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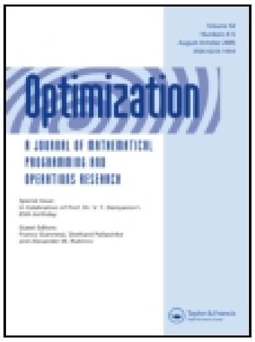

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A projection algorithm for solving pseudomonotone equilibrium problems and its application to a class of bilevel equilibria

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We propose a projection algorithm for solving an equilibrium problem (EP) where the bifunction is pseudomonotone with respect to its solution set. The algorithm is further combined with a cutting technique for minimizing the norm over the solution set of an EP whose bifunction is pseudomonotone with respect to the solution set.

Keywords: pseudomonotone equilibria; Ky Fan inequality; auxiliary subproblem principle; projection method; Armijo linesearch; bilevel equilibria

AMS Subject Classifications: 2010; 65 K10; 90 C25

1. Introduction and motivation

Let C be a non-empty closed convex subset in the Euclidean space $\boldsymbol{R}^n$ and $\Omega \subseteq \boldsymbol{R}^n$ be an open convex set containing C, and $f : \Omega \times \Omega \to \boldsymbol{R}$ be a bifunction such that $f(x, x) = 0$ for every $x \in C$. As usual, we call such a bifunction an *equilibrium bifunction*. Consider the equilibrium problem, shortly (EP)

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{EP}$$

This problem is also often called the *Ky Fan inequality* due to his contribution to this field.

EP is an important subject that recently has been considered in many research papers. It is well known [1,2] that various classes of optimization, variational inequality, fixed point, Nash equilibria in non-cooperative game theory and minimax problems can be formulated as an equilibrium problem of the form EP. There are several solution approaches that have been developed for EPs among them the projection is one of fundamental methods. It has been shown (see e.g. [3]) that the projection method, in general, is not convergent for the monotone variational inequality, which is a special case of monotone EPs. In order to obtain convergent projection algorithms, the extragradient (or double projection) algorithms have been proposed. The first extragradient method has been proposed by Korpelevich in [4] for convex optimization and saddle point problems. This method has been further extended to pseudomonotone variational inequalities and equilibrium problems.[3,5–7] To enhance convergence of double projection algorithms, recently hybrid projection-cutting algorithms have been proposed for pseudomonotone inclusions and variational inequalities.[8,9]

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Another fundamental approach to optimization, variational inequality and EPs is the Tikhonov regularization.[10] The Tikhonov regularization method was applied to pseudomonotone variational inequalities and EPs (see e.g. [11–15] and the references therein). Unlike the monotonicity case, in this case the regularized subproblems, in general, do not inherit any monotonicity property from the original problem, and therefore the existing solution methods that require monotonicity properties cannot be directly applied to solve regularized subproblems as in the case of monotone problems. However, it has been proved (see e.g. in [11]) that any Tikhonov trajectory tends to the same limit which is the projection of the starting point onto the solution set of the original pseudomonotone equilibrium problem. This result suggests that in order to obtain the limit point in the Tikhonov regularization method for pseudomonotone EPs, one can minimize the Euclidean norm over the solution set of the original pseudomonotone EP. The latter bilevel problem is a special case of mathematical programs with equilibrium constraints that have been considered intensively in recent years (see e.g. [16–20]).

The purpose of this paper is twofold. First, we extend the projection algorithm developed by Solodov and Svaiter in [9] to EP where the bifunction f is pseudomonotone on C with respect to its solution. Our extension is motivated by the fact reported in [9] that this algorithm works well for pseudomonotone variational inequality problems when the projection onto the feasible domain C is computationally expensive. Next, we combine this algorithm with a cutting technique developed in [21] to minimizing the Euclidean norm over the solution set of the EP. As mentioned before, the latter bilevel problem arises in the Tikhonov regularization method for pseudomonotone EPs.

The paper is organized as follows. The next section contains preliminaries on the Euclidean projection and EPs. The third section is devoted to presentation of the algorithm and its convergence. In section four, we describe an algorithm for minimizing the Euclidean norm over the solution set of an EP, where the bifunction is pseudomonotone with respect to its solution set. The last section is devoted to present an application of the proposed algorithm for Nash-cournot equilibrium models of electricity markets and its implementation.

2. Preliminaries

Throughout the paper, by P_C we denote the projection operator on C with the norm $\|.\|$, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \le \|y - x\| \quad \forall y \in C.$$

The following well-known results on the projection operator onto a closed convex set will be used in the sequel.

LEMMA 2.1 *Suppose that C is a nonempty closed convex set in $I\!R^n$. Then*

(i) *$P_C(x)$ is singleton and well defined for every x;*
(ii) *$\pi = P_C(x)$ if and only if $\langle x - \pi, y - \pi\rangle \le 0, \quad \forall y \in C$;*
(iii) $\|P_C(x) - P_C(y)\|^2 \le \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \quad \forall x, y \in C.$

We recall some well-known definitions on monotonicity (see e.g. [1–3,9,22]).

Definition 2.1 A bifunction $g : C \times C \to I\!R$ is said to be

(a) strongly monotone on C with modulus $\gamma > 0$, if

$$g(x, y) + g(y, x) \leq -\gamma \|x - y\|^2 \quad \forall x, y \in C;$$

(b) monotone on C if

$$g(x, y) + g(y, x) \leq 0 \quad \forall x, y \in C;$$

(c) pseudomonotone on C if

$$g(x, y) \geq 0 \Longrightarrow g(y, x) \leq 0 \quad \forall x, y \in C;$$

(d) pseudomonotone on C with respect to x^* if

$$g(x^*, y) \geq 0 \Longrightarrow g(y, x^*) \leq 0 \quad \forall y \in C.$$

We say that g is pseudomonotone on C with respect to a set S if it is pseudomonotone on C with respect to every point $x^* \in S$.

