde{t_{k_\delta}})-J(\widetilde{t_{k_\delta}})|<\frac{\delta}{2}.$$

Then, since all J_n are non-increasing, we deduce for all $n\ge n(k_\delta)$ that

$$\begin{aligned} J_n(t_n)-J(t_*) &\le J_n(\widetilde{t_{k_\delta}})-J(t_*) \\ &\le |J_n(\widetilde{t_{k_\delta}})-J(t_*)| \\ &\le |J_n(\widetilde{t_{k_\delta}})-J(\widetilde{t_{k_\delta}})|+|J(\widetilde{t_{k_\delta}})-J(t_*)|<\delta. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \sup J_n(t_n) \leq J(t_*);$$

therefore, by (2.19),

$$\lim_{n\to\infty} \sup ||u^n(t_n)|| \leq ||u(t_*)||,$$

which joined to (2.24) and (2.21) implies that $u^n(t_n) \to u(t_*)$ strongly in V_g, in contradiction with (2.23). Thus, (2.22) holds and the relative compactness of $\{u(t; \tau_n, u_{\tau_n})\}$ in V_g is proved. □

Theorem 2.2 *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H_g)$ satisfies (1.2). Then, the process $U(t,\tau)$ defined in V_g has a pullback $\mathcal{D}_\mu^{H_g,V_g}$-attractor*

$$\hat{\mathcal{A}}_{\mathcal{D}_\mu^{H_g,V_g}} = \left\{ \mathcal{A}_{\mathcal{D}_\mu^{H_g,V_g}}(t) : t \in \mathbb{R} \right\}.$$

Proof The existence of the pullback attractor is a direct consequence of Theorem 7 in [6], Lemmas 2.2 and 2.3. □

3 Exponential growth of the pullback attractor

We now prove the exponential growth of the pullback attractor

$$\hat{\mathcal{A}}_{\mathcal{D}_\mu^{H_g,V_g}} = \left\{ \mathcal{A}_{\mathcal{D}_\mu^{H_g,V_g}}(t) : t \in \mathbb{R} \right\}$$

in the spaces V_g and $H^2(\Omega, g) = \left(H^2(\Omega)\right)^2$.

Theorem 3.1 *Suppose that $f \in L^2_{loc}(\mathbb{R}; H_g)$ satisfies*

$$\sup_{s\leq 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} |f(\theta)|^2 d\theta \right) < +\infty. \tag{3.1}$$

Then

$$\lim_{t\to -\infty} \left(e^{\mu t} \sup_{v \in \mathcal{A}_{\mathcal{D}_\mu^{H_g,V_g}}(t)} ||v||^2 \right) = 0.$$

Proof First, note that $\hat{\mathcal{A}}_{\mathcal{D}_\mu^{H_g,V_g}} \in \mathcal{D}_\mu^{H_g}$. The result is now a consequence of the invariance of $\mathcal{A}_{\mathcal{D}_\mu^{H_g,V_g}}$, the second estimate in (2.3) and the tempered character of the expression (2.5). Since $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H_g)$, condition (3.1) is equivalent to

$$\sup_{s\leq t} \int_{s-1}^{s} |f(\theta)|^2 d\theta < +\infty \quad \forall t \in \mathbb{R}.$$

□

Lemma 3.1 *If $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; H_g)$ and satisfies (1.2), then for each $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_\mu^{H_g}$ there exists $\tau_1(\hat{D}, t) < t - 3$ such that*

$$|AU(r,\tau)u_\tau|^2 \leq \rho_6(t) \ \textit{for all}\ r \in [t-1, t],\ \tau \leq \tau_1(\hat{D}, t),\ \ u_\tau \in D(t),$$

where

$$\rho_6(t) = \frac{4}{\nu^2\gamma_0^2}\left(\rho_5(t) + \max_{r\in[t-1,t]} |f(r)|^2\right) + \frac{2c_1'}{\nu\gamma_0}\rho_1(t)\rho_2^2(t) + \frac{|\nabla g|_\infty \lambda_1^{1/2}}{2m_0\gamma_0}\rho_2(t), \quad (3.2)$$

and

$$\rho_5(t) = \left(\rho_4(t) + \frac{1}{\nu\gamma_0\lambda_1}\int_{t-2}^{t} |f'(\theta)|^2 d\theta\right)\exp\left(\frac{c_1^2}{\nu\gamma_0}\rho_2(t)\right). \quad (3.3)$$

