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GENERIC SINGULARITIES OF THE ATTAINABLE SET ON SURFACES WITH BOUNDARY

Hy Duc Manh
Le Quy Don Technical University
236, Hoang Quoc Viet St., Hanoi, Vietnam
hdmanh_ktqs@yahoo.com UDC 517.9

We study generic singularities of attainable sets of smooth control systems on compact connected surfaces with boundary. Bibliography: 10 *titles. Illustrations*: 8 *figures.*

1 Introduction

A control system is given by a smooth family of vector fields parametrized by a control parameter whose range is a closed smooth compact manifold of finite dimension or a disjoint union of such manifolds (possibly, with different dimensions). A *generic* system or a system in *general position* is a system in an open everywhere dense set in the space of systems equipped with the (sufficiently) smooth Whitney topology.

For a control system the *cone* of a point in the phase space is the positive linear hull of the set of admissible velocities of the system at this point and the *steep domain* is the set of all points of the phase space such that their cones do not contain the zero vector. For a generic system, in the closure of the steep domain, the boundary of the cone of a point consists of *limiting directions* of velocities of the system at this point. Generic singularities of the field of limiting directions and the attainable set on a smooth compact surface without boundary were studied in [1]–[3]. Singularities of fields of limiting directions of generic smooth dynamical inequalities on surfaces were studied in [4]. For surfaces with generic boundary singularities of the field of limiting directions were studied in [5], and generic singularities of the positive orbit of the closed starting set which lie in the interior of this orbit were investigated in [6].

In this paper, we classify singularities of attainable sets of generic systems on a smooth compact orientable surface with boundary in the case where the starting set is a smoothly embedded curve. The main results are formulated in Section 2 and are proved in Section 3.

2 The Main Results

2.1. Singularities of the field of directions on the boundary and the starting set. By the *boundary* of a surface we mean a nondegenerate zero level set of some smooth function G (i.e., everywhere on the boundary, $G = 0$ and $dG \neq 0$). Without loss of generality

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we assume that the system is defined in the domain where $G \geqslant 0$. We can reach this at each connected component of the phase space by replacing the sign of G. The domain is referred to as *feasible*.

We assume that the *starting set* (i.e., the set from which the system is allowed to start a motion) is a smooth embedding of the circle S^1 into the phase space, i.e., a smooth connected compact curve. In the case of general position, this curve may intersects the boundary, but without tangency.

On surfaces without boundary, singularities of the field of limiting directions of a generic system are stable under small perturbations of this system (cf. [1, 3]). Therefore, in the case of general position, singularities of the field of limiting directions not larger than 1, i.e., of codimension 0 and 1, can be observed on the boundary and on the starting set. For generic systems singular points of the field of limiting directions of codimension 0 are *regular points*, where the branches of the field of limiting directions are smooth, whereas singularities of the field of limiting directions of codimension 1 are *passing points*, *∂-passing points*, and *regular zero-points*.

Passing and *∂-passing points* are observed even in the case of generic bidynamical systems: at such a point, admissible velocity fields are collinear, but the differential of the angle between them differs from zero and the fields themselves are not tangent to the collinearity line and have either the same or opposite direction respectively. For systems with a large number of control parameters such phenomena are observed on limiting directions of admissible velocity fields.

Regular zero-points appear if the dimension of a control parameter is greater than 0. At such a point, the boundary of the convex hull of the set of admissible velocities contains the zero velocity, is smooth in a neighborhood of this point, and has the first order tangency with the limiting direction, whereas the limiting direction is transversal to the boundary of the steep domain.

The following assertions are useful to study singularities of the attainable set on the boundary of the surface and the starting set.

Proposition 2.1. *For a generic control system on a smooth surface with boundary each point on the boundary* (*the starting set*) *is related to one of the following five types*:

(a) *an interior point of the local transitivity zone,*

(c) *a passing point,*

(d) *a ∂-passing point,*

(e) *a regular zero-point.*

Moreover, in each of the last three cases, at this point the boundary, the starting set, and the boundary of the steep domain are transversal, whereas the limiting directions are not tangent to the boundary (*the starting set, the boundary of the steep domain*).