From the definitions, it follows that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\forall x^* \in C$.
In the sequel, we need the following blanket assumptions

(A1) $f(., y)$ is continuous on Ω for every $y \in C$;
(A2) $f(x, .)$ is lower semicontinuous, subdifferentiable and convex on Ω for every $x \in C$;
(A3) f is pseudomonotone on C with respect to the solution set S of (EP).

LEMMA 2.2 *Suppose Problem* (EP) *has a solution. Then under Assumptions* (A1)*,* (A2) *and* (A3) *the solution set S is closed, convex and*

$$f(x^*, y) \geq 0 \quad \forall y \in C \text{ if and only if } f(y, x^*) \leq 0 \quad \forall y \in C.$$

The proof of this lemma when f is pseudomonotone on C can be found, for instance, in [2,22]. When f is pseudomonotone with respect to the solution set of (EP), it can be done by the same way. So we omit it.

The following lemmas are well known from the auxiliary problem principle for EPs.

LEMMA 2.3 [23] *Suppose that G is a continuously differentiable and strongly convex function on C with modulus $\delta > 0$. Then under Assumptions* (A1) *and* (A2)*, a point $x^* \in C$ is a solution of* (EP) *if and only if it is a solution to the* EP*:*

$$\text{Find } x^* \in C : f(x^*, y) + G(y) - G(x^*) - \langle \nabla G(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C. \qquad \text{(AEP)}$$

The function

$$D(x, y) := G(y) - G(x) - \langle \nabla G(x), y - x \rangle$$

is called Bregman function. Such a function was used to define a generalized projection, called D-projection, which was used to develop algorithms for particular problems, see e.g. [24]. An important case is $G(x) := \frac{1}{2}\|x\|^2$. In this case, D-projection becomes the Euclidean one.

LEMMA 2.4 [23] *Under Assumptions* (A1), (A2), *a point* $x^* \in C$ *is a solution of Problem* (AEP) *if and only if*

$$x^* = \operatorname{argmin}\{f(x^*, y) + G(y) - G(x^*) - \langle \nabla G(x^*), y - x^* \rangle : \ y \in C\}. \tag{CP}$$

Note that, since $f(x, .)$ is convex and G is strongly convex, Problem (CP) is a strongly convex program.

For each $z \in C$, by $\partial_2 f(z, z)$ we denote the subgradient of the convex function $f(z, .)$ at z, i.e.

$$\begin{aligned}\partial_2 f(z, z) &:= \{w \in I\!R^n : f(z, y) \geq f(z, z) + \langle w, y - z \rangle \ \forall y\} \\ &= \{w \in I\!R^n : f(z, y) \geq \langle w, y - z \rangle \ \forall y\},\end{aligned}$$

and we define the halfspace H_z as

$$H_z := \{x \in I\!R^n : \ \langle g, x - z \rangle \leq 0\} \tag{2.1}$$

where $g \in \partial_2 f(z, z)$. Note that when $f(x, y) = \langle F(x), y - x \rangle$, this halfspace becomes the one introduced in [9]. The following lemma says that the hyperplane does not cut off any solution of problem (EP).

LEMMA 2.5 *Under Assumptions* (A2) *and* (A3), *one has* $S \subseteq H_z$ *for every* $z \in C$.

Proof Suppose $x^* \in S$. From $g \in \partial_2 f(z, z)$, by convexity of $f(z, .)$, it follows that

$$\langle g, x^* - z \rangle \leq f(z, x^*) - f(z, z) \leq f(z, x^*) \quad \forall y \in C.$$

Since $x^* \in S$ we have $f(x^*, z) \geq 0$. Then, by pseudomonotonicity of f with respect to x^*, it follows that $f(z, x^*) \leq 0$. Thus $\langle g, x^* - z \rangle) \leq 0$, which implies $x^* \in H_z$. □

LEMMA 2.6 *Under Assumptions* (A1) *and* (A2), *if* $\{z^k\} \subset C$ *is a sequence such that* $\{z^k\}$ *converges to* $\bar{z}$ *and the sequence* $\{g^k \in \partial_2 f(z^k, z^k)\}$ *converges to* $\bar{g}$, *then* $\bar{g} \in \partial_2 f(\bar{z}, \bar{z})$.

Proof Let $g^k \in \partial_2 f(z^k, z^k)$. Then

$$f(z^k, y) \geq f(z^k, z^k) + \langle g^k, y - z^k \rangle = \langle g^k, y - z^k \rangle \quad \forall y \in C.$$

Taking the limit as $k \to \infty$ on both sides of the above inequality, by the upper semicontinuity of $f(., y)$ with respect to the first argument, we obtain

$$f(\bar{z}, y) \geq \limsup_{k\to\infty} f(z^k, y) \geq \lim_{k\to\infty} \langle g^k, y - z^k \rangle = \langle \bar{g}, y - \bar{z} \rangle \quad \forall y \in C$$

which, together with $f(\bar{z}, \bar{z}) = 0$, implies that $\bar{g} \in \partial_2 f(\bar{z}, \bar{z})$. □

We need the following lemma.

LEMMA 2.7 [9] *Suppose that* $x \in C$ *and* $u = P_{C \cap H_z}(x)$. *Then*

$$u = P_{C \cap H_z}(\bar{x}), \ \textit{where} \ \bar{x} = P_{H_z}(x).$$

We give here a simple proof for this lemma, which is other than that in [9].