Proof As $f \in W^{1,2}_{\text{loc}}(\mathbb{R}; H_g)$, we can differentiate with respect to time in (2.8), and then multiply by $\gamma_{nj}'(s)$, and sum from $j = 1$ to n to obtain

$$\begin{aligned}
&\frac{d}{d\theta}|(u^n(\theta))'|^2 + 2\nu||(u^n(\theta))'||^2 \\
&\quad = 2\left(f'(\theta), (u^n(\theta))'\right)_g - 2\nu\left(C(u^n(\theta))', (u^n(\theta))'\right)_g - 2b\left((u^n(\theta))', u^n(\theta), (u^n(\theta))'\right) \\
&\quad \le \nu\gamma_0||(u^n(\theta))'||^2 + \frac{1}{\nu\gamma_0\lambda_1}|f'(\theta)|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||(u^n(\theta))'||^2 \\
&\qquad + 2c_1|(u^n(\theta))'|||(u^n(\theta))'||||u^n(\theta)||. \qquad (3.4)
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{d}{d\theta}|(u^n(\theta))'|^2 + 2\nu\gamma_0||(u^n(\theta))'||^2 \qquad (3.5)\\
&\le \nu\gamma_0||(u^n(\theta))'||^2 + \frac{1}{\nu\gamma_0\lambda_1}|f'(\theta)|^2 + \nu\gamma_0||(u^n(\theta))'||^2 + \frac{c_1^2}{\nu\gamma_0}|(u^n(\theta))'|^2||(u^n(\theta))||^2,
\end{aligned}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$. Then, we obtain

$$\frac{d}{d\theta}|(u^n(\theta))'|^2 \le \frac{1}{\nu\gamma_0\lambda_1}|f'(\theta)|^2 + \frac{c_1^2}{\nu\gamma_0}|(u^n(\theta))'|^2||(u^n(\theta))||^2. \quad (3.6)$$

Integrate the inequality

$$|(u^n(r))'|^2 \le |(u^n(s))'|^2 + \frac{1}{\nu\gamma_0\lambda_1}\int_{r-1}^{r}|f'(\theta)|^2 d\theta + \frac{c_1^2}{\nu\gamma_0}\int_{r-1}^{r}|(u^n(\theta))'|^2||(u^n(\theta))||^2 d\theta$$

for all $\tau \le r-1 \le s \le r$. Thus, by Gronwall's inequality,

$$|(u^n(r))'|^2 \le (|(u^n(s))'|^2 + \frac{1}{\nu\gamma_0\lambda_1}\int_{r-1}^{r}|f'(\theta)|^2 d\theta)\exp\left(\frac{c_1^2}{\nu\gamma_0}\int_{r-1}^{r}||(u^n(\theta))||^2 d\theta\right)$$

for all $\tau \le r-1 \le s \le r$.

Now, integrating this inequality with respect to s between $r-1$ and r, we obtain

$$\begin{aligned}
|(u^n(r))'|^2 \le &\left(\int_{r-1}^{r}|(u^n(s))'|^2 ds + \frac{1}{\nu\gamma_0\lambda_1}\int_{r-1}^{r}|f'(\theta)|^2 d\theta\right) \\
&\times\exp\left(\frac{c_1^2}{\nu\gamma_0}\int_{r-1}^{r}||(u^n(\theta))||^2 d\theta\right)
\end{aligned}$$

for all $\tau \leq r-1$ and any $n \geq 1$. Therefore, by (2.15) and (2.18), we deduce for any $n \geq 1$ that

$$|(u^n(r))'(r;\tau,u_\tau)|^2 \leq \rho_5(t) \text{ for all } r \in [t-1,t], \quad \tau \leq \tau_1(\hat{D},t), \quad u_\tau \in D(\tau), \tag{3.7}$$

where $\rho_5(t)$ is given by (3.3).

Finally, multiplying again in (2.8) by $\lambda_j \gamma_{nj}(r)$, and summing once more from $j=1$ to n, we obtain

$$\left((u^n(r))', Au^n(r)\right)_g + \nu|Au^n(r)|^2 + \nu\left(Cu^n(r), Au^n(r)\right)_g + b\left(u^n(r), u^n(r), Au^n(r)\right) = \left(f(r), u^n(r)\right)_g.$$

Using Lemmas 2.1 and 2.3 in [4], we have

$$\begin{aligned}\nu|Au^n(r)|^2 &= \left(f(r), Au^n(r)\right)_g - \left((u^n(r))', Au^n(r)\right)_g \\ &\quad -\nu\left(Cu^n(r), Au^n(r)\right)_g - b\left(u^n(r), u^n(r), Au^n(r)\right) \\ &\leq \left(f(r), Au^n(r)\right)_g - \left((u^n(r))', Au^n(r)\right)_g \\ &\quad + \frac{\nu|\nabla g|_\infty}{m_0}||u^n(r)|||Au^n(r)| + c_1|u^n(r)|^{1/2}||u^n(r)|||Au^n(r)|^{3/2}.\end{aligned}$$