Proposition 2.2. *For a generic control system on a smooth surface with boundary the branch of the field of limiting directions may touch the boundary* (*or the starting set*) *only at its regular point with first order tangency, but not at its common point of the boundary and the starting set* (*the boundary respectively*).

We omit the proof of Propositions 2.1 and 2.2 since it immediately follows from the Thom transversality theorem [7] and the results of [1].

A point z is called a *point with first order* (*second order*) tangency if this point lies in steep domain and a phase curve of some of the branches of the field of limiting directions touches at this point the boundary (the starting set respectively) with first order tangency.

2.2. Structure of the boundary of the attainable set. We choose an orientation of surfaces (phase spaces) and denote by L_1 and L_2 the fields of minimal and maximal limiting directions respectively.

It is clear that for any starting set the boundary of the attainable set lies in the union of the closure of the domain and the boundary of the surface. Consider a point z of this boundary and denote by $\eta_i(z)$ ($\eta_i^+(z)$ and $\eta_i^-(z)$ respectively) the limiting line passing trough the point z of the branch L_i of the field of limiting directions (outgoing from z and incoming to z respectively). We denote by $O^+(S)$ and $O^-(S)$ (or O^+ and O^- if no misunderstanding arises) the positive and negative orbits of a subset S of the phase space respectively.

A family of limiting lines of the branch of the field of limiting directions of a control system is said to be *structurally stable* if the corresponding family of limiting lines of any sufficiently close system is transformed to the original family by a homeomorphism of the phase space that is close to the identity mapping.

We begin with the case where the starting set S does not intersect the boundary of the surface. In this case, singularities of the boundary of the attainable set on the boundary of the surface are studied in the same way as in [6]. Moreover, the set of singular lines (counterparts of separatrices and cycles of usual vector fields; cf. [6]) should include limiting lines that are

(a) tangent to the starting set and

(b) outgoing from the points of intersection of the surface boundary and the starting set.

We will see that these new singular lines can lie on the boundary of the attainable set. The following assertion holds.

Proposition 2.3. *For a generic control system on a surface with boundary the set of singular limiting lines is structurally stable.*

This means that singular limiting lines of a generic system and any sufficiently close system may be mapped into each other by a homeomorphism of the phase space that is close to the identity mapping, i.e., for a generic system the set of these limiting lines is rough (cf. [8]–[10]). We omit the proof of Proposition 2.3 since it is similar to that in [2].

The following two theorems describe the structure of the boundary of the attainable set at points lying in the starting set, outside this set, and on its boundary.

Theorem 2.1. *If the starting set S is a compact smoothly embedded curve on a compact orientable surface M with boundary, then for a generic control system the germ of the boundary of the attainable set O^+ at every point z of S that does not lie on the boundary is one of the following three germs*:

($s1$) (S, z),

($s2$) $(\eta_i^-(z) \cup (S \cap \partial O^+), z)$, *where z is not a point of second-type tangency.*

($s3$) $(\eta_i^+(z) \cup (S \cap \partial O^+), z)$, *where $\eta_i^+(z)$ is tangent to S with first order tangency.*

This assertion is a counterpart of the corresponding result in [3].

Theorem 2.2. *If the starting set S is a generic smoothly embedded compact curve on a compact orientable surface M with boundary, then for a generic control system the germ of the* *boundary of the attainable set* O^+ *at every point* z *of the intersection of the starting set with the boundary is one of the following five germs*:

($s4$) $(\partial M, z)$,

($s5$) $((S \cap \partial O^+) \cup (\partial M \cap O^+), z)$,

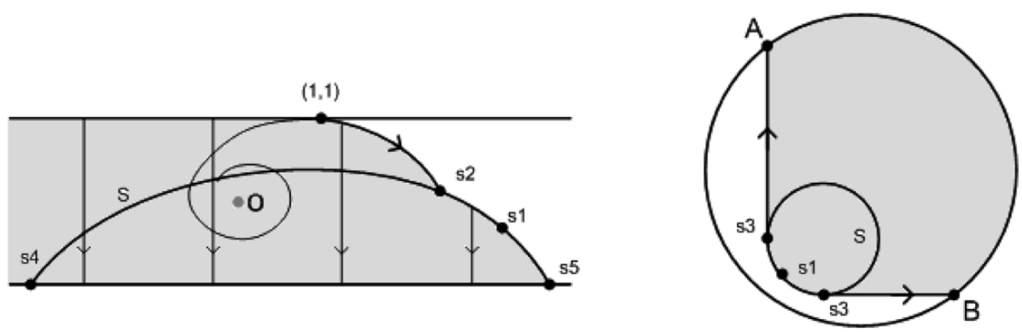
($s6$) $(\eta_i^+(z) \cup (\partial M \cap O^+), z)$,