Proof Let $w = P_{C \cap H_z}(\bar{x})$. We show that $w = u$. Indeed, suppose contradiction that $w \neq u$, then by the property of the projection onto a closed convex set, we have $\|\bar{x} - w\| < \|\bar{x} - u\|$. By Pythagoras's theorem, $\|x - u\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - u\|^2$ and $\|x - w\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - w\|^2$. Combining with $\|x - u\| < \|x - w\|$ we obtain $\|\bar{x} - u\| < \|\bar{x} - w\|$, which contradicts to $\|\bar{x} - w\| < \|\bar{x} - u\|$. □

3. A projection algorithm for EPs

The following algorithm can be considered as an extension of Solodov-Svaiter's algorithm [9] to Problem (EP).

Algorithm 1. Pick $x^0 \in C$ and choose two parameters $\eta \in (0, 1)$, $\rho > 0$. At each iteration $k = 0, 1, \ldots$ having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min \left\{ f(x^k, y) + \frac{1}{\rho}\Big[G(y) - G(x^k) - \langle \nabla G(x^k), y - x^k\rangle\Big] : \ y \in C \right\} \qquad \mathrm{CP}(x^k)$$

to obtain its unique solution y^k.

If $f(x^k, y^k) + \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] \geq 0$, terminate: x^k is a solution of (EP). Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m satisfying

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k : \\ \langle g^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] \\ \text{with } \ g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \tag{3.1}$$

Step 3. Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}$, $g^k := g^{k,m_k}$. Take

$$C_k := \{x \ \in C : \langle g^k, x - z^k\rangle \leq 0\}, \ x^{k+1} := P_{C_k}(x^k), \tag{3.2}$$

and go to Step 1 with k is replaced by $k + 1$.

Remark 3.1

(i) If the algorithm terminates at Step 1, i.e.

$$f(x^k, y^k) + \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] \geq 0,$$

then

$$f(x^k, y) + \frac{1}{\rho}\Big[G(y) - G(x^k) - \langle \nabla G(x^k), y - x^k\rangle\Big] \geq 0 \quad \forall y \in C.$$

Thus, by Lemma 2.3, x^k is a solution to (EP).

(ii) $g^k \neq 0 \ \forall k$, indeed, at the begining of Step 2, $x^k \neq y^k$. By the Armijo linesearch rule and δ-strong convexity of G, we have

$$\begin{aligned} \langle g^k, x^k - y^k\rangle &\geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] \\ &\geq \frac{\delta}{\rho}\|x^k - y^k\|^2 > 0. \end{aligned}$$

(iii) To implement the linesearch rule, at each iteration k, for a positive integer number m, one can check the inequality

$$\langle g^{k,m}, x^k - y^k \rangle \geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k \rangle\Big]$$

with any $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$. If this inequality is satisfied, we are done. Otherwise, one increases m by one and check again the inequality with $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ for the new m. As we will show in Lemma 3.1 below that, for each iteration k, there exists an integer number $m > 0$ such that the inequality in the linesearch rule is satisfied for every $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$. So, to implement the linesearch rule, one needs to know only one subgradient.

Now we are going to analyse the validity and convergence of the algorithm. Our proofs are based on the proof scheme in [9] (see also [25]).

LEMMA 3.1 *Under Assumptions* (A1), (A2)*, the linesearch rule* (3.1) *is well-defined in the sense that, at each iteration k, there exists an integer number* $m > 0$ *satisfying the inequality in* (3.1) *for every* $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$*, and if, in addition Assumption* (A3) *is satisfied, then for every solution* x^* *of* (EP)*, one has*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - \bar{x}^k\|^2 - \left(\frac{\eta_k \delta}{\rho \|g^k\|}\right)^2 \|x^k - y^k\|^4 \quad \forall k, \tag{3.3}$$

where $\bar{x}^k = P_{H_{z^k}}(x^k)$.

Proof First, we prove that there exists a positive integer m_0 such that

$$\langle g^{k,m_0}, x^k - y^k \rangle \geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k \rangle\Big]$$
$$\forall g^{k,m_0} \in \partial_2 f(z^{k,m_0}, z^{k,m_0}).$$

Indeed, suppose by contradiction that, for every positive integer m and $z^{k,m} = (1-\eta^m)x^k + \eta^m y^k$ there exists $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ such that

$$\langle g^{k,m}, x^k - y^k \rangle < \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k \rangle\Big].$$

Since $z^{k,m} \to x^k$ as $m \to \infty$, by Theorem 24.5 in [26], the sequence $\{g^{k,m}\}_{m=1}^{\infty}$ is bounded. Thus we may assume that $g^{k,m} \to \bar{g}$ for some $\bar{g}$. Taking the limit as $m \to \infty$, from $z^{k,m} \to x^k$ and $g^{k,m} \to \bar{g}$, by Lemma 2.6, it follows that $\bar{g} \in \partial_2 f(x^k, x^k)$ and

$$\langle \bar{g}, x^k - y^k \rangle \leq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k \rangle\Big]. \tag{3.4}$$

Since $\bar{g} \in \partial_2 f(x^k, x^k)$, we have

$$f(x^k, y^k) \geq f(x^k, x^k) + \langle \bar{g}, y^k - x^k \rangle = \langle \bar{g}, y^k - x^k \rangle.$$

Combining with (3.4) yields

$$f(x^k, y^k) + \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k \rangle\Big] \geq 0,$$

which contradicts to the fact that

$$f(x^k, y^k) + \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] < 0.$$

Therefore, the linesearch is well defined.