Using Cauchy inequality, we obtain

$$\begin{aligned}\nu|Au^n(r)|^2 &\leq \left(f(r), Au^n(r)\right)_g - \left((u^n(r))', Au^n(r)\right)_g + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|Au^n(r)|^2 \\ &\quad + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{4m_0}||u^n(r)||^2 + c_1|u^n(r)|^{1/2}||u^n(r)|||Au^n(r)|^{3/2},\end{aligned}$$

or

$$\begin{aligned}\nu\gamma_0|Au^n(r)|^2 &\leq (f(r), Au^n(r))_g - ((u^n(r))', Au^n(r))_g \\ &\quad + c_1|u^n(r)|^{1/2}||u^n(r)|||Au^n(r)|^{3/2} + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{4m_0}||u^n(r)||^2,\end{aligned}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$.

Using Cauchy inequality and Young inequality, we have

$$\begin{aligned}\nu\gamma_0|Au^n(r)|^2 &\leq \frac{\nu\gamma_0}{8}|Au^n(r)|^2 + \frac{2}{\nu\gamma_0}|f(r)|^2 \\ &\quad + \frac{\nu\gamma_0}{8}|Au^n(r)|^2 + \frac{2}{\nu\gamma_0}|(u^n(r))'|^2 + \frac{\nu\gamma_0}{4}|Au^n(r)|^2 \\ &\quad + c_1'|u^n(r)|^2||u^n(r)||^4 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{4m_0}||u^n(r)||^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\nu\gamma_0}{2}|Au^n(r)|^2 \leq& \frac{2}{\nu\gamma_0}\left(|f(r)|^2 + |(u^n(r))'|^2\right) \\ &+ c_1'|u^n(r)|^2||u^n(r)||^4 + \frac{\nu|\nabla g|_\infty\lambda_1^{1/2}}{4m_0}||u^n(r)||^2\end{aligned} \tag{3.8}$$

for all $\tau \leq r$. Thus, since in particular $f \in C(\mathbb{R}; H_g)$, from (2.11), (2.15), and (3.7), we deduce for any $n \geq 1$ that

$$|Au^n(r;\tau,u_\tau)|^2 \leq \rho_6(t) \quad \text{for all } r \in [t-1,t], \quad \tau \leq \tau_1(\hat{D},t), \quad u_\tau \in D(\tau), \tag{3.9}$$

where $\rho_6(t)$ is given by (3.2).

The result now is a consequence of Lemma 11.2 in [15] and (3.9), taking into account the well-known facts that $u^n(.;\tau,u_\tau)$ converges weakly to $u(.;\tau,u_\tau)$ in $L^2(t-1,t;V_g)$, and $u(.;\tau,u_\tau) \in C([t-1,t];V_g)$. □

Now, we can obtain a result about exponential growth in $H^2(\Omega, g)$ of the pullback attractor.

Theorem 3.2 *Suppose that* $f \in W^{1,2}_{\text{loc}}(\mathbb{R}; H_g)$ *satisfies* (3.1)*, and moreover*

$$\lim_{t\to-\infty}\left(e^{\mu t}\int_{t-1}^{t}|f'(\theta)|^2 d\theta\right) = 0, \tag{3.10}$$

and

$$\lim_{t\to-\infty}\left(e^{\mu t}|f(t)|^2\right) = 0. \tag{3.11}$$

Then

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_\mu^{H_g},V_g}(t)}||v||^2_{H^2(\Omega,g)}\right) = 0.$$

Proof Observe that

$$|f(r)| \leq |f(r-1)| + \left(\int_{t-1}^{t}|f'(\theta)|^2 d\theta\right)^{1/2} \quad \text{for all } r \in [t-1,t].$$

Thus, taking into account (3.10) and (3.11), we get the result from the invariance of $\hat{\mathcal{A}}_{\mathcal{D}_\mu^{H_g},V_g}$, Lemma 3.1, (2.4), (2.5), and (2.7). □

Remark 3.1 In fact, we obtain the exponential growth in V_g and $H^2(\Omega, g)$ for any family $\hat{D} \in \mathcal{D}_\mu^{H_g}$ invariant with respect to process $U(t,\tau)$, not necessary to be a pullback attractor.

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