($s7$) $(\eta_i^+(z) \cup (S \cap \partial O^+), z)$,

($s8$) $(\eta_1^+(z) \cup \eta_2^+(z), z)$,

where, at each singular point of type ($s6$), ($s7$), *and* ($s8$), *the limiting line* $\eta_i^+(z)$ *is not tangent to the boundary of the surface,* $i = 1, 2$.

Example 2.1. Suppose that a bidynamical system is given by the velocity fields $(x+y, -x+y)$ and $(0, -1)$ in the strip $|y| \leqslant 1$ on the plane $\mathbb{R}^2_{x,y}$, the starting set S is the circle of radius $3/2$ with center $(1, -2)$, and the motion can be started only with its part lying in this strip. The attainable set is presented by the shaded domain in Figure 1 (a), and the boundary point $(1, 1)$ is a point with first order tangency. The germ of this boundary O^+ has type ($s2$) at the point, where the limiting line outgoing from this point with tangency intersects the starting set, and type ($s5$) at the point where the circle intersects the boundary in the domain $x \geqslant 0$. Points between the singular points ($s2$) and ($s5$) have type ($s1$), whereas the point of intersection of the starting set with the boundary in the domain $x \leqslant 0$ provides a singularity of type ($s4$).



(a) (b)

FIGURE 1. Singular points on the boundary of the starting set

Example 2.2. For a bidynamical system given by the velocity fields $(0, 1)$ and $(1, 0)$ in the disk $(x-2)^2 + (y-2)^2 \leqslant 8$ on the plane $\mathbb{R}^2_{x,y}$ the positive orbit O^+ of the circle of radius 1 with center $(1, 1)$ is presented by the shaded domain in Figure 1 (b). The points $(0, 1)$ and $(1, 0)$ are points with second order tangency, and, at these points, the germ of the boundary O^+ has type ($s3$). Points of the starting set lying between two singular points of type ($s3$) have type ($s1$). The points $A = (0, 4)$ and $B = (4, 0)$ are also singular points (cf. [6]).

Remark 2.1. Theorems 2.1 and 2.2 generalize the results of [6] for systems on a compact connected orientable surface with boundary to the case where the closed starting set does not lie in the interior of its positive orbit.

2.3. Singularities of the attainable set. A point of the boundary of the attainable set that lies on the starting set is called a *singular point of type* $i = 0 \operatorname{div} 2$ if the closure of this set in a neighborhood of this point coincides with the corresponding set $y \geqslant g(x)$ (and also $y \leqslant g(x)$ for $i = 1$) with the value of the function g indicated in the second row of Table 1 under a suitable choice of smooth local coordinates x, y with the origin at this point.

TABLE 1. Singularities of the boundary of the attainable set

i	0	1	2
$g(x)$	0	$\|x\|$	$x\|x\|$

Theorem 2.3. *If the starting set is a smoothly embedded compact curve on a compact orientable surface with boundary, then for a generic control system the attainable set at each point of its boundary on the starting set has one of singularities of type* $i = 0 \operatorname{div} 2$. *Moreover, the germ of the orbit itself at this point has one of the corresponding singularities indicated in the second column of Table* 2*, whereas the germ of its boundary has one of types indicated in the third column of Table* 2 *respectively.*

TABLE 2. Singularities of the attainable set

Type	Singularity of O^+	Type of point of ∂O^+
0	$y \geqslant 0$	(s1) , (s4)
1	a) $y \geqslant \|x\|$	(s5), (s6), (s7), (s8)
	b) $y \leqslant \|x\|$	(s2)
	c) $\{y \leqslant -x,\ x \leqslant 0\} \cup \{y < x,\ x \geqslant 0\}$	(s2)
2	$y \geqslant x\|x\|$	(s3)

Remark 2.2. All singularities indicated in Table 2 are realized by a system and the starting set which are stable under small perturbations for any range of U, $\#U \geqslant 2$. We note that the complete list of generic singularities of the boundary of the attainable set on the surface with boundary without taking into account the nature of their appearance is not larger than the combined list of singularities presented in [6] and [3]. However, in the case of surfaces with boundary, the nature of their appearance is much richer.