Now we prove (3.3). For simplicity of notation, let $d^k := x^k - y^k$, $H_k := H_{z^k}$. Since $x^{k+1} = P_{C\cap H_k}(\bar{x}^k)$ and $x^* \in S$, by Lemma 2.5, $x^* \in C \cap H_k$, then

$$\|x^{k+1} - \bar{x}^k\|^2 \leq \langle x^* - \bar{x}^k, x^{k+1} - \bar{x}^k\rangle$$

which together with

$$\|x^{k+1} - x^*\|^2 = \|\bar{x}^k - x^*\|^2 + \|x^{k+1} - \bar{x}^k\|^2 + 2\langle x^{k+1} - \bar{x}^k, \bar{x}^k - x^*\rangle$$

implies

$$\|x^{k+1} - x^*\|^2 \leq \|\bar{x}^k - x^*\|^2 - \|x^{k+1} - \bar{x}^k\|^2. \tag{3.5}$$

Replacing

$$\bar{x}^k = P_{H_k}(x^k) = x^k - \frac{\langle g^k, x^k - z^k\rangle}{\|g^k\|^2} g^k$$

into (3.5) we obtain

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2-\|x^{k+1}-\bar{x}^k\|^2-2\langle g^k, x^k-x^*\rangle\frac{\langle g^k, x^k - z^k\rangle}{\|g^k\|^2}+\frac{\langle g^k, x^k - z^k\rangle^2}{\|g\|^2}.$$

Substituting $x^k = z^k + \eta_k d^k$ into the last inequality, we get

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\leq \|x^k-x^*\|^2 - \|x^{k+1}-\bar{x}^k\|^2 + \left(\frac{\eta_k\langle g^k, d^k\rangle}{\|g^k\|}\right)^2 - \frac{2\eta_k\langle g^k, d^k\rangle}{\|g^k\|^2}\langle g^k, x^k - x^*\rangle \\ &= \|x^k - x^*\|^2 - \|x^{k+1} - \bar{x}^k\|^2 - \left(\frac{\eta_k\langle g^k, d^k\rangle}{\|g^k\|}\right)^2 - \frac{2\eta_k\langle g^k, d^k\rangle}{\|g^k\|^2}\langle g^k, z^k - x^*\rangle.\end{aligned}$$

In addition, by the Armijo linesearch rule, using the δ-strong convexity of G we have

$$\langle g^k, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle\nabla G(x^k), y^k - x^k\rangle\Big] \geq \frac{\delta}{\rho}\|x^k - y^k\|^2.$$

Note that $x^* \in H_k$ we can write

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - \bar{x}^k\|^2 - \left(\frac{\eta_k\delta}{\rho\|g^k\|}\right)^2\|x^k - y^k\|^4$$

as desired. □

THEOREM 3.1 *Suppose that Problem* (EP) *admits a solution and that f is jointly continuous on Ω. Then under Assumptions* (A2), (A3) *the sequence $\{x^k\}$ generated by Algorithm 1 converges to a solution of* (EP).

Proof Let x^* be any solution of (EP). By Lemma 3.1,

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - \bar{x}^k\|^2 - \left(\frac{\eta_k\delta}{\rho\|g^k\|}\right)^2\|x^k - y^k\|^4,$$

which implies that the sequence $\{\|x^k - x^*\|\}$ is nonincreasingly convergent. Thus, we can deduce that the sequences $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ are bounded. Taking the limit on both sides of (3.3), we get

$$\lim_{k\to\infty} \eta_k \|x^k - y^k\| = 0. \tag{3.6}$$

We will consider two distinct cases:

Case 1 $\limsup_{k\to\infty} \eta_k > 0$. Then there exists $\bar{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}\ \forall i$, and, by (3.6), one has

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{3.7}$$

Since $\{x^k\}$ is bounded, we may assume that x^{k_i} converges to some $\bar{x}$ as $i \to \infty$. From (3.7), $y^{k_i} \to \bar{x}$ as $i \to \infty$, and therefore $z^{k_i} \to \bar{x}$. By definition of y^{k_i}, we have

$$\begin{aligned} &f(x^{k_i}, y) + \frac{1}{\rho}\Big[G(y) - G(x^{k_i}) - \langle \nabla G(x^{k_i}), y - x^{k_i}\rangle\Big] \\ &\quad\geq f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}\Big[G(y^{k_i}) - G(x^{k_i}) - \langle \nabla G(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big] \quad \forall y \in C. \end{aligned}$$

Letting $i \to \infty$, by strong convexity of G and continuity of f, ∇G, we obtain in the limit that

$$\begin{aligned} &f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - \bar{x}\rangle\Big] \\ &\quad\geq f(\bar{x}, \bar{x}) + \frac{1}{\rho}\Big[G(\bar{x}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{x} - \bar{x}\rangle\Big] \end{aligned}$$

Hence

$$f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - \bar{x}\rangle\Big] \geq 0 \quad \forall y \in C$$

which means that $\bar{x}$ is a solution of (EP). Applying (3.3) with $x^* = \bar{x}$, we see that the sequence $\{\|x^k - \bar{x}\|\}$ converges. Since $\|x^{k_i} - \bar{x}\| \to 0$, we can conclude that the whole sequence $\{x^k\}$ converges to $\bar{x} \in S$.