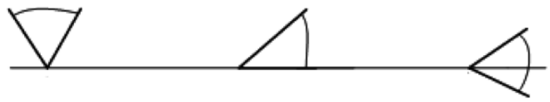
3 Singularities of Boundary of Attainable Set

In this section, we prove the main results of the paper. We first classify generic germs of the boundary of the attainable set and then find the corresponding normal forms.

Proof of Theorem 2.1. Consider a point z in the intersection of the boundary of the closure of the positive orbit O^+ and the starting set. We first prove Theorem 2.1 for points in the steep domain. By Proposition 2.1, in the steep domain, each point of the starting set is either a regular point of each branch of the field of limiting direction or a passing point (one of the branches if $\#U \geqslant 3$ and both branches if $\#U = 2$).

We consider only the case $\#U > 2$ since the arguments are similar for $\#U = 2$.

Let z be a regular point. We choose a smooth local system of coordinates with the origin at z in such a way that the starting set goes to the x-axis and one of admissible velocities has positive first component. In the case of general position, there are three essentially different locations of the cone of this point (Figure 2 (a)–(c)).

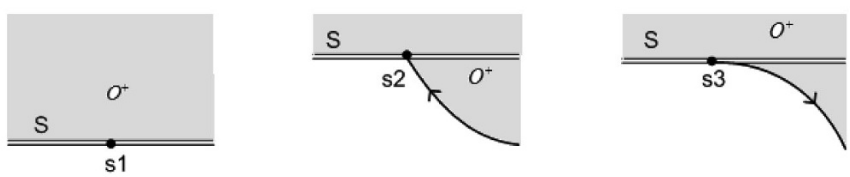


(a) (b) (c)

FIGURE 2. The cone of a regular point on the starting set in the steep domain

It is easy to see that the point z lies inside O^+ in case (c). This fact contradicts the condition $z \in \partial O^+$. Therefore, case (c) is impossible.

In case (a), three situations are possible. In the first one, each limiting line incoming to the point z does not lie in the closure of the attainable set. Then the germ of the boundary of the attainable set at this point has type ($s1$) (Figure 3 (a)). In the second situation, only one limiting line incoming to z lies in the closure and, in this case, the germ of this boundary at this point has type ($s2$) (Figure 3 (b)). Finally, in the third situation, both limiting lines incoming to z lie in the closure of the attainable set. But, in this case, the point z lies inside O^+, which contradicts the assumption that the point belongs to the boundary of the attainable set. Hence case (a) is impossible.



(a) (b) (c)

FIGURE 3. A point of the boundary of the starting set

In case (b), for a generic system z is a point with second order tangency. By Proposition 2.3, the limiting line incoming to this point without tangency with the starting set does not lie on the boundary of O^+. Two situations are possible: in a neighborhood of z, the limiting line touching the starting set lies in the half-plane $y \leqslant 0$ or in the half-plane $y \geqslant 0$. In the first situation, the germ has type ($s3$) at this point (Figure 3 (c)), whereas, in the second one, the point z lies in the interior of the attainable set which means that case (b) is impossible.

Further, if a point z belongs to the boundary of the steep domain, then, in the case of general position, it is either a ∂-passing point or a regular zero-point. However, in both cases, the point lies in the local transitivity zone (cf. [1]). Therefore, it cannot lie on the boundary of the orbit O^+ and, consequently, both situations are impossible.

Finally, if z is an interior point of the local transitivity zone, then, as above, it lies in the interior of the orbit O^+ and, consequently, this case is impossible. Theorem 2.1 is proved. ☐

Proof of Theorem 2.2. Let z lie in the intersection of the starting set and the boundary. By the Thom transversality theorem, In the case of general position, the starting set does not intersect the boundary of the surface at points of codimension 1 on the boundary, i.e., at points with first order tangency and at passing (∂-passing) points or regular zero-points. Therefore, z is not related to the above-listed types.

In the case of general position, the starting set and the boundary of the surface transversally intersect at z. We choose local coordinates with the origin at z in such a way that the boundary of the surface becomes the x-axis, the starting set goes to the y-axis, and the feasible domain goes to the half-plane $y \geqslant 0$.

If a point z belongs to the local transitivity zone, then the germ of the boundary of O^+ at this point has type ($s4$) (Figure 5 (a)). Otherwise, it is a regular point of each branch of the filed of limiting directions since, by Proposition 2.3 and the transversality theorem, the singular litiming lines of a generic system are not incoming to this point.

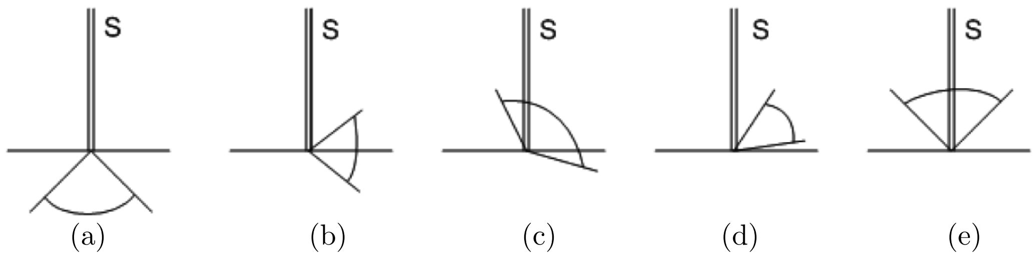


FIGURE 4. A point of the orbit boundary on the boundary of the surface

There are five essentially different cases of the location of the cone of this point relative to the tangents to the starting set and the boundary ∂M at this point (Figure 4 (a)–(e) (we note that in the case of general position, these tangents are not limiting directions).

In the first case, the cone of the point z "is directed towards" the interior of the complement to the feasible domain. The following two typical cases are possible: the direction vector $(0, -1)$ lies or not in the interior of the cone. In the second case, the cone lies in the third or fourth quadrant and the germ of the boundary O^+ at this point has type ($s5$) (Figure 5 (b)). In the first case, it has type ($s4$) (Figure 5 (a)).

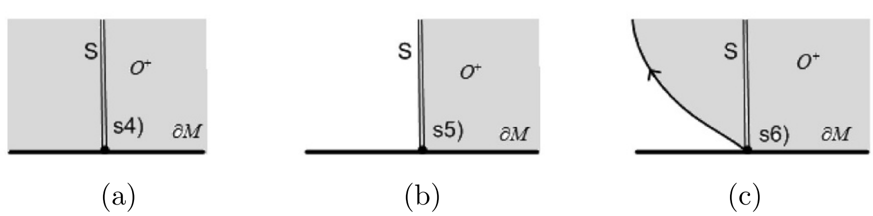


FIGURE 5. A point of the orbit boundary on the boundary of the starting set

In cases (b) and (c), the first limiting direction "is directed towards" the fourth quadrant, whereas the others "are directed towards" the first and third quadrant respectively. By Proposition 2.2, the limiting directions tangent neither the boundary of the surface nor the starting set. It is clear that if the second limiting direction lies in the first quadrant, then the germ of the boundary of O^+ at this point has type ($s5$). If the second limiting direction lies in the second quadrant, then the germ of the boundary of O^+ at this point has type ($s6$) (Figure 5 (c)).