Case 2 $\lim_{k\to\infty} \eta_k = 0$. According to the algorithm, we have

$$z^k = (1 - \eta_k)x^k + \eta_k y^k.$$

As before, we may assume that the subsequence $\{x^{k_i}\} \subset \{x^k\}$ converges to some point $\bar{x}$. By the same arguments as above we see that the sequence $\{y^k\}$ is bounded. Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^{k_i}\}$ converges to some point $\bar{y}$. From the definition of y^{k_i} we can write

$$\begin{aligned} &f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}\Big[G(y^{k_i}) - G(x^{k_i}) - \langle \nabla G(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big] \\ &\quad\leq f(x^{k_i}, y) + \frac{1}{\rho}\Big[G(y) - G(x^{k_i}) - \langle \nabla G(x^{k_i}), y - x^{k_i}\rangle\Big], \quad \forall y \in C. \end{aligned}$$

Taking the limit as $i \to \infty$, by lower semicontinuity of $f(.,.)$ and upper semicontinuity of $f(., y)$ we have

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big]$$
$$\leq f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - \bar{x}\rangle\Big] \quad \forall y \in C. \tag{3.8}$$

In the other hand, by the Armijo linesearch rule (3.1), for $m_{k_i} - 1$, there exists $g^{k_i, m_{k_i}-1} \in \partial_2 f(z^{k_i, m_{k_i}-1}, z^{k_i, m_{k_i}-1})$ such that

$$\langle g^{k_i, m_{k_i}-1}, x^{k_i} - y^{k_i}\rangle < \frac{1}{\rho}\Big[G(y^{k_i}) - G(x^{k_i}) - \langle \nabla G(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big]$$

Since $z^{k_i, m_{k_i}-1} \to \bar{x}$ as $i \to \infty$, by Theorem 24.5 in [26] we have that the sequence $\{g^{k_i, m_{k_i}-1}\}$ is bounded. Combining this fact with lemma 2.6 that we may assume that $g^{k_i, m_{k_i}-1} \to \bar{g} \in \partial_2 f(\bar{x}, \bar{x})$, and thus the above inequality becomes

$$\langle \bar{g}, \bar{x} - \bar{y}\rangle \leq \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big]. \tag{3.9}$$

From $\bar{g} \in \partial_2 f(\bar{x}, \bar{x})$ follows $f(\bar{x}, y) \geq f(\bar{x}, \bar{x}) + \langle \bar{g}, y - \bar{x}\rangle$ $\forall y \in C$. In particular, $\langle \bar{g}, \bar{x} - \bar{y}\rangle \geq -f(\bar{x}, \bar{y})$. Combining with (3.9), we get

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big] \geq 0. \tag{3.10}$$

From (3.8) and (3.10), we have

$$0 \leq f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - x^{k_i}\rangle\Big] \quad \forall y \in C,$$

which implies that $\bar{x}$ is a solution of (EP). Now we can apply (3.3) with $x^* = \bar{x}$, by the same arguments as above, we can conclude that the whole sequence $\{x^k\}$ converges to $\bar{x} \in S$. $\square$

4. Application to minimizing the Euclidean norm with pseudomomotone equilibrium constraints

In this section, we combine Algorithm 1 with a cutting technique in order to obtain an algorithm for solving the following optimization problem

$$\min\{\|x - x^g\| : x \in S\}, \tag{BP}$$

where $x^g \in C$ is given (plays the role of a guess solution) and S is the solution set of Problem (EP). It is well known that under Assumptions (A1), (A2) and (A3), the solution set S of (EP) is a closed convex set. We emphasize that the main difficulty in Problem (BP) is that its feasible domain S, although is convex, it is not given explicitly as in a standard mathematical programming problem. In the sequel, we always suppose that Assumptions (A1), (A2) and (A3) are satisfied.

Algorithm 2. Take $x^1 := x^g \in C$ and choose parameters $\rho > 0$, $\eta, \sigma \in (0, 1)$.
At each iteration $k = 1, 2, ..$ having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min\left\{f(x^k, y) + \frac{1}{\rho}\Big[G(y) - G(x^k) - \langle\nabla G(x^k), y - x^k\rangle\Big] :\ y \in C\right\} \qquad \mathrm{CP}(x^k)$$

to obtain its unique solution y^k.

If $x^k = y^k$, take $u^k := x^k$ and go to Step 4.

Step 2. Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k : \\ \langle g^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle\nabla G(x^k), y^k - x^k\rangle\Big] \\ \text{with } \ g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \tag{4.1}$$

Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}$, $g^k = g^{k,m}$.

Step 3. Take $u^k := P_{C_k}(x^k)$, where

$$C_k := \{x \in C : \langle g^k, x - z^k\rangle \leq 0\}. \tag{4.2}$$

Step 4. Define the two polyhedral convex sets

$$B_k := \{x : \|u^k - x\| \leq \|x^k - x\|\}, \tag{4.3}$$

$$D_k := \{x : \langle x - x^k, x^g - x^k\rangle \leq 0\} \tag{4.4}$$

and compute

$$x^{k+1} := P_{A_k}(x^g) \tag{4.5}$$

where $A_k := B_k \cap D_k \cap C$. Repeat iteration k with k is replaced by $k+1$.

The following lemma shows that u^k is closer to the solution set S than x^k. More precisely,

LEMMA 4.1 *Suppose that $u^k = P_{C_k}(x^k)$, then*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \left(\frac{\eta_k \delta}{\rho\|g^k\|}\right)^2 \|x^k - y^k\|^4 \ \forall x^* \in S,\ \forall k. \tag{4.6}$$

Proof The proof of this lemma can be done similarly as the proof of Lemma 3.1. So we give here only a sketch. For simplicity of notation, let $H_k := H_{z^k}$ and $d^k := x^k - y^k$.

Since $u^k = P_{C\cap H_k}(\bar{x}^k)$ and $x^* \in C \cap H_k$, by Lemma 2.5 one has

$$\|u^k - \bar{x}^k\|^2 \leq \langle u^k - \bar{x}^k, u^k - x^*\rangle$$

from which it follows that

$$\|u^k - x^*\|^2 \leq \|\bar{x}^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2. \tag{4.7}$$

Replacing

$$\bar{x}^k = P_{H_k}(x^k) = x^k - \frac{\langle g^k, x^k - z^k\rangle}{\|g^k\|^2} g^k$$

into (4.7), we obtain

$$\|u^k - x^*\|^2 \le \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - 2\langle g^k, x^k - x^*\rangle \frac{\langle g^k, x^k - z^k\rangle}{\|g^k\|^2} + \frac{\langle g^k, x^k - z^k\rangle^2}{\|g\|^2}.$$

Substituting $x^k = z^k + \eta_k d^k$ in to the last inequality we get

$$\|u^k - x^*\|^2 \le \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 + \left(\frac{\eta_k\langle g^k, d^k\rangle}{\|g^k\|}\right)^2 - \frac{2\eta_k\langle g^k, d^k\rangle}{\|g^k\|^2}\langle g^k, x^k - x^*\rangle$$

$$= \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \left(\frac{\eta_k\langle g^k, d^k\rangle}{\|g^k\|}\right)^2 - \frac{2\eta_k\langle g^k, d^k\rangle}{\|g^k\|^2}\langle g^k, z^k - x^*\rangle.$$

In addition, by the Armijo linesearch rule and δ-strong convexity of G, we have

$$\langle g^k, x^k - y^k\rangle \ge \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big] \ge \frac{\delta}{\rho}\|x^k - y^k\|^2.$$

Note that $x^* \in H_k$ we can write

$$\|u^k - x^*\|^2 \le \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \left(\frac{\eta_k\delta}{\rho\|g^k\|}\right)^2\|x^k - y^k\|^4. \qquad \square$$

THEOREM 4.1 *Under the assumptions of Theorem* 3.1*, the sequences* $\{x^k\}$ *and* $\{u^k\}$ *converge to the unique solution of Problem* (BP).

Proof From Lemma 4.1, it follows that $\|u^k - x^*\| \le \|x^k - x^*\|$ for every k and $x^* \in S$. Hence, by the definition of B_k, $S \subseteq B_k$ for every k. Furthermore, by induction, we can see that $S \subseteq D_k$ for every k. Thus, $S \subseteq A_k := B_k \cap D_k \cap C$.

On the other hand, by definition of D_k, we have $x^k = P_{D_k}(x^g)$ for every k. Since $x^{k+1} \in D_k$, one has

$$\|x^k - x^g\| \le \|x^{k+1} - x^g\| \quad \forall k.$$

Thus, $\lim \|x^k - x^g\|$ exists and therefore the sequence $\{x^k\}$ is bounded.

Now we show that the sequence $\{x^k\}$ is asymptotically regular, i.e. $\|x^{k+1} - x^k\| \to 0$ as $k \to \infty$. Indeed, since $x^k \in D_k$ and $x^{k+1} \in D_k$, by convexity of D_k, $\frac{x^{k+1}+x^k}{2} \in D_k$. Then from $x^k = P_{D_k}(x^g)$, by using the strong convexity of the function $\|x^g - .\|^2$, we can write

$$\begin{aligned}
\|x^g - x^k\|^2 &\le \left\|x^g - \frac{x^{k+1} + x^k}{2}\right\|^2 \\
&= \left\|\frac{x^k - x^g}{2} + \frac{x^{k+1} - x^g}{2}\right\|^2 \\
&= \frac{1}{2}\|x^g - x^{k+1}\|^2 + \frac{1}{2}\|x^g - x^k\|^2 - \frac{1}{4}\|x^{k+1} - x^k\|^2
\end{aligned}$$

which implies that

$$\frac{1}{2}\|x^{k+1} - x^k\|^2 \le \|x^g - x^{k+1}\|^2 - \|x^g - x^k\|^2.$$

Note that $\lim \|x^k - x^g\|$ exists, we obtain $\|x^{k+1} - x^k\| \to 0$.

On the other hand,

$$\begin{aligned}\|x^k - u^k\| &= \|x^k - x^{k+1} + x^{k+1} - u^k\| \\ &\leq \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\|.\end{aligned}$$

Since $x^{k+1} \in B_k$, by definition of B_k, $\|x^{k+1} - u^k\| \leq \|x^k - x^{k+1}\|$. Thus, we have

$$\|x^k - u^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - x^k\|,$$

which together with $\|x^{k+1} - x^k\| \to 0$ implies $\|u^k - x^k\| \to 0$ as $k \to \infty$.

Next we show that any cluster point of the sequence $\{x^k\}$ is a solution to Problem (EP). Indeed, let $\bar{x}$ be any cluster point of $\{x^k\}$. For simplicity of notation, without loss of generality, we may assume that x^k converges to $\bar{x}$. We consider two distinct cases:

Case 1 $u^k = x^k$ at Step 1 for infinitely many k. In this case, clearly, $\bar{x}$ solves (EP).

Case 2 $u^k = x^k$ at Step 1 for only a finitely many k. Then, according to the algorithm, we may assume that $u^k = P_{C_k}(x^k)$ for every k. Applying Lemma 4.1 for some $x^* \in S$, we have

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \left(\frac{\delta\eta_k}{\rho}\right)^2 \|x^k - y^k\|^4 \quad \forall k,$$

which implies

$$\left(\frac{\delta\eta_k}{\rho}\right)^2 \|x^k - y^k\|^4 \leq \big(\|x^k - x^*\| - \|u^k - x^*\|\big)\big(\|x^k - x^*\| + \|u^k - x^*\|\big).$$

By using the triangle inequality $\|x^k - x^*\| - \|u^k - x^*\| \leq \|x^k - u^k\|$, we get

$$\left(\frac{\delta\eta_k}{\rho}\right)^2 \|x^k - y^k\|^4 \leq \big(\|x^k - u^k\|\big)\big(\|x^k - x^*\| + \|u^k - x^*\|\big) \quad \forall k.$$

Since $\{u^k\}$, $\{x^k\}$ are bounded and $\|u^k - x^k\| \to 0$ as $k \to \infty$, taking the limit in both sides of the last inequality we obtain $\lim_k \eta_k \|x^k - y^k\| = 0$.