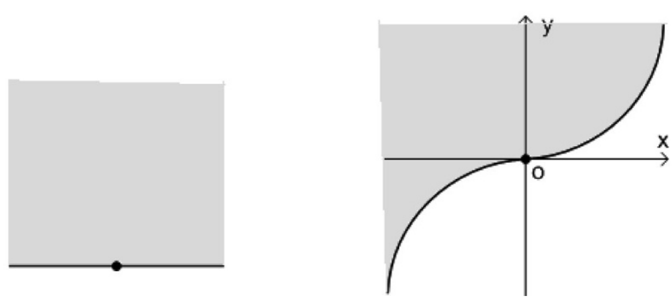
(a) (b)

FIGURE 6. A point of the orbit boundary on the boundary of the starting set

In case (d), the cone at z "is directed towards" the first quadrant. In this case, the germ of the boundary of O^+ at the point z has type ($s7$) (Figure 6 (a)).

Finally, in case (e), the cone at a point z "is directed towards" the feasible domain; moreover, one limiting direction "is directed towards" the first quadrant, whereas the other "is directed towards" the second quadrant. It is clear that the germ of the boundary O^+ at this point has type ($s8$) (Figure 6 (b)). Theorem 2.2 is proved. □

Proof of Theorem 2.3. For a generic system the germ of the boundary of O^+ at each point z of the starting set and on the intersection of the boundary of the surface and the starting set has one of the types indicated in Theorems 2.1 and 2.2. We consider these cases.



(a) (b)

FIGURE 7. A singular point of type 0 and 2

If, at a point z, the germ of the boundary of O^+ has type ($s1$) or ($s4$), then the germ of $(\partial O^+, z)$ is C^∞-diffeomorphic to the germ at the zero set of type 0 as indicated in Table 1 since the boundary of the surface and the starting set are smooth. Consequently, in this case, the germ of the orbit O^+ at this point also has singularity of type 0 as indicated in Table 2 (Figure 7 (a)).

By the smoothness of the boundary of the surface and the starting set, as well as by the transversality, at a point of type ($s5$), the germ $(\partial O^+, z)$ is C^∞-diffeomorphic to the germ of a set of type 1 indicated in Table 1 at zero, whereas the germ of the orbit O^+ at this point has singularity of type 1a) indicated in Table 2. At points of type ($s6$), ($s7$), ($s8$), the germ $(\partial O^+, z)$ is C^∞-diffeomorphic to the germ of a set of type 1 in Table 1 at zero since the limiting lines passing through this point are transversal and each of these limiting lines is transversal to the boundary of the surface. The germ of the orbit O^+ at these points has singularity of type 1a) as indicated in Table 2 (Figure 8 (a)).

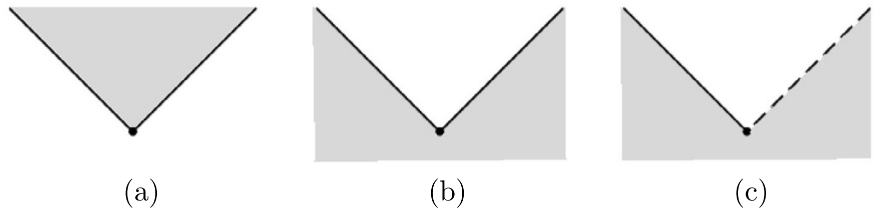


FIGURE 8. A singular point of type 1

As above, at a point of type ($s2$), the germ $(\partial O^+, z)$ is C^∞-diffeomorphic to the germ of a set of type 1 in Table 1 at zero and the germ of the orbit O^+ at this point has singularity of type 1b) in Table 2. In particular, if the limiting line incoming to this point does not lie on the boundary of O^+, then the germ of the orbit O^+ has singularity of type 1c) in Table 2 (Figure 8 (b)–(c) respectively).

Finally, if z is a singular point of type ($s3$), then the germ $(\partial O^+, z)$ is C^∞-diffeomorphic to the germ of a set of type 2 as indicated in Table 1 at zero since the limiting line outgoing from this point is tangent to the starting set with first order tangency, whereas this set is smooth. It is obvious that the germ of the orbit O^+ at this point also has singularity of type 2 as indicated in Table 2 (Figure 7 (b)). Theorem 2.3 is proved. □

Example 3.1. For the above bidynamical systems on the plane $\mathbb{R}^2_{x,y}$ with the same starting set the attainable set is presented by the shaded domain in Figure 1 (a)–(b). On the boundary of the attainable set, one can see singularities of types 0, 1, and 2.

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