We distinguish two distinct cases:

Case 1 $\limsup_k \eta_k > 0$. In this case, there exist subsequences $\{x^{k_i}\} \subseteq \{x^k\}$, $\{y^{k_i}\} \subseteq \{y^k\}$ such that $\lim_i \|x^{k_i} - y^{k_i}\| = 0$. Thus, four sequences $\{x^{k_i}\}$, $\{y^{k_i}\}$, $\{u^{k_i}\}$, $\{z^{k_i}\}$ converge to the same point $\bar{x}$. By the same arguments as above, we can see that $\bar{x}$ solves (EP) and that these sequences converge to $\bar{x}$.

Case 2 $\lim_k \eta_k = 0$. Since $\{x^k\}$ is bounded and y^k is the unique solution of problem

$$\min\left\{ f(x^k, y) + \frac{1}{\rho}\Big[G(y) - G(x^k) - \langle \nabla G(x^k), y - x^k\rangle\Big] :\ y \in C \right\}. \qquad \mathrm{CP}(x^k)$$

whose objective function is lower semicontinuous, by the Berge Maximum Theorem ([27] Theorem 19), the sequence $\{y^k\}$ is bounded too. Moreover, the solution of Problem CP(x), as a function of x, is continuous ([27] Theorem 19). Then, without loss of generality, we may assume that y^k converges to some $\bar{y}$. Using again the fact that y^k solves CP(x^k), we obtain in the limit that

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big]$$
$$\leq f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - \bar{x}\rangle\Big], \quad \forall y \in C. \tag{4.8}$$

On the other hand, by the Armijo linesearch rule, for $m_k - 1$, there exists $g^{k,m_k-1} \in \partial_2 f(z^{k,m_k-1}, z^{k,m_k-1})$, such that

$$\langle g^{k,m_k-1}, x^k - y^k\rangle < \frac{1}{\rho}\Big[G(y^k) - G(x^k) - \langle \nabla G(x^k), y^k - x^k\rangle\Big].$$

Taking the limit as $k \to \infty$, we see that z^{k,m_k-1} converges to $\bar{x}$, g^{k,m_k-1} converges to some $\bar{g} \in \partial_2 f(\bar{x}, \bar{x})$, and the last inequality becomes

$$\langle \bar{g}, \bar{x} - \bar{y}\rangle \leq \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big]. \tag{4.9}$$

Since $\bar{g} \in \partial_2 f(\bar{x}, \bar{x})$ we get

$$f(\bar{x}, y) \geq f(\bar{x}, \bar{x}) + \langle \bar{g}, y - \bar{x}\rangle \quad \forall y \in C.$$

In particular, $\langle \bar{g}, \bar{x} - \bar{y}\rangle \geq -f(\bar{x}, \bar{y})$. Combining with (4.9) we obtain

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[G(\bar{y}) - G(\bar{x}) - \langle \nabla G(\bar{x}), \bar{y} - \bar{x}\rangle\Big] \geq 0. \tag{4.10}$$

From (4.8) and (4.10) follows

$$0 \leq f(\bar{x}, y) + \frac{1}{\rho}\Big[G(y) - G(\bar{x}) - \langle \nabla G(\bar{x}), y - \bar{x}\rangle\Big] \quad \forall y \in C,$$

which implies that $\bar{x}$ is a solution of EP. Then, from $\|u^k - x^k\| \to 0$, we can conclude that every limit point of $\{u^k\}$ is also a solution to EP.

Finally, we show that $\{x^k\}$ converges to $s := P_S(x^g)$. To this end, let x^* be any cluster point of $\{x^k\}$. Then, there exists a subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to x^*$ as $j \to \infty$. By the definition of s and $x^* \in S$, one has

$$\|s - x^g\| \leq \|x^* - x^g\| = \lim_j \|x^{k_j} - x^g\| \leq \limsup_k \|x^k - x^g\| \leq \|s - x^g\|$$

where the last inequality follows from the fact that $x^{k+1} = P_{A_k}(x^g)$ and $s \in S \subseteq A_k$ for every k. Hence $\lim \|x^k - x^g\| = \|s - x^g\| = \|x^* - x^g\|$. Since $x^* \in S$, $s = P_S(x^g)$ and the projection of x^g onto S is unique, we have $x^* = s$, and therefore $x^k \to s$ as $k \to \infty$. Then, from $\|x^k - u^k\| \to 0$, it follows that $u^k \to s$ as $k \to \infty$. □

An important special case of EPs is the variational inequality of the form

$$\text{Find } x^* \in C : \langle F(x^*), x - x^*\rangle \geq 0 \quad \forall x \in C, \tag{VI}$$

where $C \subseteq I\!R^n$ is closed, convex and $F : C \to I\!R^n$. Suppose that F is continuous and pseudomonotone on C with respect to every solution of Problem (VI). We recall that F is pseudomonotone on C with respect to $x^* \in C$, if

$$\langle F(x^*), x^* - y\rangle \le 0 \Rightarrow \langle F(y), y - x^*\rangle \ge 0 \quad \forall y \in C,$$

Let

$$f(x, y) := \langle F(x), y - x\rangle. \tag{4.11}$$

It has been shown (see e.g. [22] page 65 and [20]) that a point x^* is a solution of Problem (VI) if and only if it is a solution of the EP(C, f) with f defined by (4.11). Since F is continuous, f is continuous too. In addition, it is easy to see that if F is pseudomonotone on C with respect to $x^* \in C$, then f is pseudomonotone with respect to x^* on C. So Algorithm 2 can be applied to this case.

5. Numerical examples

In this section, we apply Algorithm 1 to solve an equilibrium model arising from Nash-Cournot oligopolistic EPs of electricity markets. This model has been investigated in some research papers (see e.g. [28,29]). To test the algorithm, we take the example in [29]. In this example, there are n^c companies, each company i may possess I_i generating units. Let x denote the the vector whose entry x_i stands for the the power generating by unit i. Following [28,29] we suppose that the price p is a decreasing affine function of the σ with $\sigma = \sum_{i=1}^{n^g} x_i$ where n^g is the number of all generating units, that is

$$p(x) = 378.4 - 2\sum_{i=1}^{n^g} x_i = p(\sigma).$$

Then the profit made by company i is given by

$$f_i(x) = p(\sigma)\sum_{j\in I_i} x_j - \sum_{i\in I_i} c_j(x_j).$$

where $c_j(x_j)$ is the cost for generating x_j. As in [29] we suppose that the cost $c_j(x_j)$ is given by

$$c_j(x_j) := \max\{c_j^0(x_j), c_j^1(x_j)\}$$

with

$$c_j^0(x_j) := \frac{\alpha_j^0}{2}x_j^2 + \beta_j^0 x_j + \gamma_j^0, \quad c_j^1(x_j) := \alpha_j^1 x_j + \frac{\beta_j^1}{\beta_j^1 + 1}\gamma_j^{-1/\beta_j^1}(x_j)^{(\beta_j^1+1)/\beta_j^1},$$

Table 1. The lower and upper bounds for the power generation of the generating units and companies.

Com.	Gen.	$x^g_{\min}$	$x^g_{\max}$	$x^c_{\min}$	$x^c_{\max}$
1	1	0	80	0	80
2	2	0	80	0	130
2	3	0	50	0	130
3	4	0	55	0	125
3	5	0	30	0	125
3	6	0	40	0	125

Table 2. The parameters of the generating unit cost functions.

Gen.	α_j^0	β_j^0	γ_j^0	α_j^1	β_j^1	γ_j^1
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 3. Results computed with some starting points and regularization parameters.

Iter(k)	τ	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	Cpu(s)
0	0.1	0	0	0	0	0	0	
160		46.6543	32.1476	15.0017	21.7795	12.4989	12.4982	24.9914
0	0.5	0	0	0	0	0	0	
273		46.6588	32.1428	15.0101	21.5109	12.6344	12.6331	35.9894
0	0.9	0	0	0	0	0	0	
338		46.6595	32.1195	15.0333	21.1765	12.8010	12.7992	50.5911
0	0.1	30	20	10	15	10	10	
113		46.6518	32.1343	15.0112	21.6789	12.5487	12.5491	17.0353
0	0.5	30	20	10	15	10	10	
191		46.6599	32.1230	15.0300	21.5192	12.6299	12.6299	24.9602
0	0.9	30	20	10	15	10	10	
225		46.6599	32.0659	15.0862	21.2464	12.7656	12.7657	31.8398

where $\alpha_j^k, \beta_j^k, \gamma_j^k$ $(k = 0, 1)$ are given parameters.

Let $x_j^{\min}$ and $x_j^{\max}$ be the lower and upper bounds for the power generating by the unit j. Then the strategy set of the model takes the form

$$C := \left\{ x = (x_1, ..., x^{n^g})^T : x_j^{\min} \le x_j \le x_j^{\max} \quad \forall j \right\}.$$

Let us introduce the vector $q^i := (q_1^i, ..., q_{n^q}^i)$ with

$$q_j^i := 1, \text{ if } j \in I_i, \text{ and } q_j^i = 0, \text{otherwise},$$

and then define

$$A := 2\sum_{i=1}^{n^c}(1 - q_j^i)(q^i)^T, \quad B := 2\sum_{i=1}^{n^c} q^i (q^i)^T, \tag{5.1}$$

$$a := -387.4 \sum_{i=1}^{n^c} q^i, \quad c(x) := \sum_{j=1}^{n^g} c_j(x_j). \tag{5.2}$$

Then the oligopolistic equilibrium model under consideration can be formulated by the following EP (see [29] Lemma 7):

$$x^* \in C : f(x, y); = \left(\left(A + \frac{3}{2}B\right)x + \frac{1}{2}By + a\right)^T (y - x) + c(y) - c(x) \geq 0 \quad \forall y \in C.$$

We test Algorithm 1 for this problem with corresponds to the first model in [28] where three companies ($n^c = 3$) are considered, and the parameters are given in Tables 1 and 2

We implement Algorithm 1 in Matlab R2008a running on a Laptop with Intel(R) Core(TM) i3CPU M330 2.13GHz with 2GB Ram with regularization function $G(x) = \|x\|^2$ and parameter $\tau = \frac{1}{\rho}$. To terminate the Algorithm, we use the stopping criteria $\frac{\|x^{k+1}-x^k\|}{\max\{\|x^k\|,1\}} \leq \epsilon$ with a tolerance $\epsilon = 10^{-4}$. The computational results are reported in Table 3 with some starting points and regularization parameters.

Table 3 shows that the number of iterations and computational time depend crucially on the regularization parameters and starting points.

6. Conclusion

We have extended a projection algorithm developed in [9] to EPs where the bifunctions are pseudomonotone with respect to the solution sets. We then have combined the proposed algorithm with a cutting technique to develop a hybrid projection-cutting algorithm for minimizing the norm over the solution set of an EP whose bifunction is pseudomonotone with respect to its solution set. The latter bilevel problem arises from the Tikhonov regularization method for pseudomonotone EPs. We have tested a proposed algorithm on a Nash-Cournot oligopolistic equilibrium model of electricity markets. Some computed numerical results are reported.

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